A contribution to the theory of fertility and economic development

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Abstract

The aim of this research is to develop a theory for explaining economic development in a (neoclassical) growth model with endogenous fertility. The economy is comprised of overlapping generations of rational and identical individuals with preferences towards material consumption and the number of children (that directly enters the parent's lifetime utility function), and identical competitive firms producing with a constant-returns-to-scale technology with no externalities. From a theoretical perspective, the distinguishing feature of this work is that endogenous fertility *per se* is able to explain the existence of low and high development regimes. It then provides alternative reasons (history driven or expectations driven) why some countries enter a development trajectory with high GDP and low fertility and others experience under-performances with low GDP and high fertility. The model is also able to reproduce fertility fluctuations and explain the U.S. baby busts and baby booms observed in the last century.

Keywords Economic development; Endogenous fertility; Local and global indeterminacy; OLG model

JEL Classification C61; C62; J1, J22; O41

1 INTRODUCTION

"...if the theory is complicated, it's wrong." Richard P. Feynman

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The present research is intended to explain the reasons why some countries develop whereas others are entrapped in stagnation or poverty. For doing this, it considers a parsimonious (neoclassical) economic growth model with overlapping generations (OLG), rational and *identical* agents, and no externalities except those intrinsic to the assumption of endogenous fertility.[1] The work directly enters the debate about history versus self-fulfilling expectations as in the seminal articles of Krugman (1991) and Matsuyama (1991). This is because the assumption of a Constant-Inter-temporal-Elasticity of Substitution (CIES) utility function, where the number of children directly enters as a consumption good, and the assumption of perfect foresight make fertility able to be a source of indeterminacy. Then, it introduces a novel (*utility-driven*) mechanism in the OLG literature of neoclassical growth confirming the empirical findings of Mankiw et al. (1992), Hall and Jones (1999), Palivos (1995), Moe (1998) and the subsequent literature, for which there exist different convergence groups of (developed and developing or less developed) countries all around the world for different reasons.[2] The article provides a new theoretical reason why economies starting with similar initial conditions have experienced different development trajectories [this is the case, for instance, of South Korea and the Philippines, an example mentioned in Lucas (1993) whose initial conditions were similar at the beginning of the sixties] or initially poorer (resp. richer) economies have entered a phase of sustained economic development (resp. under-development) with larger values of GDP per capita and lower fertility rates (resp. smaller values of GDP per capita and higher fertility rates). Countries belonging to the former group are, for instance, several European economies if we take the end of World War II as a starting point, whereas in the latter group of countries there may be found some Latin American economies by considering the same initial time period. From a theoretical perspective, the present contribution complements the work of Palivos (1995), who provides similar insights in a continuous-time optimal growth model with infinite horizon optimising agents encompassing the child quantity-quality trade-off. In his model, in fact, the possibility of multiple steady states and problems of coordination failures are *production-driven* (the production set should be non-convex: the net marginal product of capital should not be monotonically decreasing in capital). The work now proceeds by discussing the main

[1] An additional child causes an increase in future output but it also reduces the capital-labour ratio. As parents do not take these externalities into account in a decentralised setting, the number of children emerging in the market economy may be different from the socially optimal one. This result, however, emerges a direct consequence of the "dynamic inefficiency" outcome of the OLG model. The works of van Groezen et al. (2003), van Groezen and Meijdam (2008) and Fanti and Gori (2012) provide some solutions to these external effects of children in OLG economies with public pensions.

[2] According with Mankiw et al. (1992), rich and poor countries perform differently because of differences in physical capital accumulation and educational attainments, whereas Hall and Jones (1999) find that differences in output per worker between rich and poor countries are essentially caused by differences in social infrastructures (i.e., quality of institutions and government policies). Palivos (1995) considers endogenous fertility in an infinite horizon optimal growth model and finds evidence for the existence of distinct convergence groups of countries: one with high GDP and low fertility; another with low GDP and high fertility.

motivations and the links to the related literature.

Demographic variables (fertility, longevity, migration rates) are recognised to play a preeminent role as determinants of long-term macroeconomic outcomes of nations. The interaction between demography and income is the object of a growing body of studies from both empirical and theoretical perspectives [Kirk (1996); Fogel (2004); Galor (2005, 2011, 2012); Hall and Jones (2007); Lorentzen et al. (2008); Cervellati and Sunde (2011, 2013); Ashraf and Galor (2011, 2013); Fiaschi and Fioroni (2014); Livi-Bacci (2017)]. Human beings in Western countries have experienced tremendous improvements in both the standard of living and quality of life in the past two centuries, and currently there is no clear consensus on which were the main sources of this development, i.e., human capital accumulation [Glaeser et al. (2004)] and/or quality of institutions [Acemoglu et al. (2001)], and then on the different policy consequences that these alternative explanations may cause. The influence of longevity and fertility on economic growth (a concept referred to the growth of an economic variable such as GDP per capita) and development (a multi-dimensional phenomenon related - amongst other things - to fertility behaviours, life expectancy, quality of institutions, happiness, poverty, the distribution of income and so on) has led several economists to consider them as endogenous variables and tackle this issue in models that - since the pioneering works of Leibenstein (1957) and Becker (1960) that have originated the so-called New Home Economics - have given rise to the Unified Growth Theory [Galor and Weil (2000); Galor and Moav (2002, 2004); Elgin (2012)]. This theory aims at explaining the process of economic growth and development on the basis of the interaction between technology (endogenous technological progress) and human capital formation by showing that endogenous fertility [Galor and Weil (2000)] and endogenous human evolution [Galor and Moav (2002)] are relevant causes of the demographic transition (that is, a transition from stagnation to growth is accompanied by a demographic shift from high to low birth and death rates [Jones and Tertilt (2006); Cervellati et al. (2016)]. Cervellati and Sunde (2005) have also provided an additional explanation of such a transition based on the interaction amongst life expectancy, human capital and endogenous technological progress.[3] A distinctive feature of this literature was to have built on a unified theoretical and empirical framework where explaining the process of development across nations and continents. This process is usually divided in three phases: 1) Malthusian epoch (a long time period that ends up almost to 1750 AC). 2) Post-Malthusian regime (1750-1870). 3) Modern growth regime (1870-today). A common feature of the works belonging to this literature is to have substantially modified the standard OLG model by including several additional ingredients, such as human capital accumulation and adult mortality. Except some of the seminal contributions of Galor and his coauthors, which tend to emphasise the importance of fertility and child mortality in the process of the economic and demographic transition, the mechanisms leading towards a phase of pre-industrialisation to a phase intensive industrial production are essentially adult-

[3]While the number of children (quantity) becomes too costly, parents may decide to invest more in the quality of a small number of (better educated and healthy) children.

mortality-driven [Cervellati and Sunde (2005, 2011, 2015); Fiaschi and Fioroni (2014)]. In these more recent works, fertility is basically introduced in order to explain the phases of the demographic transition, but it does not represent the trigger for the transition amongst the various stages of development.

On the side of the theories of endogenous fertility - that try to explain economic reasons why families have children - there are plenty of pieces of research in the economic literature. According to some of these theories, the number of children works exactly out as a material good for parents, and thus benefits and costs of having children can actually be explained through the standard consumer demand theory: parents are selfish and the number of children directly enters their utility (child quantity). This way of modeling fertility can then be viewed as a weak form of altruism towards children or pure egoism [Galor and Weil (1996); Zhang and Zhang (1998)]. Other forms of altruism towards children have actually been studied later on. Specifically, Becker and Barro (1988), Barro and Becker (1989) and Benhabib and Nishimura (1989) have concentrated on models where parents derive utility from consumption and the utility of their offspring (pure altruism), whereas Andreoni (1989) and Strulik (2004) have introduced a peculiar form of impure altruism towards children for which parents draw utility from both child quantity and child quality. A motive for having children different from those mentioned above is represented by the old-age security hypothesis [Bental (1989); Cigno (1992); Raut and Srinivasan (1994); Boldrin and Jones (2002)]. In this case, children act as an investment good for parents because they provide them support during old age. From this point of view, historically fertility declines follow the rise in pension systems, especially in Europe.[4]

