Curvature and Uniqueness of Equilibrium

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Abstract: Let $E(r)$ be the equilibrium manifold associated to a pure exchange smooth economy with fixed total resources r. We endow $E(r)$ with the metric induced from its ambient space, $S \times \Omega(r)$, where S denotes the set of normalized prices and $\Omega(r)$ represents the space of endowments. We show that, in the case of two commodities and an arbitrary number of agents, the curvature K of $E(r)$ is zero if and only if there is uniqueness of equilibrium for every economy $\omega \in \Omega(r)$. Hence $E(r)$ is a plane in $S \times \Omega(r)$ parallel to $\{0\} \times \Omega(r)$.

Keywords: Equilibrium manifold, uniqueness of equilibrium, Gaussian curvature.

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1 Introduction

It is the intimate relationship between geometric and economic properties, together with a reasonable level of abstraction, which represents one of the distinctive marks of general equilibrium theory. This is particular true for the equilibrium manifold approach.

The equilibrium manifold is defined as the set of pairs of prices and endowments such that the aggregate excess demand function is equal to zero. Due to the strong connection between mathematical formalization and equivalent economic properties, the investigation of its geometric properties has led to several and important contributions in economic theory, where the standard smooth exchange setup has played the role of benchmark model which can be powerfully extended to encompass more realistic and complex contexts.

Since the equilibrium manifold is not a simple collection of data, its shape significantly matters. This further stimulates the investigation of its geometric properties and, in particular, those properties which are intrinsic, i.e. which do not depend on the ambient space (such as, e.g., the Riemannian metric studied by [5]).

In this paper we study the curvature, an important intrinsic differential geometry concept. We have been inspired by a result by Balasko (see Theorem 2.2) which, roughly speaking, states that the equilibrium manifold is flat (i.e. it has zero curvature) if there is uniqueness of equilibrium. A very natural question is whether a zero curvature implies uniqueness. Due to the difficulty of the computations involved, we have only analyzed the case with two goods and an arbitrary number of agents, where the equilibrium manifold is an hypersurface, i.e. it has codimension one with respect to its ambient space. In our main result, Theorem 4.1, we show that a zero curvature implies uniqueness of equilibrium. Moreover, as a by-product of our proof, we also show that it is sufficient to compute the curvature in a measure zero set of the space of the economies in order to have a global information on the uniqueness. We believe that the curvature issue may deserve further attention and investigation because it is intimately related to the multiplicity of equilibria and the geodesic flow and ergodic theory. This could hopefully open original ways to address the equilibrium selection problem. This paper is organized as follows. Section 2 recalls the economic setup and Section 3 recalls the standard mathematical concepts used in Section 4, where our main result is proved.

2 Geometric implications of uniqueness of equilibrium

The economic setup is represented by a pure exchange smooth economy with l goods and m consumers. By smooth (see [2, Chapter 2]) is meant that consumer i's preferences, $i = 1, 2, \dots, m$, are represented by a smooth utility function $u_i : \mathbb{R}^l \to \mathbb{R}$, where: (1) the first order derivatives are all strictly positive, and (2) the quadratic form $y^t D^2 u_i(x) y$, where $D_i^2 u_i(x)$ denotes the Hessian matrix restricted to the tangent plane to the indifference surface, is negative definite. The set of normalized prices is defined by

$$S = \{p = (p_1, \dots p_l) \in \mathbb{R}^n \mid p_j > 0, j = 1, \dots, l,\ p_l = 1\}$$

and the set $\Omega = (\mathbb{R}^l)^m$ denotes the space of endowments $\omega = (\omega_1, \ldots, \omega_m)$, $\omega_i \in \mathbb{R}^l$. The problem of maximizing the smooth utility function $u_i : \mathbb{R}^l \to \mathbb{R}$ subject to the budget constraint $p \cdot \omega_i = w_i$ gives the unique solution $f_i(p, w_i)$, i.e., consumer's i demand. The *equilibrium manifold* E is the closed set of the pairs $(p, \omega) \in S \times \Omega$, which satisfy the following equilibrium equations:

$$\sum_{i=1}^{m} f_i(p, p \cdot \omega_i) = \sum_{i=1}^{m} \omega_i.$$

By introducing the following two smooth mappings:

- the map $\phi : S \times \Omega \to S \times \mathbb{R}^m \times \mathbb{R}^{(l-1)(m-1)}$ defined by

  $$(p, \omega_1 \ldots, \omega_m) \mapsto (p, p \cdot \omega_1, \ldots, p \cdot \omega_m, \bar{\omega}_1, \ldots, \bar{\omega}_{m-1}),$$

  where $\bar{\omega}_i$ denotes the first $l-1$ components of ω_i, for $i = 1, \ldots, m-1$;

- the map $\theta : S \times \mathbb{R}^m \times \mathbb{R}^{(l-1)(m-1)} \to S \times \Omega$ defined by

  $$(p, w_1, \ldots, w_m, \bar{\omega}_1, \ldots, \bar{\omega}_{m-1}) \mapsto (p, \bar{\omega}_1, \omega_1^l, \bar{\omega}_2, \omega_2^l, \ldots, \bar{\omega}_{m-1}, \omega_{m-1}^l, \omega_m)$$

  where $\omega_i^l = w_i - p_1\omega_i^1 - \ldots - p_{l-1}\omega_i^{l-1}$, for $i = 1, \ldots, m-1$ and $\omega_m = \sum_{i=1}^m f_i(p, w_i) - \sum_{i=1}^{m-1} \omega_i$;

Balasko shows [2, p.73-74] that the composition mapping $\phi \circ \theta$ is the identity mapping and that the equilibrium manifold E is the image of the mapping θ. By [2, Lemma 3.2.1] E is a smooth submanifold of $S \times \Omega$, globally diffeomorphic to $S \times \mathbb{R}^m \times \mathbb{R}^{(l-1)(m-1)} = \mathbb{R}^{lm}$, i.e. $\phi_{|E} \cong \mathbb{R}^{lm}$.

In order to better understand the geometric structure of E, the following two subsets of E are introduced: the set of *no-trade equilibria* $T = \{(p, \omega) \in E |\, f_i(p, p \cdot \omega_i) = \omega_i,\ i = 1, \ldots, m\}$ and *the fiber* associated with $(p, w_1, \ldots, w_m) \in S \times \mathbb{R}^m$, which is defined as the set of pairs $(p, \omega) \in S \times \Omega$ such that:

- $p \cdot \omega_i = w_i$ for $i = 1, \ldots, m$;
- $\sum_i \omega_i = \sum_i f_i(p, w_i)$.

By defining the two smooth maps $f : S \times \mathbb{R}^m \to S \times \mathbb{R}^{lm}$, where $f(p, w_1, \ldots, w_m) = (p, f_1(p, w_1), \ldots, f_m(p, w_m))$, and $\phi_{Fiber} : E \to S \times \mathbb{R}^m$, where $\phi_{Fiber}(p, \omega_1, \ldots, \omega_m) = (p, p \cdot \omega_1, \ldots, p \cdot \omega_m)$, because $f(S \times \mathbb{R}^m) = T \subset E$ and $\phi_{Fiber} \circ f$ is the identity mapping, by applying [2, Lemma 3.2.1], Balasko shows [2, Proposition 3.3.2] that T is a smooth submanifold of E diffeomorphic to $S \times \mathbb{R}^m$.

By construction, every fiber associated with $(p, w_1, \ldots, w_m)$ is a subset of E which is the inverse image of $(p, w_1, \ldots, w_m)$ via the mapping ϕ_{Fiber}. It is intuitively clear that while holding $(p, w_1, \ldots, w_m)$ fixed and letting ω varying along the fiber, there are not any nonlinearities which may arise from the aggregate demand. In fact the fiber is a linear submanifold of E of dimension $(l-1)(m-1)$ [2, Proposition 3.4.2].

Since every fiber contains only one no-trade equilibrium [2, Proposition 3.4.3], the equilibrium manifold E can be thought as a disjoint union of fibers parametrized by the no-trade equilibria T via the mapping $\phi_{|E} : E \to S \times \mathbb{R}^m \times \mathbb{R}^{(l-1)(m-1)}$: for a fixed $(p, w_1, \ldots, w_m) \in S \times \mathbb{R}^m$, each fiber is parametrized by $\bar{\omega}_1, \ldots, \bar{\omega}_{m-1}$. By letting $(p, w_1, \ldots, w_m)$ varing in $S \times \mathbb{R}^m$, we obtain the bundle structure of the equilibrium manifold. Finally, the *natural projection* $\pi : E \to \Omega$ is the smooth map defined by the restriction to E of the projection $pr : S \times \Omega \to \Omega$, $(p, \omega) \mapsto \omega$.
For convenience, we sum up in the following Figure 1 these geometric relationships.