There are several contributions analysing problems of economic development in growth models with endogenous fertility that do not strictly belong to the Unified Growth Theory literature. These works aim at explaining the reasons why some countries experience high values of GDP and low fertility rates, whereas others remain entrapped in a situation where GDP is low and fertility is high. In most cases, scholars use the OLG framework, as it represents a natural basis where including demographic variables, although there exist some works in continuous-time growth models with infinite horizon optimising agents [Wang et al. (1994); Palivos (1995); Palivos and Scotese (1996); Palivos et al. (1997); Yip and Zhang (1997)]. Within the class of OLG models with finite lived individuals, which is our reference framework, we mention here the works of Galor and Weil (1996), Blackburn and Cipriani (2002), de la Croix and Doepke (2003, 2004), Varvarigos and Zakaria (2013), Fanti and Gori (2014), Nakamura and Seoka (2014) and Nakamura and Mihara (2016) that come to light several distinct reasons for the existence multiple stationary equilibria. All the these models share the same characteristic: they have modified the standard OLG framework substantially. In particular, in the works of Blackburn and Cipriani (2002), Varvarigos and Zakaria (2013), Fanti and Gori (2014) and Nakamura and Mihara (2016), aiming at explaining the processes of the economic and demographic transitions

[4]See Ehrlich and Lui (1997) for a well developed survey on different theories and motives for having children.

in models where children are viewed as a consumption good, the main determinant that generates multiple development regimes is adult mortality (alternatively driven by education, private and public health spending together or public health spending alone). The main findings are that poor countries tend to have high fertility and mortality rates and a low level of GDP per capita, according to the empirical evidence on the demographic transition. Differently, de la Croix and Doepke (2003, 2004) and concentrate on fertility differentials (between rich and poor individuals) affecting the accumulation of human capital in models where parents face a quantity-quality trade-off, pointing out that an increase in inequality lowers average education, increases fertility and reduces economic growth [de la Croix and Doepke (2003)] and the importance of endogenous fertility in the financing of educational policies for inequality and growth [de la Croix and Doepke (2004)]. Private education may produce higher growth when human capital inequality between rich and poor individuals is little, whereas public education may promote growth and reduce fertility differential in economies with a larger human capital inequality. In a subsequent work where differential fertility matters, Nakamura and Seoka (2014) find that what makes possible escaping from poverty poor people is a competition between the accumulation of human capital by the rich and the accumulation of children by the poor people. Finally, Galor and Weil (1996) examine the relationship between fertility and economic growth by including *gender differences*. The reduction in fertility and the increase in output growth in their model is due to a threefold effect: an increase in capital per worker causes a raise in women's relative wages. This produces a reduction in fertility because of the increase in the relative cost of raising children. The reduction in fertility eventually favours the increase in capital per worker. Multiple development regimes are possible because of the positive effect on the rate of output growth caused by women joining the labor force.

The present article adds additional explanations (history driven or expectations driven) for the existence of a high regime of development (the accumulation of capital is high and fertility rates are low), which resembles the Modern growth regime, and a low regime of development (the accumulation of capital is low and fertility rates are high), which resembles the Malthusian regime, to the OLG literature. This result is obtained in a very basic framework with a constant-returns-to-scale technology and without externalities. This is worth to be mentioned especially because - depending on some key preference parameters - the model is able to give rise to coordination failures. In this case, in fact, agents know that there exist multiple stationary states but do not know how to coordinate themselves to avoid Pareto dominated outcomes. This problem may be overcome only with an element of self-fulfilling expectations. This result is uncommon in the OLG literature without externalities and, to the best of our knowledge, it is the first time that indeterminacy is fertility-driven instead of labour-supply-driven, as instead is usual in the presence of endogenous labour and externalities [Cazzavillan et al. (1998); Cazzavillan (2001); Cazzavillan and Pintus (2006)].

The rest of the article proceeds as follows. Section 2 develops a simplified version of the model of child quantity and time cost of children of Galor and Weil (1996), where - different from them - rational *homogeneous* individuals have a lifetime CIES utility function, which includes log-preferences as a sub case, and identical firms produce by employing a standard neoclassical technology with constant returns to scale (we avoid distinguishing between physical labour and mental labour as it unnecessarily complicates the analysis). Section 3 characterises the conditions for the existence of stationary equilibria and then studies the equilibrium dynamics of the model by clarifying the mail theoretical findings with numerical simulations (global analysis). It also details some distinct development scenarios the economy is able to capture. Section 4 outlines the conclusions. An Appendix provides some mathematical results useful throughout the main text.

2 THE MODEL

This section builds on a modified (simplified) version of Galor and Weil (1996). Specifically, it considers a general equilibrium closed economy with endogenous fertility and (neoclassical) growth describing the characteristics of consumers and producers. There exists a single commodity that can be consumed or accumulated. There are no external effects on both the consumers' side (expect those intrinsically related to the assumption of endogenous fertility) and producers' side.

The OLG closed economy under scrutiny is populated by a continuum of perfectly rational and identical individuals of measure N_t per generation [Diamond (1965)], where $t = 0, 1, 2, ...$ is the time index. The life of the typical agent is divided into childhood and adulthood. As a child, an individual does not make economic decisions. He is assumed to spend time in the parent's household and to consume resources directly from him. There is no child mortality. As an adult, an individual works and takes care of children when he is young, and retires when he is old. The N_t young members of generation t are then economically active and overlap for one period (youth) with N_{t-1} old individuals that belong to generation $t-1$ and for one period (old-age) with N_{t+1} young individuals that belong to generation $t+1$.

When young, an individual is endowed with 2 units of time.[5] We assume that raising children is time consuming.[6] This is a reasonable assumption especially if one wants to describe labour markets in developed countries, where female participation rates (and opportunity costs) are higher than in developing and underdeveloped ones [World Bank (2013)]. Specifically, the child bearing technology requires an exogenous fraction $q < 2$ of parent's time endowment to raise a child. It represents parent's foregone earnings as the time cost for bearing a single child increases. Then, by letting $n_t > 0$ be the number of children at time t, qn_t is the time needed to care for n_t descendants of a parent that belongs to generation t. This assumption directly follows Galor and Weil (1996), who consider a model of child quantity. It implies that child

[5]By considering a time endowment larger than one allows obtaining a growth factor of population larger than, smaller than or equal to one, i.e. a population that grows, decreases or it is stationary over time.

[6]For empirical evidence about the assumption of time cost of children see Guryan et al. (2008).

rearing proportionally reduces the time endowment of parents (the time required to care for children cannot be spent working), i.e. the marginal time cost of children is constant.[7] The remaining share $\ell_t = 2 - qn_t > 0$ of time is supplied to firms in exchange for wage w_t per unit of labour.

Individuals consume only in the second period of their life, i.e. they behave as pure life-cyclers. This assumption is quite usual in the OLG literature at least since Woodford (1984) and Reichlin (1986). It has been used, amongst others, by Galor and Weil (1996), Grandmont et al. (1998), Antoci and Sodini (2009), Gardini et al. (2009), Gori and Sodini (2014) and Matsuyama et al. (2016) in models dealing with issues different than the present one. The budget constraint of the young individual representative of generation t is $s_t = w_t \ell_t$, i.e. labour income is entirely saved (s_t) to consume one period later.[8] When old, an individual retire and consumption (C_{t+1}) is constrained by the amount of resources saved when young plus expected interest accrued from time t to time $t+1$, so that $C_{t+1} = R^e_{t+1} s_t$ where R^e_{t+1} is the expected interest factor, which will become the realised interest factor at time $t+1$. Therefore, the lifetime budget constraint can be expressed as follows:

$$C_{t+1} = R^e_{t+1} w_t (2 - qn_t), \tag{1}$$

where $qn_t < 2$ must hold to satisfy the individual time endowment.

An adult individual of generation t has preferences towards the number of children when young and consumption when old. The way of modelling children as a desirable good that directly enters parent's utility is similar to Eckstein and Wolpin (1985), Eckstein et al. (1988) and Galor and Weil (1996). Under this assumption, parents are selfish and give birth to children not for being supported when they will be old or to enjoy their well-being but exclusively to increase their own utility. The lifetime utility index of the individual representative of generation t is described by a twice continuously differentiable utility function $U_t(n_t, C_{t+1})$. We specify this function by using the following Constant Inter-temporal Elasticity of Substitution (CIES) formulation:

$$U_t(n_t, C_{t+1}) = \frac{n_t^{1-\gamma}}{1-\gamma} + \frac{C_{t+1}^{1-\sigma}}{1-\sigma}, \tag{2}$$

where $\gamma > 0$ ($\gamma \neq 1$) and $\sigma > 0$ ($\sigma \neq 1$) measure the constant elasticity of marginal utility with respect to fertility and consumption, respectively. This functional form of a utility function (general isoelastic specification) is aimed for generality. With this formulation, consumption

[7] This is actually a quite standard assumption in the literature. It represents a difference with respect to Palivos (1995), who finds the possibility of multiple equilibria in a continuous-time model of optimal (exogenous) growth with endogenous fertility when the net rate of return on capital is non-decreasing, i.e. when there are economies of scale in raising children (implying that the marginal time cost of children is decreasing) or due to the shape of the term nk in the resource constrain of the economy.

[8] With this formulation for the budget constraint, the only input required to care about a child is time (a fraction of time spent on child-caring cannot be spent working), so that the opportunity cost of children is proportional to the wage per unit of labour.

and children are substituted inter-temporally with a (constant) elasticity different from one. In the particular case $\gamma = 1$ and $\sigma = 1$, the expression in (2) boils down to $U_t(n_t, C_{t+1}) = \ln(n_t) + \ln(C_{t+1})$ and consumption and children are substituted inter-temporally with an elasticity equal to one. The assumption that, in general, consumption and children are substituted over time with an elasticity different from one is crucial for the results shown later in this article. With the formulation for lifetime utility as those expressed in (2), with no young material consumption, $1/\sigma$ (resp. $1/\gamma$) may be interpreted as proxy for measuring the (constant) inter-temporal elasticity of substitution in consumption (resp. children). An increase in σ (resp. γ) causes a decline in the marginal utility of material consumption (resp. fertility) when C_{t+1} (resp. n_t) increases. This in turn implies that an individual unwillingly accepts to deviate from a consumption pattern that guarantees consumption smoothing (lower willingness to substitute intertemporally). Empirical evidence [Hall (1988); Jones and Schoonbroodt (2010); Havranek et al. (2015)] finds that the inter-temporal elasticity of substitution in consumption is consistently smaller than one ($\sigma > 1$), whereas $1/\gamma$ is a measure for the the elasticity of inter-generational substitution between consumption and (the number of) children. Córdoba and Ripoll (2015) find that this index is significantly larger than one ($\gamma < 1$), meaning that the consumption of material goods and the consumption of children tend to be substitutes over time.

By taking factor prices as given, the individual representative of generation t maximises the expression in (2) - with respect to n_t and C_{t+1} - subject to (1) and $n_t < 2/q$. However, by substituting out (1) in (2) the maximisation problem can be reduced to

$$\max_{n_t \in (0,2/q)} \left\{ \frac{n_t^{1-\gamma}}{1-\gamma} + \frac{\left[R^e_{t+1} w_t (2 - q n_t)\right]^{1-\sigma}}{1-\sigma} \right\}. \tag{3}$$

Therefore, the first order conditions for an interior solution are given by:

$$n_t^{-\gamma} = q (R^e_{t+1} w_t)^{1-\sigma} (2 - q n_t)^{-\sigma}. \tag{4}$$

Eq. (4) implies that at the optimum the marginal utility of an extra child should be equal to the (indirect) marginal utility of consumption. Basically, it tells us how much consumption to give up when old to consume one more child when young by keeping utility unchanged. The expression in (4) can also be rewritten as follows:

$$\frac{n_t^{\gamma}}{(2 - q n_t)^{\sigma}} = \frac{(R^e_{t+1} w_t)^{\sigma-1}}{q}. \tag{5}$$

The formulation in (5) allows us to clarify the effects (at the individual level) of a change in wage income on the demand for children in the case of a CIES utility function. In particular, the left-hand side of (5) is an increasing function of n_t. An increase in the wage causes a twofold effect. On the one hand, it implies that children become more costly relative to material consumption. Then, at the optimum, an individual wants to substitute the consumption of children when young for the consumption of material goods when old (substitution effect) through this channel.

On the other hand, it also implies that an individual gets richer as the value of his overall time endowment increases (income effect). As an individual always offers a positive amount of his time endowment to firms (labour supply), what eventually determines the sign of the change in the demand for children following a wage increase is the sign of the income effect. In particular, the optimal number of children is a normal (resp. an inferior) good when $\sigma > 1$ (resp. $\sigma < 1$), thus producing a positive (resp. negative) income effect. Therefore, an increase in wage income increases (resp. reduces) the demand for children through this channel. When $\sigma = 1$ children are neither normal nor inferior goods (the substitution and income effects cancel each other out in that case) and the demand for children is independent of the (capitalised) wage income. Definitely, if children are a normal good ($\sigma > 1$) the substitution effect and the income effect are of opposite sign and the final effect of a change in wage income on the demand for children is a priori uncertain. In contrast, if children are an inferior good ($\sigma < 1$) the substitution effect and the income effect are both negative so that the demand for children reduces when the wage increases (this is in accord with the Beckerian tradition).[9] Therefore, a CIES utility function is able per se to provide a reason why individual fertility reacts differently to changes in wage income. This adds a novel utility-driven mechanism that can potentially explain the historical pattern of the demographic transition. The analysis of the relationship between the number of children and GDP per young person will be clarified later in this article in both cases $\sigma > 1$ and $\sigma < 1$ when we will account for the macroeconomic (general equilibrium) effects of the model (the interest factor and the wage rate will depend on both the capital stock per young person and the fertility rate).

Firms are identical and act competitively on the market. Different from the work of Raut and Srinivasan (1994), that takes into account a production function with a Hicks-neutral total factor productivity affected by the size of the working population, we assume that at time t firms produce a homogeneous good (Y_t) by combining capital (K_t) and labour (L_t) by means of the following Cobb-Douglas technology with constant returns to scale:

$$Y_t = AF(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha}, \tag{6}$$

where $0 < \alpha < 1$ is the output elasticity of capital and $A > 0$ is a constant production scaling parameter that weights technological progress (Total Factor Productivity). Profits are given by $AK_t^{\alpha} L_t^{1-\alpha} - w_t L_t - R_t K_t$. The temporary equilibrium condition in the labour market at time t is given by $L_t = \ell_t N_t = (2 - qn_t)N_t$, i.e. amount of labour hired by firms is equal to the mass

[9] With this regard, see the recent empirical work of Córdoba and Ripoll (2015). This result can be clearly ascertained by imposing the restriction $\gamma = \sigma$. In this particular case, in fact, from the first order condition in (5) one can get the unique closed-form expression for the optimal value of individual fertility, which is the following: $n_t = \frac{2}{q\left[1+q^{\frac{1-\sigma}{\sigma}}\left(R_{t+1}^e w_t\right)^{\frac{1-\sigma}{\sigma}}\right]}$. Although this simplification allows getting explicit expressions for fertility and consumption, it does not represent a good approximation to characterise all the development scenarios the model is able to generate. Then, given also the different empirical estimates on the two parameters, we will continue to study the model by keeping γ and σ at different values.

of young individuals of generation t times the fraction of time they spend working, which is a function of the fertility rate because of the assumption of time cost of children. An increase in the time devoted to child rearing activities reduces the time an individual can work. By assuming full depreciation of capital, a unit price of output and taking factor prices as given, profits maximisation by the representative firm implies that the wage and the interest factor are equal to the marginal product of labour and the marginal product of capital, respectively, that is:

$$w_t = (1-\alpha)Ak_t^{\alpha}(2-qn_t)^{-\alpha}, \tag{7}$$

$$R_t = \alpha Ak_t^{\alpha-1}(2-qn_t)^{1-\alpha}, \tag{8}$$

where $k_t := K_t/N_t$ is the stock of capital per young person.

The market-clearing condition in the capital market is determined by equating aggregate investment and aggregate saving and it is given by $K_{t+1} = S_t = s_t N_t$. As $N_{t+1} = n_t N_t$ determines the evolution of population, equilibrium implies:

$$k_{t+1} = \frac{s_t}{n_t}, \tag{9}$$

where $s_t = w_t(2-qn_t)$ and n_t is determined by the individual first order conditions.