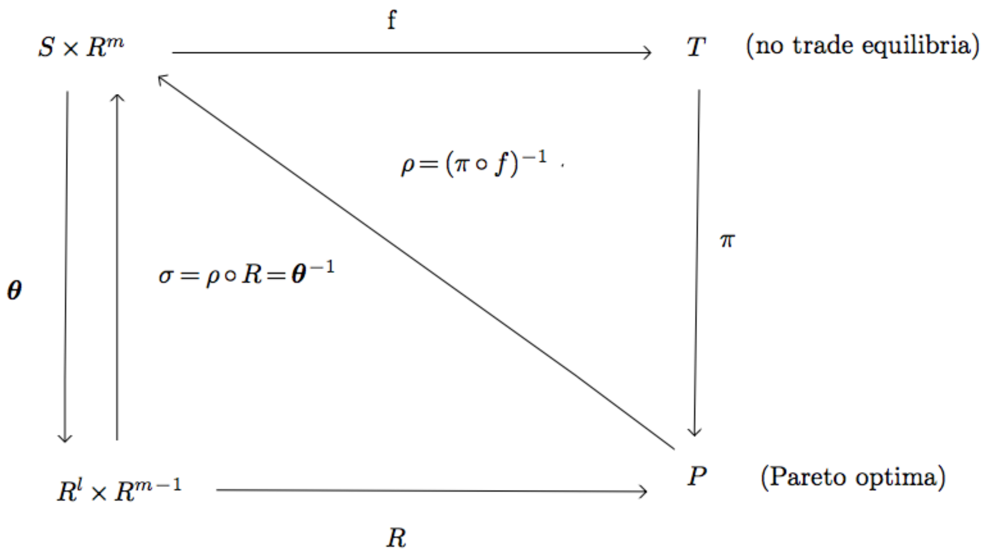

Figure 1: Geometric construction.

The map $\rho : P \to S \times \mathbb{R}^m$ is defined by $(x_1, \ldots, x_m) \mapsto (g(x), g(x) \cdot x_1, \ldots, g(x) \cdot x_m)$, where $g(x) = (\frac{\partial u_1}{\partial x^1}, \ldots, \frac{\partial u_1}{\partial x^l})(x_1)$ is the price supporting the allocation $x = (x_1, \ldots, x_m)$. The map R associates to $(r, u_1^*, \ldots, u_{m-1}^*)$ the Pareto optimum representing the unique solution to the problem of maximizing $u_m(x_m)$ subject to $\sum_i x_i = r$ and $u_i(x_i) \geq u_i^*$, $i = 1, 2, \ldots, m-1$. The map $\boldsymbol{\theta} : S \times \mathbb{R}^m \to R^l \times R^{m-1}$ associates to (p, w) $(\sum_i f_i(p, w_i), u_1(f_1(p, w_1)), \ldots, u_{m-1}(f_{m-1}(p, w_{m-1}))$.

If total resources are fixed, the equilibrium manifold is defined as

$$E(r) = \{(p, \omega) \in S \times \Omega(r) \mid \sum_{i=1}^{m} f_i(p, p \cdot \omega_i) = r\},$$

where $r \in \mathbb{R}^l$ is the vector that represents the total resources of the economy and $\Omega(r) = \{\omega \in \mathbb{R}^{lm} \mid \sum_{i=1}^m \omega_i = r\}$.

Let $B(r) = \{(p, w_1, \ldots, w_m) \in S \times \mathbb{R}^m | \sum_{i=1}^m f_i(p, w_i) = r\}$ be the set of *price-income equilibria* (see [2, Definition 5.1.1]). We have that $\phi(E(r)) = B(r) \times \mathbb{R}^{(l-1)(m-1)}$. $B(r)$ is a submanifold of $S \times \mathbb{R}^m$ diffeomorphic to $\mathbb{R}^{m-1}$ [2, Corollary 5.2.4] and the equilibrium manifold $E(r)$ is a submanifold of $S \times \Omega(r)$ diffeomorphic to $\mathbb{R}^{l(m-1)}$ [2, Corollary 5.2.5]. The following result shows the connection between the curvature of $B(r)$ and the uniqueness of equilibrium.