Then, by using (4), (7), (8), (9) and knowing that individuals have perfect foresight, so that $R_{t+1} = \alpha Ak_{t+1}^{\alpha-1}(2-qn_{t+1})^{1-\alpha}$, the dynamics of the economy is characterised by the following two-dimensional map:

$$M : \begin{cases} k_{t+1} &= Q_1(k_t, n_t) := \dfrac{A(1-\alpha)k_t^{\alpha}(2-qn_t)^{1-\alpha}}{n_t} \\ n_{t+1} &= Q_2(k_t, n_t) := \frac{1}{q}\left(2 - k_t^{\frac{-\alpha^2}{1-\alpha}}(2-qn_t)^{-\alpha+\frac{1}{(1-\sigma)(1-\alpha)}} n_t^{-1-\frac{\gamma}{(1-\sigma)(1-\alpha)}} B\right) \end{cases}, \tag{10}$$

where

$$B := A^{\frac{-(1+\alpha)}{1-\alpha}}(1-\alpha)^{\frac{-\alpha}{1-\alpha}}\alpha^{\frac{-1}{1-\alpha}}q^{-\frac{1}{(1-\sigma)(1-\alpha)}}. \tag{11}$$

The former equation in (10) describes the evolution of capital per young person from time t to time $t+1$ and the latter equation describes the evolution of fertility from time t to time $t+1$. We recall that the capital stock k_t is a state variable and fertility n_t is a control variable. We also note that as $k_0 := K_0/N_0$, the initial value of the control variable (n_0) does not affect the initial value of the state variable (k_0). As can be seen from map M, the assumptions of CIES preferences and perfect foresight are crucial as they contribute to determine a dynamic expression for the number of children, which instead is absent in the case of log-utility (a case for which it is not necessary to specify any expectations formation mechanism about the future interest factor). In fact, if $\gamma = 1$ and $\sigma = 1$ fertility is constant and given by $n = 1/q$ so that the dynamics of the economy is characterised by the uni-dimensional map $k_{t+1} = q(1-\alpha)Ak_t^{\alpha}$, from which one can get the unique (globally asymptotically stable) stationary equilibrium $k^* = [q(1-\alpha)A]^{\frac{1}{1-\alpha}}$, as in a standard OLG model à la Diamond (1965).

3 EXISTENCE OF STATIONARY EQUILIBRIA AND EQUILIBRIUM DYNAMICS

This section begins with the analysis of the model by identifying the stationary equilibria of the map. By the first equation of map M we have that at the stationary state it must hold

$$k = h(n) := (2 - qn) \left(\frac{A(1-\alpha)}{n} \right)^{\frac{1}{1-\alpha}}. \tag{12}$$

The steady-state values of the fertility rate are solutions of the following equation:

$$g(n) := \frac{1}{q} \left(2 - n^{\frac{2\alpha - 1}{(1-\alpha)^2} + \frac{\gamma}{(\sigma-1)(1-\alpha)}} (2 - qn)^{-\frac{\alpha}{1-\alpha} - \frac{1}{(\sigma-1)(1-\alpha)}} D \right) = n, \tag{13}$$

where $D := B\left[(1-\alpha) A\right]^{-\frac{\alpha^2}{(1-\alpha)^2}}$. The expression in (12) introduces a negative relationship between n and k at the stationary state. This implies that larger values of the capital stock are related to lower values of the fertility rate. This is important to be stressed as under the assumption that children are a normal good ($\sigma > 1$) the model overcomes the paradox between individual choices and macro behaviour. In fact, empirical evidence shows the existence of a positive relationship between wage income and the number of children (positive income effect) at the individual level, whereas - at an aggregate level - larger values of GDP are associated with lower fertility rates in the last stages of the economic and demographic transition. Although it is not an easy task assessing the effects of changes in fertility on growth, as population variables change endogenously along the process of economic development, there exists evidence confirming the importance of fertility declines for explaining GDP growth [Jones and Tertilt (2006); Sinding (2009); Ashraf et al. (2013)].

In order to characterise the number of equilibria, we now study the behaviour of g when $n \to 0^+$ and when $n \to (2/q)^-$.

Lemma 1 *Let*

$$\tilde{\gamma} := \frac{(1-2\alpha)(\sigma-1)}{1-\alpha}, \tag{14}$$

be a threshold value of γ. *(1) If* $\sigma > 1$ *and* $\gamma > \tilde{\gamma}$ *or if* $\sigma < 1$ *and* $\gamma < \tilde{\gamma}$ *then* $\lim_{n \to 0^+} g(n) = 2/q$. *(2) If* $\sigma > 1$ *and* $\gamma < \tilde{\gamma}$ *or if* $\sigma < 1$ *and* $\gamma > \tilde{\gamma}$ *then* $\lim_{n \to 0^+} g(n) = -\infty$. *(3) If* $\sigma > 1$ *then* $\lim_{n \to (2/q)^-} g(n) = -\infty$. *(4) If* $\sigma < 1$ *then* $\lim_{n \to (2/q)^-} g(n) = 2/q$. *(5) In addition,* g *has a critical point*

$$n_{crit} := \frac{(4\sigma - 2\gamma - 4)\,\alpha - 2\sigma + 2\gamma + 2}{\left[\alpha^2 (\sigma - 1) + (\sigma - \gamma)\,\alpha - \sigma + \gamma\right] q}, \tag{15}$$

in the interval $(0, 2/q)$ *if and only if* $\gamma < \tilde{\gamma}$. *(6) If*

$$\left[(1-\alpha)\,\gamma + 2\alpha\sigma - 2\alpha - \sigma + 1\right]\left[(1-\alpha)\,\gamma + 3\alpha\sigma - 2\alpha - 2\sigma + 1\right](\sigma - 1) > 0 \tag{16}$$

then one or two inflection points for g *can exist in the interval* $(0, 2/q)$.

Proof. Results (1)-(4) are obtained by studying the sign of the exponents of the terms n and $2-qn$ in the expression $g(n)$ in (13). Results (5) and (6) follow by the study of $g'(n)$, by noting that its sign coincides with the sign of the following first degree polynomial:

$$p(n) := q\left[\sigma(\alpha^2+\alpha-1)-(\alpha-1)\gamma-\alpha^2\right]n+(2\sigma-1)(1-2\alpha)-2\gamma(1-\alpha), \tag{17}$$

and by the study of $g''(n)$, by noting that its sign coincides with the sign of the following second degree polynomial:

$$P(n) := P_2n^2+P_1n+P_0, \tag{18}$$

where

$$P_2 := \left[\left(\sigma-\frac{1}{3}\gamma-\frac{2}{3}\right)\alpha-\frac{2}{3}\sigma+\frac{1}{3}\gamma+\frac{1}{3}\right]\left[(\sigma-1)\,\alpha^2+(\sigma-\gamma)\alpha-\sigma+\gamma\right]q^2, \tag{19}$$

$$P_1 := \left[\frac{4}{3}q(\alpha\gamma-2\alpha\sigma+2\alpha-\gamma+\sigma-1)\right]\left[\alpha\gamma-3\alpha\sigma+2\alpha-\gamma+2\sigma-1\right], \tag{20}$$

$$P_0 := -\left\{\frac{8}{3}\left[(\sigma-1)\alpha^2+(-4\sigma+\gamma+4)\alpha+2\sigma-\gamma-2\right]\right\}\left[\left(\sigma-\frac{1}{2}\gamma-1\right)\alpha-\frac{1}{2}\sigma+\frac{1}{2}\gamma+\frac{1}{2}\right]. \tag{21}$$

■

Remark 2 *From an empirical point of view, $1-2\alpha>0$ as $\alpha<0.5$ generally holds [Krueger (1999); Gollin (2002); Jones (2004)]. Values of the capital share in income larger than 0.5 in this kind of models may make sense by considering a broader concept of capital including human components [see Chakraborty (2004) and the literature cited therein]. However, all numerical simulations presented in this work adopt the usual notion of physical capital and make use of a standard value around 0.33. These experiments are presented for illustrative purposes and do not serve as calibration outcomes that are indeed possible.*

From Lemma (1) the proposition characterising the existence and number of stationary equilibria for map M follows.

Proposition 3 *[Existence and number of stationary equilibria]. (1) If $\sigma>1$ and $\gamma>\tilde{\gamma}$ then there exists a unique interior fixed point. (2) If $\sigma>1$ and $\gamma<\tilde{\gamma}$ [this case is meaningful only when $\alpha<1/2$] then there exists a threshold value $\tilde{A}>0$ such that for $A<\tilde{A}$ there exist zero interior fixed points, whereas for $A>\tilde{A}$ there exist two interior fixed points. (3) If $\sigma<1$ and $\gamma>\tilde{\gamma}$ then there exists a unique interior fixed point. (4) If $\sigma<1$ and $\gamma<\tilde{\gamma}$ [this case is meaningful only when $\alpha>1/2$] then there exists a threshold value $\bar{A}>0$ such that for $A<\bar{A}$ there exist two interior fixed points, whereas for $A>\bar{A}$ there exist no interior fixed points.*

Proof. We separate the proof with respect to the cases introduced in the statement of the proposition. (1) By Lemma 1 it is possible to show that $g'(n)$ has constant and negative sign in the interval $(0, 2/q)$. In fact, given $\hat{\gamma} := [\alpha^2(\sigma-1)+\sigma(\alpha-1)]/(\alpha-1)$ we have that if $\gamma < \hat{\gamma}$ then $p(n)$ defines a negatively sloped linear function that vanishes at a point $n < 0$, whereas if $\gamma > \hat{\gamma}$ then $p(n)$ defines a positively sloped linear function that vanishes at a point $n > 2/q$. (2) By Lemma 1, g has an interior maximum point $n_{\max} := n_{crit}$ by the study of $g''(n)$ it follows that no inflection points do exist and then g is always concave. (3) It is easy to show that g is increasing in the interval $(0, 2/q)$ and $\lim_{n \to 0^+} g(n) = -\infty$, and $g(2/q) = 2/q$ and $g'(2/q) = 0$ (then, the the graph of g lies above the 45 degree line). To verify that g is concave we consider two cases. If $\tilde{\gamma} < \gamma < \frac{(3\sigma-2)\alpha+1-2\sigma}{\alpha-1}$ then the discriminant of the polynomial in (18) is negative so that there do not exist interior inflection points and $g''(n) \leq 0$ in the interval $(0, 2/q)$. If $\gamma > \frac{(3\sigma-2)\alpha+1-2\sigma}{\alpha-1}$ then the roots of the polynomial in (18) are located outside the interval $(0, 2/q)$ and $g''(n) \leq 0$ in the interval $(0, 2/q)$. (4) Function g always admits a minimum point $n_{\min}$ in the interval $(0, 2/q)$. First, define the following threshold value of γ: $\bar{\gamma} := (\alpha^2 - 4\alpha + 2)(\sigma - 1)/(1 - \alpha)$, where $\bar{\gamma} < \tilde{\gamma}$. Second, to inquire about the number of fixed points, it is convenient to distinguish between two cases. If $\bar{\gamma} < \gamma < \tilde{\gamma}$ then g is decreasing and convex in the interval $(0, n_{\min})$, where $n_{\min} := n_{crit}$; it is increasing and convex in the interval $(n_{\min}, 2/q)$; it is increasing and concave in the interval $(f_1, 2/q)$, where $f_1 \in (n_{\min}, 2/q)$ is the unique inflection point; it eventually ends up at point $2/q$ with $g(2/q) = 2/q$ and $g'(2/q) = 0$. If $\gamma < \bar{\gamma}$ then g has two inflection points f_1 and f_2 in the interval $(0, 2/q)$. Function g is decreasing and concave in the interval $(0, f_1)$; it is decreasing and convex in the interval $(f_1, n_{\min})$; it is increasing and convex in the interval $(n_{\min}, f_2)$; it is increasing and concave in the interval $(f_2, 2/q)$ with $g(2/q) = 2/q$ and $g'(2/q) = 0$. ■

The geometry of existence and number of stationary states of map M, outlined in Proposition 3, is illustrated in Panels (a)-(d) of Figure 1. The stationary states of the map are the intersection points of $g(n)$ with the 45° degree line. Depending on the parameter configurations there exist either uniqueness (Panels (a) and (c)) or multiplicity (Panels (b) and (d)). In the case of multiple equilibria, the Total Factor Productivity parameter plays a crucial role in determining the gap between the two states. Interestingly, this is in line with the result of the endogenous lifetime model of Chakraborty (2004).

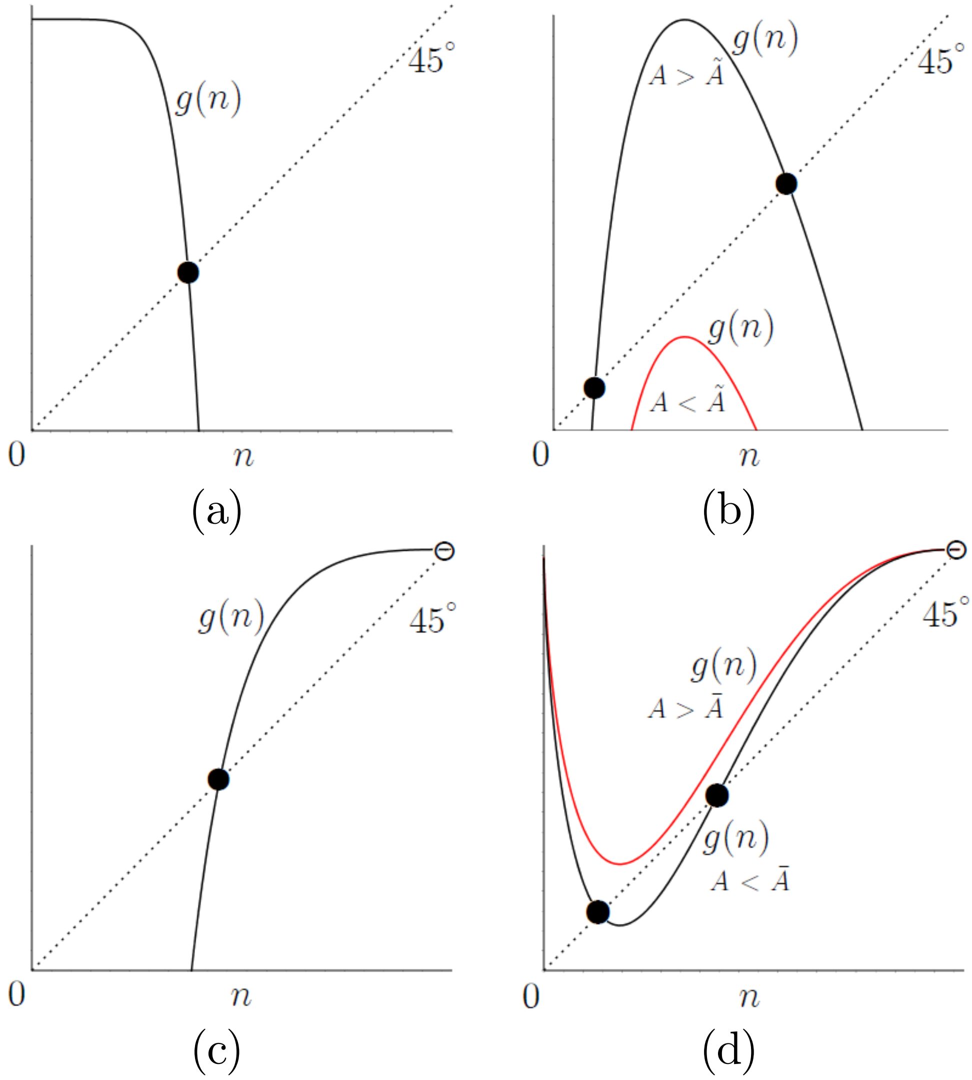

Figure 1. Geometry of existence and number of stationary states (denoted by the black point) of map M as detailed in Proposition 3. (a) Case 1: $\sigma > 1$ and $\gamma > \tilde{\gamma}$. There is a unique stationary state. (b) Case 2: $\sigma > 1$ and $\gamma < \tilde{\gamma}$ [this case is meaningful only when $\alpha < 1/2$]. If $A < \tilde{A}$ there are no stationary states (red curve). If $A > \tilde{A}$ there are two stationary states (black curve) (c) Case 3: $\sigma < 1$ and $\gamma > \tilde{\gamma}$. There is a unique stationary state. (d) Case 4: $\sigma < 1$ and $\gamma < \tilde{\gamma}$ [this case is meaningful only when $\alpha > 1/2$]. If $A < \bar{A}$ there are two stationary states (black curve). If $A > \bar{A}$ there are no stationary states (red curve). In Case 3 and Case 4, $n = 2/q$ (denoted by the empty circle) is not a stationary state of the map. However, it can play an important role for the dynamics of the model, as is shown later in this article.