Theorem 2.1 [2, p. 262 Proposition 7.Ann.1] *There is uniqueness of equilibrium for every economy $\omega \in \Omega(r)$ if and only if the manifold $B(r)$ embedded in $S \times \mathbb{R}^{m-1}$ is perpendicular to $S \times (0)$. The manifold $B(r)$ then becomes equal to $\{p\} \times \mathbb{R}^{m-1}$.*

Notice that a flat $B(r)$ (i.e. zero curvature) does not imply that the equilibrium manifold has zero curvature. Moreover $B(r)$ is not a submanifold of $E(r)$, since it "lives" in the *dual space* of the set of price-income vectors [2, Chapter 7]. Its counterpart, in the space of economies $\Omega(r)$, is the set of Pareto optima, which is diffeomorphic to T, a submanifold of $E(r)$. Hence there is not apparently any equivalence relationship between the curvature of the equilibrium manifold and uniqueness of equilibrium. To the best of our knowledge, the following result is the only known connection.

Theorem 2.2 [2, p. 188 Theorem 7.3.9 part (2)] *If for every $\omega \in \Omega(r)$ there is uniqueness of equilibrium, the equilibrium correspondence is constant: The equilibrium price vector p associated with ω does not depend on ω.*

Therefore, according to this result, if there is uniqueness of equilibrium, the equilibrium manifold is a plane and, hence, its sectional curvature vanishes. The aim of this paper is to show that the converse of Theorem 2.2 also holds, if $l = 2$ (see Theorem 4.1).

3 Mathematical preliminaries

Some economists may not be unacquainted with differential geometry. Even if it is not possibile to make this paper self contained, we hope that the reader may benefit from reading this section, where some standard facts of differential geometry are recalled. Our main reference is [4].

A subset $M^k \subset \mathbb{R}^n$ is a *regular surface* of dimension k, $k \leq n$, if for every $p \in M$ there exists a neighborhood V of $p \in M$ in $\mathbb{R}^n$ and a mapping $x : U \subset \mathbb{R}^k \to M \cap V$ of an open set $U \subset \mathbb{R}^k$ onto $M \cap V$ such that x is a differentiable homeomorphism and the differential $dx_q : \mathbb{R}^k \to \mathbb{R}^n$ is injective for all $q \in U$. If $n - k = 1$, M is called *hypersurface*.

The visual intuition in $\mathbb{R}^3$ of the previous definition is that a surface $S = M^2$ can be seen as the result of glueing together deformed pieces of planes in such a way that for every point $p \in S$, the tangent plane $T_p(S)$[1] can be defined. The *Gaussian curvature*

[1]The vector subspace $dx_q(\mathbb{R}^2) \subset \mathbb{R}^3$ represented by all tangent vectors to S at $x(q) = p$.

at a point $x(q) = p \in S$ measures of how much the surface "departs" from its tangent plane. Since a plane is defined by a point p and normal vector attached to it, the rate of change of this normal vector provides a measure of how fast the surface departs from its tangent plane in a neighborhood of p.

To formalize this concept in local coordinates, let $x : U \subset \mathbb{R}^2 \to S$ be a parametrization of a surface S and denote by $\{x_u, x_v\}$ the basis of $T_p(S)$ where $dx_q(e_1) = x_u$, $dx_q(e_v) = x_v$ and $\{e_1, e_2\}$, is the canonical basis in $\mathbb{R}^2$. Since, by the definition of regular surface, we have that vector product $x_u \wedge x_v \neq 0$, we can define for every point $p \in x(U)$ a map N which associates to p the unit normal vector at p, i.e.

$$N(q) = \frac{x_u \wedge x_v}{|x_u \wedge x_v|}(q).$$

N is a differentiable field of unit normal vectors. The regular surface is called *orientable* if admits N on its whole surface. The choice of N is called an *orientation*.

The map $N : S \to S^2$ is called the *Gauss map*. Its differential dN_p is a linear transformation of the tangent space of S (because $T_{N(p)}S^2$ can be identified with the parallel plane $T_p(S)$) and provides the rate of change of how the surface pulls away from $T_p(S)$ in a neighborhood of p. This measure depends on the direction in which we move away from $p \in S$. More precisely, let $\alpha = x(u(t), v(t))$ be a curve in S such that $\alpha(0) = p$ and $w = \alpha'(0)$. The dot product $II_p(w) = - < dN_p(w), w >$ is called *the second fundamental form* and its value represents the *normal curvature* along a direction w at p. If we restrict ourselves to directions represented by unit vectors, then the continuous map II_p defined on the unit circle admits a maximum k_1 and a minimum k_2, $k_1 \geq k_2$, called the *principal curvatures* at p, with *principal directions* e_1 and e_2, respectively. As an example, in a plane or in a sphere k_1 is equal to k_2: all directions are extremals for the normal curvature.