Let us compare now a situation where there exist two stationary states under the assumption that children are a normal good ($\sigma > 1$). Corresponding to a stationary state with a lower value of capital accumulation (less developed economy) individuals have a lower wage than a stationary state with a larger value of capital per young person (developed economy). This means that ceteris paribus (i.e. given the same interest factor) in a context of underdevelopment individuals choose to have less children than individuals actually do in a developed economy. However, this is only part of the story. In fact, associated with a lower value of capital accumulation there is higher factor of interest in comparison with an economy with a larger value of the capital stock. This induces individuals to increase the number of children as these are a normal good and the interest factor is an element that helps capitalising wage income over time. The ultimate outcome of these opposite effects is to generate a higher level of fertility in an economy with the lowest long-term stock of capital. Interestingly, without introducing exotic assumptions regarding the behaviour of agents this theory can explain the empirical behavior on the demand for children both at microeconomic and macroeconomic levels.

Of course, the existence of one or more stationary states is economically meaningful only if there exist trajectories that can lead an economy to converge towards them. The following results provide a classification of the equilibrium dynamic properties of map M.

Lemma 4 *If (a) $\sigma < 1$ and $\gamma > \sigma$ or (b) if $\sigma > 1$ and $\gamma > \frac{\sigma(2-qn_{ss})}{2-qn_{ss}}$ then the determinant of the Jacobian matrix associated with map M is positive, where n_{ss} is the generic stationary state value of n.*

Proposition 5 *[Local stability of stationary equilibria]. Under the hypotheses of Lemma 4, if the graph of g at n_{ss} intersects the 45° lines from below we have that (k_{ss}, n_{ss}) is a saddle, where k_{ss} is the generic stationary state value of k obtained by the expression in (12).*

Proposition 6 *If $\sigma < 1$ and $\gamma > -\frac{(1-\sigma)(\alpha^2 qn_{ss}-2\alpha+2)+qn_{ss}\sigma}{2-qn_{ss}}$ then if the graph of g at n_{ss} intersects the 45° lines from above we have that (k_{ss}, n_{ss}) is not a saddle.*

Proof. The proof of Lemma 4 and Propositions 5 and 6 are in the Appendix. ■

In the light of previous results, we can give an insight about the stability of equilibrium points in the different cases outlined in Figure 1. Specifically, under the assumptions introduced in the propositions, the left-located stationary state in Panel (b) identifies a saddle point. This means that given an initial condition on the stock of capital, there exists a unique choice on the control variable that brings the economy to lie on the trajectory converging towards it. The same result holds for the unique stationary state identified in Panel (c) and for the right-located one of Panel (d). A result similar to the one outlined in the comparison between the stationary equilibria discussed above holds (at least locally) on the stable manifold of the different saddles detailed in the various scenarios, defining a development path with k and n being negatively correlated between them. Instead, nothing can be said in the case detailed in Panel (a). In fact, in this case equilibrium dynamics can have different properties. In order to clarify the outcome in this case, Figure 2 shows the possibility that the equilibrium is locally indeterminate. This means that although this is the unique stationary state of the model, then for a given initial condition of the state variable there exist infinite choices on the control variable (the number of children) that may lead towards it.

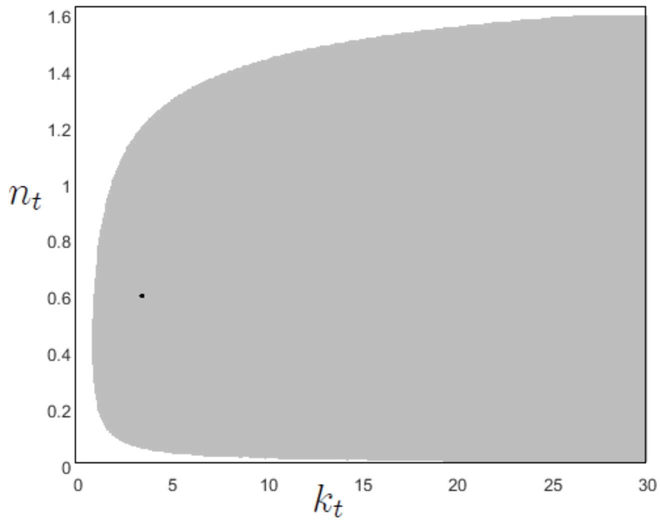

Figure 2. Parameter set: $\alpha = 0.354$, $\sigma = 4.01993$, $A = 1.61$, $q = 0.94$ and $\gamma = 1.4485$. Local indeterminacy of the unique attractor of the map. The grey-coloured region is the basin of attraction of the attractor. The white area is the region of unfeasible trajectories.

When the hypotheses of the previous results are violated, the classification of the stationary states of map M is quite cumbersome from an analytical point of view, with conditions that are very difficult to be interpreted economically. However, it is interesting to note that unlike most of the results of the literature on indeterminacy, this model can produce an outcome for which there exists a unique stationary state and infinite trajectories corresponding to the same initial condition on k leading towards it so that problems of coordination failures may arise. These trajectories are characterised by different values of capital accumulation and fertility choices. More details are given in the Appendix.

As is shown in Proposition 3, there exists cases with respect to which there are multiple stationary states. This makes it possible to have also global indeterminacy. This implies that the model is able to generate distinct development trajectories leading towards different long-term values of capital per young person (state variable) and fertility (choice variable). For instance, with the following parameter values (which are plausible values also from an empirical point of view) $\alpha = 0.33$, $\sigma = 4.7$, $q = 0.53$ (which represents almost the 30 per cent of the time endowment of parents for the caring of children), $A = 1.545$, $\gamma = 0.183$ we obtain two stationary states. One of these two states represents the under-development outcome (low GDP and high fertility), i.e. the low development regime, and its coordinate values are given by $(k^*, n^*) = (1.23, 1.13)$. The other, instead, represents the paradigm for developed countries (high GDP and low fertility), i.e. the high development regime, and its coordinate values are given by $(k^{**}, n^{**}) = (1.96, 0.88)$. The long-term low development regime is a locally indeterminate fixed point, whereas the long-term high development regime is a saddle (see Figure 3).

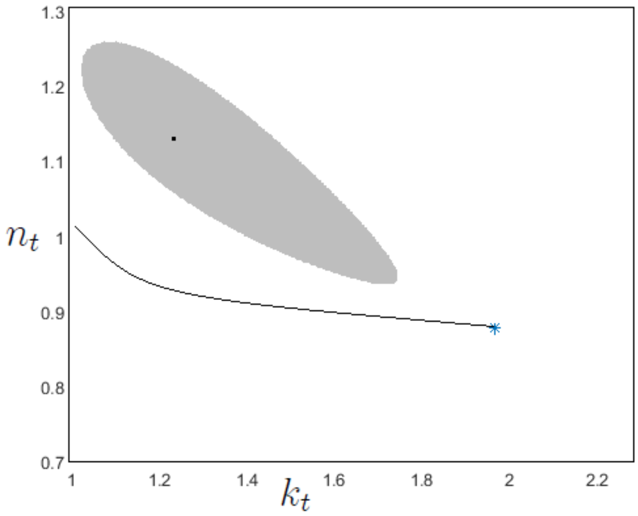

Figure 3. Parameter set: $\alpha = 0.33$, $\sigma = 4.7$, $q = 0.53$, $A = 1.545$, $\gamma = 0.183$. Global indeterminacy. The grey-coloured region is the basin of attraction of (k^*, n^*). The black line represents an approximation of a branch of the stable manifold of the saddle (k^{**}, n^{**}) on which an economy converges towards the developed state.

On the existence of feasible trajectories with $n_t \to 2/q$ and $k_t \to 0$ when $\sigma < 1$. By exploring the two equations of map M, it is not possible to have feasible trajectories such that $n_t \to 2/q$ and $k_t \to 0$ when $\sigma < 1$. In this case, point $(0, 2/q)$ is an attractor of the system despite the map is not defined on such a point. This event is shown through numerical simulations in the example of Figure 4. Panel (a) depicts the basin of attraction (grey-coloured) of the attractor $(0, 2/q)$, the red point, whereas the boundary of the basin is defined by the stable manifold of the interior stationary state, i.e. the saddle (black) point (k^*, n^*). The white region in the figure represents the space of initial conditions for which trajectories become unfeasible after a finite number of iterations. A feature of Figure 4(a) is that the system is globally indeterminate as there exist two distinct long-run outcomes (only one of which is an interior fixed point of map M) attainable given the same initial level of capital per young person and there exists an infinite number of trajectories leading towards the locally indeterminate state $(0, 2/q)$, which represents a poverty trap scenario with a low level of capital and high fertility, and there exists a unique (saddle) path on which the economy converges to the interior stationary state, which represents a paradigm of developed countries with a high level of capital and low fertility. Converging towards on or the other scenario is a matter of individuals' choices about fertility. This is a typical expectations driven outcome that can bring problems of coordination failures.

In fact, U evaluated at $(0, 2/q)$ is smaller than U evaluated at (k^*, n^*) meaning that (k^*, n^*) Pareto dominates $(0, 2/q)$ but individuals can choose to coordinate themselves on the Pareto dominated equilibrium. This holds because individuals, by expecting a very low return on capital, tend to increase the amount of time devoted to the caring of children so that their number increases approaching its upper bound $(2/q)$, thus causing an increasingly reduction in the accumulation of capital. Panel (b) of Figure 4 shows two typical trajectories leading towards the poverty trap outcome.

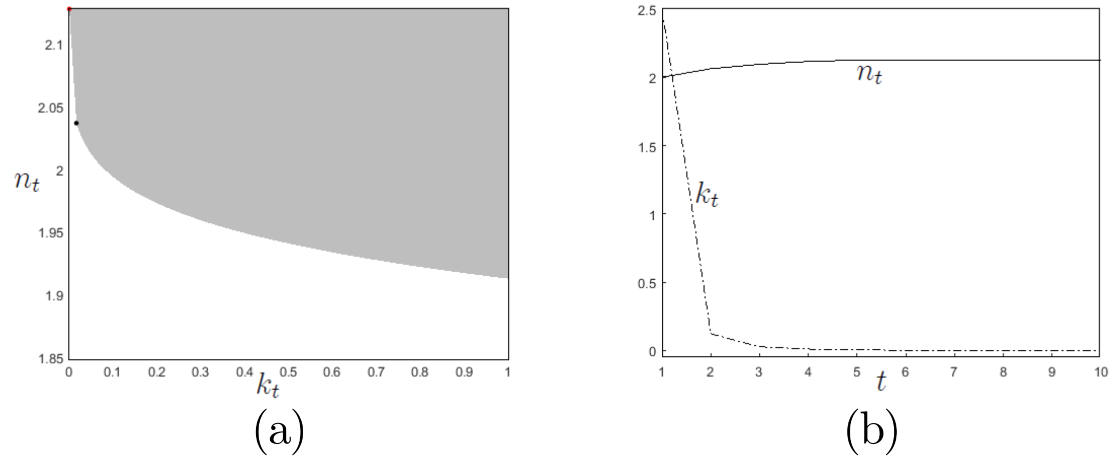

Figure 4. Parameter set: $\alpha = 0.354$, $\sigma = 0.3$, $A = 1.07$, $q = 0.94$, $\gamma = 0.004$. (a) Basin of attraction (depicted in grey) of $(0, 2/q)$ (the red point) and the boundary of the basin that defines the stable manifold of the saddle (black) point (k^*, n^*). (b) Time series of n_t and k_t of a trajectory approaching towards $(0, 2/q)$.

Another result the model is able to reproduce is given by (endogenous) fertility fluctuations around the high equilibrium that are in line with the baby busts and baby booms observed the last century in U.S. There are two recent articles analysing the reasons why fertility fluctuates over time. We refer to the works of Doepke et al. (2015) and Jones and Schoonbroodt (2016). The former wishes to assess the effects of the shock of World War II on subsequent baby boom in U.S. (following the historical decrease in fertility due to working of demographic transition forces). They consider a model where women can choose labour supply, the number of children and when having children and there exists an interaction amongst subsequent cohorts. Then, they perform quantity experiments to explain the post-war increase in fertility on the basis of a drop in labour force participation of young women (whose wages declined in that period) because of the increase in competition caused by the higher participation of older women (and the persisted high demand of female labour after the end of World War II). Younger women then exited the labour market and started having children. The latter work, instead, considers a general equilibrium model with endogenous fertility and dynastic altruism showing essentially that fertility and the opportunity cost of children in U.S. are pro-cyclical. Our theory simplifies the framework substantially and it is able to give an explanation of these fluctuations within a typical neoclassical set up. The main mechanism for fetility fluctuations is similar to the one developed by Doepke et al. (2015). This is because in our model an increase in the number of children directly reduces both the labour supply and labour productivity.

4 CONCLUSIONS

"The concept of development is by no means unproblematic. The different problems underlying the concept have become clearer over the years on the basis of conceptual discussions as well as from insights emerging from empirical work" [Sen (1991), p. 23]. For a very long time in human history, the number of births and deaths were almost equivalent and, therefore, the world total population was relatively stable. During this extended period, income per person remained fairly constant or it grew slowly. After the process commonly known as Industrial Revolution, in newly industrialised countries mortality started declining first, and then, after an initial stage of stability, fertility followed a declining trend as well. In these phases, total population started increasing together with income. Then, Europe faced a long age of dramatic social, political and institutional changes that subsequently spread to other countries all over the world.

Economic development is a long-term involved phenomenon that includes social, institutional, economic and demographic changes across nations and continents. The present work has treated development on the side of economic and demographic matters. Why do some countries experience high levels of GDP and low fertility and others low levels of GDP and high fertility? Standard one-sector models of neoclassical growth often conclude that economies with similar technologies will converge towards a common stationary-state equilibrium even if the initial conditions are very different. This is the main result of the Solow (1956) set up implying the well-known result that poorer countries will tend to growth faster (during the transition towards the steady state) than richer ones. This finding is also shared by several works dealing with the basic OLG model as well as the infinite horizon optimising agent model in the literature with exogenous fertility and endogenous fertility. However, it is widely accepted that there exist persistent differences in the level of real activity and fertility rates amongst several groups of countries all around the world [e.g., Mankiw et al. (1992); Hall and Jones (1999); Palivos (1995); Jones and Tertilt (2006); Cervellati et al. (2016)]. This kind of models, therefore, is not able to explain these macroeconomic and demographic differences, so that the above as well as other similar questions are likely to remain unanswered within the basic neoclassical growth set up or the basic endogenous growth one. This unsatisfactory result of the growth literature has led several economists to (strongly) modify these frameworks in several ways trying to building on more suitable theories in either settings in the cases of both exogenous fertility [Azariadis and Drazen (1990)] and endogenous fertility [Becker and Barro (1988); Barro and

Becker (1989); Becker et al. (1990); Palivos (1995); Galor and Weil (1996)]. The literature has then grown rapidly leading to what is commonly known as the Unified Growth Theory (as discussed in the introduction), where the main factors explaining the demographic and economic transitions are generally child mortality/fertility (surviving children), adult mortality, human capital accumulation and so on.[10] However, at the time of writing it is still difficult to find theories where (endogenous) fertility per se does represent the trigger for the transition amongst the various stages of development. The present article has intended to fill that gap by using a very parsimonious OLG model of neoclassical growth with finite lived individuals. The works most closely related to the present one are Palivos (1995) and Galor and Weil (1996). The former contribution has introduced endogenous fertility (child quantity and child quality) in a continuous-time neoclassical optimal growth set up with infinite lived individuals , finding a *production-driven* channel through which fertility choices may be a source of multiple steady states and coordination failures (when the net marginal product of capital is a non-monotonic function in the capital stock). The latter one, instead, has emphasised the importance of *gender differences* in wage income in explaining the existence of multiple paths of economic development in an OLG model with child quantity.[11] Differently, this work has shown that an economy à la Galor and Weil (1996) with *homogeneous individuals* and identical firms producing with a constant-returns-to-scale technologies does not converge towards a common steady state. It has then introduced a new *utility-driven* mechanism (related to the working of the inter-temporal elasticity of substitution in consumption) through which fertility choices are a source of global indeterminacy. The model has entered the debate about history versus self-fulfilling expectations and it has provided reasons why economies with different initial conditions (history matters) or, alternatively, similar or the same initial condition(s) (expectations matter) in capital and fertility converge towards different long-term equilibria. In the latter case, sustained economic development or under-development are a matter of global indeterminacy.

The work had the ambition of trying to giving an answer to the question raised by Jones et al. (2008) in the title of their work: "Fertility Theories: Can They Explain the Negative Fertility-Income Relationship?", in a very basic macroeconomic general equilibrium set up and without using special assumptions.