Since dN_p is a self-adjoint operator , i.e. $< dN_p(w_1), w_2 >=< w_1, dN_p(w_2) >$ for all $w_1, w_2 \in T_p(S)$, by standard results of linear algebra ([3, Appendix to Chapter 3]), there exists an orthonormal basis $\{e_1, e_2\}$ of $T_p(S)$ such that e_i and k_i, $i = 1, 2$, are, respectively, the eigenvectors and eigenvalues of dN_p. The determinant of dN_p, given by the product of the principal curvatures, k_1k_2 is called the *Gaussian curvature* $K(p)$.

We now write the curvature formula using the coordinate system, i.e. the parametrization of S. Let w be a vector belonging to $T_p(S)$ which can be thought as a tangent vector to a curve $\alpha(t) = x(u(t), v(t))$ such that $\alpha(0) = x(u_0, v_0) = p$. Observe that the natural standard product in $\mathbb{R}^3$ naturally induces in $T_p(S)$ a symmetric bilinear form, called the *first fundamental form* and denoted by $I_p : T_p(S) \to \mathbb{R}$. In the standard basis $\{x_u, x_v\}$, we have $I_p(w) =< w, w >_p= I_p(\alpha'(0) =< x_uu' + x_vv', x_uu' + x_vv' >= E(u')^2 + 2Fu'v' + G(v')^2$, where $E =< x_u, x_u >$, $F =< x_u, x_v >$ and $G =< x_v, x_v >$. Hence the associated matrix of I_p is

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

In order to express the matrix of dN_p in the basis $\{x_u, x_v\}$, consider the tangent vectors $N_u = dN_p(x_u)$ and $N_v = dN_p(x_v)$. Since N_u and N_v belong to $T_p(S)$, one can write $N_u = a_{11}x_u + a_{21}x_v$, $N_v = a_{12}x_u + a_{22}x_v$ and $dN(w) = N_u u' + N_v v'$, i.e. the associated matrix (a_{ij}) of dN_p is given by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Finally, we have that $II_p(w) = -< dN(w), w >= -< N_u u' + N_v v', x_u u' + x_v v' >$. If we let $e = -< N_u, x_u >=< N, x_{uu} >$, $f = -< N_u, x_v >=< N, x_{uv} >=< N, x_{vu} = -< N_v, x_u >$ and $g = -< N_v, x_v >=< N, x_{vv} >$, we can express II_p in the basis $\{x_u, x_v\}$ as the matrix:

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

As a straightforward computation, we get

$$\begin{aligned} -f &=< N_u, x_v >= a_{11}F + a_{21}G \\ -f &=< N_v, x_u >= a_{12}E + a_{22}F \\ -e &=< N_u, x_u >= a_{11}E + a_{21}F \\ -g &=< N_v, x_v >= a_{12}F + a_{22}G \end{aligned}$$

or, in matrix form,

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = -\begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

Hence we have that the Gaussian curvature is

$$K = det(a_{ij}) = \frac{eg - f^2}{EG - F^2}.$$

The above formula can be rewritten as follows. Recalling that

$$e =< x_{uu}, N >=< x_{uu}, \frac{x_u \wedge x_v}{|x_u \wedge x_v|} >= \frac{det(x_{uu}, x_u, x_v)}{|x_u \wedge x_v|}$$

and, similarly,

$$f = \frac{det(x_{uv}, x_u, x_v)}{|x_u \wedge x_v|}$$

and

$$g = \frac{det(x_{vv}, x_u, x_v)}{|x_u \wedge x_v|},$$

we can write

$$K = \frac{det(x_{uu}, x_u, x_v)det(x_{vv}, x_u, x_v) - det(x_{uv}, x_u, x_v)^2}{(||x_u||^2||x_v||^2 - < x_u, x_v >^2)^2} \tag{1}$$

3.1 Parametrization and curvature of $E(r)$ for $l = m = 2$.

We recall that $\phi(E(r)) = B(r) \times \mathbb{R}^{(l-1)(m-1)}$ and the manifold $B(r)$ is diffeomorphic to $\{r\} \times \mathbb{R}^{m-1}$ through the map θ. When $l = m = 2$, $E(r)$ is a surface and

$$B(r) = \{(p, w_1, w_2) \mid f_1(p, w_1)+f_2(p, w_2) = r)\} = \{(p, w_1) \mid f_1(p, w_1)+f_2(p, pr-w_1) = r\}$$

is globally diffeomorphic to $\mathbb{R}$. Hence , there exists a diffeomorphism

$$\varphi : \mathbb{R} \to B(r), t \mapsto \varphi(t) = (p(t), w_1(t)).$$

This diffeomorphism is given by the inverse of the restriction to $B(r)$ of the indirect utility function

$$\hat{u}_1 : S \times \mathbb{R} \to \mathbb{R}, \hat{u}_1(p, w_1) = u_1(f_1(p, w_1))$$

of the first consumer. Therefore a parametrization of

$$E(r) = \{(p, \omega_1^1, \omega_1^2) \mid f_1^1(p, p\omega_1^1 + \omega_1^2) + f_2^1(p, p(r_1 - \omega_1^1) + r_2 - \omega_1^2) = r_1\}$$

is given by:

$$\Phi : \mathbb{R}^2 \to E(r), (t, \omega_1^1) \mapsto (p(t), \omega_1^1, \omega_1^2(t) = w_1(t) - p(t)\omega_1^1).$$

Set $v = \omega_1^1$, $\beta(t) = (p(t), 0, w_1(t))$, $\delta(t) = (0, 1, -p(t))$ then

$$\Phi(t, v) = \beta(t) + v\delta(t).$$

The fact that Φ is indeed a parametrization implies that

$$W = \|\Phi_t \wedge \Phi_v\| \neq 0$$

By applying formula (1) we get the Gaussian curvature K of $E(r)$ with respect to the metric induced by the Euclidean metric of $S \times \Omega(r)$:

$$K(t, v) = -\frac{p'(t)^4}{W^4}.$$

Therefore $K \leq 0$. Moreover K vanishes if and only if $-(p'(t))^4 = 0$ i.e. p is a constant, say C. It follows that the map $t \mapsto w_1(t)$ is a diffeomorphism and hence

$$E(r) = \{(w, v) \in \mathbb{R}^2 \ |(C, v, w - Cv)\}$$

is the horizontal plane $p = C$ in $S\times\Omega(r)$, namely a horizontal plane parallel to $\{0\}\times\Omega(r)$. This shows that, in an economy with two agents and two commodities, the zero curvature of $E(r)$ implies uniqueness of equilibrium. This represents a particular case of our main result, Theorem 4.1.

Remark 3.1 Observe that the assumption that the Gaussian curvature of a surface be zero does not imply that the surface is a plane! In fact (see [3, Theorem p. 408]), a complete surface in $\mathbb{R}^3$ with zero Gaussian curvature is a plane or a cylinder where "... a cylinder is a regular surface S such that through each point p there passes a unique line $R(p) \subset S$ (the generator through p) which satisfies the condition that if $q \neq p$, then the lines $R(p)$ and $R(q)$ are parallel or equal" [3, p. 408].

3.2 Hypersurfaces

For abstract manifolds M of higher dimensions, the computation of the curvature operator is more involved but our analysis and computations are quite simplified because we are considering hypersurfaces. Moreover, for our purposes we will only need the *sectional curvature*.

Let M be a submanifold immersed in $\mathbb{R}^n$. Observe that the Euclidean metric on $\mathbb{R}^n$ naturally induces a *Riemannian metric* on M, by applying to the subspace $T_p(M)$ the standard inner product of $T_p(\mathbb{R}^n) = \mathbb{R}^n$. A smooth *vector field* Y on M is a smooth map $Y : M \to TM$, where the *tangent bundle* of M is the set $TM = \{(p, v)|\, p \in M,\, v \in T_pM\}$.

In terms of the basis $\{\frac{\partial}{\partial i}\}$ associated to the parametrization $x : U \subset \mathbb{R}^k \to M$, the vector field $Y(p)$ can be expressed as

$$Y(p) = \sum_{i=1}^{k} a_i(p) \frac{\partial}{\partial x_i} \,.$$

This notation suggests the idea of Y as an operator whose domain is $\mathcal{D}$, the set of differentiable functions on M. In fact, if f belongs to $\mathcal{D}$, $f : M \to \mathbb{R}$, we can write

$$(Yf)(p) = \sum_{i=1}^{k} a_i(p) \frac{\partial f}{\partial x_i}(p) \,,$$

which means to take the directional derivative ∇f of any smooth function f in the direction $Y(p)$.