Of course, we are aware that ours is a toy model and preferences about fertility may depend on culture, beliefs and social norms specifically related to institutions or ethnic groups (often followed by linguistic and religious contours, that also affect choices about contraception), and that these elements should therefore be included as endogenous variables in the analysis. However, the goal was been to keep the model as simple as possible to bring to light several

[10]See also the bright synthesis of Azariadis (1993).

[11]Actually, rich and educated mothers tend to have less children than poor and unskilled ones. In addition, the income of fathers tend to positively affect fertility, while the income of mothers tend to negatively affect fertility. The importance of differential fertility and inequality for economic development is well addressed by de la Croix and Doepke (2003, 2004).

possible outcomes of a neoclassical growth set up that have remained until now unexplored. These extensions will be included in future articles belonging to our research agenda.

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Conflict of Interest The authors declare that they have no conflict of interest.

APPENDIX

For the sake of completeness, we analyse here some mathematical details of map M not discussed in the main text.

On the feasible region of map M. Map M is defined on a sub set of the non-negative orthant. In fact, given $n_t > 0$ and $k_t > 0$, in order to have $n_{t+1} > 0$ and $k_{t+1} > 0$ it must hold that

$$k_t > \left[\frac{(2 - qn_t)^{-\alpha - \frac{1}{(\sigma-1)(1-\alpha)}} n_t^{-1+\frac{\gamma}{(\sigma-1)(1-\alpha)}}}{2B} \right]^{\frac{1-\alpha}{\alpha^2}}. \tag{22}$$

Depending on the parameter setting we have three different cases.

1) If $\sigma > 1$ and $\gamma > (\sigma - 1)(1 - \alpha)$ then the region defined by the inequality in (22) is described by the grey area in Panel (a) of Figure A.1.

2) If $\sigma > 1$ and $\gamma < (\sigma - 1)(1 - \alpha)$ then the region defined by the inequality in (22) is described by the grey area in Panel (b) Figure A.1.

3) If $\sigma < 1$ then the region defined by the inequality in (22) is described by the grey area in Panel (c) of Figure A.1.

The regions detailed above ensure the possibility of computing a single iterate. However, in order to have well-defined forward dynamics, the trajectory generated by a generic initial condition must be bounded in this region for every iterate. For this reason, the economically meaningful trajectories analysed in the main text actually lie on in a smaller region than the one shown in the three panels of Figure A.1.

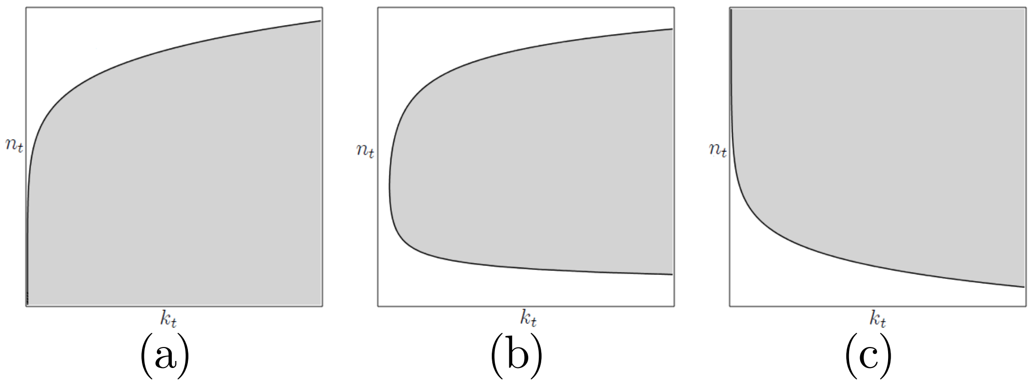

Figure A.1. Feasible region (grey-coloured) of map M depending on the parameter setting.

On the non-existence of feasible trajectories with vanishing n_t. If $\sigma > 1$ and $\gamma > (\sigma-1)(1-\alpha)$ then we can rule out the existence of feasible trajectories with $n_t \to 0$. In fact, if there were feasible trajectories such that $n_t \to 0$ then $k_t \to +\infty$, as can be ascertained by looking at the first equation of map (10). However, this would not be consistent with the second equation of that system, which describes the dynamics of fertility, as $k_t \to +\infty$ would imply $n_t \to 2/q$.

On the non-existence of feasible trajectories with an unbounded growth of k_t. If $\sigma > 1$ and $\gamma > (\sigma-1)(1-\alpha)$ then we can rule out the existence of feasible trajectories with $k_t \to +\infty$. In fact, if there were feasible trajectories such that $k_t \to +\infty$ then $n_t \to 2/q$, as can be ascertained by looking at the second equation of map (10). However, this would not be consistent with the first equation of that system, which describes the dynamics of the stock of capital, as when $n_t \to 2/q$ for a high enough value of k_t one would get $k_{t+1} < k_t$.

On the non-existence of feasible trajectories with an unbounded growth of k_t and $n_t \to 2/q$. Consider a feasible trajectory with $n_t \to 2/q$. Then, for a value of t sufficiently large it holds that $k_{t+1} = A(1-\alpha)k_t^\alpha(2-qn_t)^{1-\alpha} < A(1-\alpha)k_t^\alpha \to k^* = [(1-\alpha)A]^{\frac{1}{1-\alpha}}$.

On the non-existence of feasible trajectories with $n_t \to 2/q$ when $\sigma > 1$. We can rule out the existence of feasible trajectories with $n_t \to 2/q$ in the case $\sigma > 1$. If there were feasible trajectories such that $n_t \to 2/q$ then by the first equation in (10) $k_t \to 0$. However, this would not be consistent with the second equation in (10) as $k_t \to 0$ would imply $n_{t+1} < 0$.

Proof of Lemma 4 and Propositions 5 and 6. Results about stability of the stationary states follows by the study of Jacobian matrix evaluated at a generic state (k_{ss}, n_{ss}). The Jacobian matrix is the following:

$$J(k_{ss}, n_{ss}) := \begin{pmatrix} J_{1,1} & J_{1,2} \\ J_{2,1} & J_{2,2} \end{pmatrix}, \tag{23}$$

where

$$J_{11} := \alpha > 0, \tag{24}$$

$$J_{1,2} := -\left[(1-\alpha)A\right]^{\frac{1}{1-\alpha}} n^{\frac{\alpha-2}{1-\alpha}} (2-\alpha q n_{ss}) < 0, \tag{25}$$

$$J_{2,1} := \frac{\alpha^2 k_{ss}^{\frac{\alpha^2}{\alpha-1}} (2-qn_{ss})^{\frac{(\alpha^2-\alpha)(1-\sigma)+1}{(1-\sigma)(1-\alpha)}} n_{ss}^{\frac{(1-\alpha)(\sigma-1)-\gamma}{(1-\sigma)(1-\alpha)}} B}{q(1-\alpha)k_{ss}} > 0, \tag{26}$$

$$J_{2,2} := \frac{J_{2,1} k_{ss} \left\{\left[\alpha^2(\sigma-1)-\sigma\right] q n_{ss} + 2(\sigma-1)(1+\alpha) - (2-qn_{ss})\gamma\right\}}{\alpha^2(\sigma-1)(2-qn_{ss})n_{ss}}. \tag{27}$$

In particular, we note that $g(n) = v(h(n), n)$ from which $g'(n) = v'_k(h(n), n)h'_n(n) + v'_n(h(n), n)$. At a stationary state such expression becomes:

$$v'_k(k_{ss}, n_{ss}) \frac{\frac{\partial Q_1}{\partial n}\Big|_{(k,n)=(k_{ss},n_{ss})}}{1 - \frac{\partial Q_1}{\partial k}\Big|_{(k,n)=(k_{ss},n_{ss})}} + v'_n(k_{ss}, n_{ss}). \tag{28}$$

Corresponding at an intersection from below (resp. above) of the graph of g with the 45° line we have that the expression in (28) is greater than (resp. smaller than) one. Rearranging terms, we have that such expression identifies the condition for which $Det(J(k_{ss}, n_{ss})) - Tr(J(k_{ss}, n_{ss})) + 1 < 0$. Results follows from by identifying the sign of $Det(J(k_{ss}, n_{ss}))$, where

$$sgn\left\{Det(J(k_{ss}, n_{ss}))\right\} = sgn\left\{\frac{2 + (\gamma-\sigma)(2-qn)}{q(2-qn)(1-\sigma)}\right\} \tag{29}$$

and by identifying a sufficient condition for which $J_{22} > 0$ (a condition that guarantees $Tr(J(k_{ss}, n_{ss})) > 0$).

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