For each $p \in M$, $T_p(\mathbb{R}^{n+1}) = T_p(M) \oplus T_p(M)^{\perp}$, a vector $v \in T(\mathbb{R}^{n+1})$ can be splitted into a tangential component $v^T \in T_p(M)$ and a normal component $v^N \in T_p(M)^{\perp}$. Since the vector field $\frac{dY}{dt}$ is not necessarily tangent to M, we define the *covariant derivative* $\frac{DY}{dt}$ of the tangent vector field Y on M as the projection of $\frac{dY}{dt}$ on $T_p(M)$. Similarly to what we have done for (Yf) we can define a rate of change of the vector field Y in the direction of a tangent vector X_p by denoting $\nabla_{X_p} Y = (\frac{DY}{dt})_{t_0}$ along any curve $\alpha(t) \in M$ with $\alpha(t_0) = p$ and $\alpha' = X_p$.

A tangent vector field on M can be locally extended on its ambient space (we will use the bar notation when we refer to the ambient space). Let $\{\frac{\partial}{\partial_i}, N_1, \dots N_{n-k}\}$ be the extended basis of $T_p(\mathbb{R}^n)$, where N_j denote the normal vectors for M. Following the notation above, we have $\nabla_X Y = (\bar{\nabla}_{\bar{X}} \bar{Y})^T$.
Denote by

$$B(X, Y) = \bar{\nabla}_{\bar{X}} \bar{Y} - \nabla_X Y$$

a local vector field on $\mathbb{R}^n$ normal to M. One can show that the map B is bilinear and symmetric. It follows that the mapping $H_\eta : T_pM \times T_pM \to \mathbb{R}$ defined by

$$H_\eta(x, y) =< B(x, y), \eta >,\; x, y \in T_pM,\; \eta \in (T_pM)^{\perp}$$

is a symmetric bilinear form. The second fundamental form of f at p along the normal vector η is the quadratic form

$$II_\eta(x) = H_\eta(x, x)$$

to which it is associated *the shape operator*, a linear self-adjoint operator $S_\eta : T_pM \to T_pM$, defined by

$$< S_\eta(x), y >= H_\eta(x, y) =< B(x, y), \eta > .$$

If N is a local extension of the vector η normal to M, then $S_\eta(x) = -(\bar{\nabla}_x N)^T$. By the symmetry of S_η, there exists an orthonormal basis $\{e_i\}$ of $T_p(M)$ with eigenvalues λ_i, such that $S(e_i) = \lambda_i e_i$. The e_i and λ_i are called the principal directions and principal curvatures of the immersion $f : M \to \mathbb{R}^n$. Let $\{x, y\}$ be a basis of σ, a two-dimensional subspace of $T_p(M)$. The segments of the set of geodesics that start at p and are tangent to σ determine, in a normal neighborhood of M at p, a dimension-two submanifold S of M. The Gaussian curvature of S is called the *sectional curvature* of σ at p. If M is an hypersurface, the sectional curvature of M is given by

$$K(e_i, e_j) = \lambda_i \lambda_j. \tag{2}$$

4 Main result

In this section we consider an economy with two goods and an arbitrary number of consumers. In this case the equilibrium manifold is an hypersurface. We establish a connection between the uniqueness of equilibrium and the curvature of $E(r)$. More precisely, we show that the price is unique if and only if $E(r)$ has zero curvature.

Theorem 4.1 *Let $l = 2$. A necessary and sufficient condition for a unique equilibrium price is that the curvature of $E(r)$ is zero.*

Proof: The implication that uniqueness of equilibrium implies that the curvature of $E(r)$ is zero is proved by Balasko in Theorem 2.2. We need to show the other implication, which is our main result. Let n be $m - 1$. The manifold $B(r)$ is globally diffeomorphic to $\mathbb{R}^n$ via a diffeomorphism

$$\varphi : \mathbb{R}^n \to B(r), t = (t_1, \dots, t_n) \mapsto (p(t), w_1(t), \dots, w_n(t)).$$

By setting $\alpha_j = \omega_j^1$, a parametrization of $E(r)$ is given by:

$$\Phi : \mathbb{R}^{2n} \to E(r), (t, \alpha_1, \dots, \alpha_n) \mapsto (p(t), \alpha_1, w_1(t) - p(t)\alpha_1, \dots, \alpha_n, w_n(t) - p(t)\alpha_n).$$

Setting $p_{t_j} = \frac{\partial p}{\partial t_j}, w_{jt_k} = \frac{\partial w_j}{\partial t_k}, j, k = 1, \dots, n$, we get:

$$\Phi_{\alpha_j} = (0, \dots, 0, \underbrace{1}_{2j}, \underbrace{-p}_{2j+1}, 0, \dots, 0). \tag{3}$$

$$\Phi_{t_j} = (p_{t_j}, 0, w_{1t_j} - p_{t_j}\alpha_1, \dots, 0, w_{nt_j} - p_{t_j}\alpha_n) \tag{4}$$

and $\{\Phi_{t_1}, \dots, \Phi_{t_n}, \Phi_{\alpha_1}, \dots, \Phi_{\alpha_n}\}$ is a basis of $T_xE(r)$ at $x = \Phi(t, \alpha_1, \dots, \alpha_n) \in E(r)$.

Consider the $(n+1)\times n-$Jacobian matrix of the map φ

$$J\varphi = \begin{pmatrix} p_{t_1} & \dots & p_{t_n} \\ w_{1t_1} & \dots & w_{1t_n} \\ \vdots & \dots & \vdots \\ w_{nt_1} & \dots & w_{nt_n} \end{pmatrix}.$$

and the determinants of its $n\times n$ submatrixes

$$A_0 = \begin{pmatrix} w_{1t_1} & \dots & w_{1t_n} \\ \vdots & \dots & \vdots \\ w_{nt_1} & \dots & w_{nt_n} \end{pmatrix}, \ A_j = \begin{pmatrix} p_{t_1} & \dots & p_{t_n} \\ w_{1t_1} & \dots & w_{1t_n} \\ \vdots & \dots & \vdots \\ \hat{w_{j}}_{t_1} & \dots & \hat{w_{j}}_{t_n} \\ \vdots & \dots & \vdots \\ w_{nt_1} & \dots & w_{nt_n} \end{pmatrix}.$$

where A_j is obtained by deleting the $(j+1)$-row from the matrix $J\varphi$.

By setting

$$A = \begin{pmatrix} w_{1t_1} - p_{t_1}\alpha_1 & \dots & w_{1t_n} - p_{t_n}\alpha_1 \\ \vdots & \dots & \vdots \\ w_{nt_1} - p_{t_1}\alpha_n & \dots & w_{nt_n} - p_{t_n}\alpha_n \end{pmatrix}$$

it is easily seen that

$$\nu(t,\omega_1^1,\dots,\omega_n^1) = (A, -A_1p, -A_1, A_2p, A_2, \dots, (-1)^j A_j p, (-1)^j A_j)$$

is a nonvanishing normal vector field of $E(r)$, i.e. $\nu(t,\alpha_1,\dots,\alpha_n) \in (T_xE(r))^\perp$ at each $x = \Phi(t,\alpha_1,\dots,\alpha_n) \in E(r)$, where, with a slight abuse of notation, we also denote by A, A_j, $j = 1,2,\dots$ the determinants of the respective matrices. (The fact that ν does not vanish follows by the fact that $A(t,0,\dots,0) = A_0$ and $A_0^2 + A_1^2 + \dots + A_n^2 \neq 0$ which follows by the fact that the rank of φ is n for all $t \in \mathbb{R}^n$).

Therefore a unit normal vector $N = \frac{\nu}{\|\nu\|}$ at the point $x = \Phi(t,\alpha_1,\dots,\alpha_n) \in E(r)$ is given by:

$$N = B^{-\frac{1}{2}}(A, -A_1p, -A_1, A_2p, A_2, \dots, (-1)^n A_n p, (-1)^n A_n), \tag{5}$$

where

$$B = A^2 + (1+p^2)\sum_{j=1}^n A_j^2. \tag{6}$$

Let $x_0 = \Phi(\bar{t},0,0\dots,0)$ (i.e. for $t = \bar{t}$, $\alpha_1 = \dots = \alpha_n = 0$) and consider the $n+1$-dimensional subspace $V_i \subset T_{x_0}E(r)$, $i = 1,\dots n$, spanned by the vectors $\overline{\Phi}_{\alpha_1},\dots,\overline{\Phi}_{\alpha_n},\overline{\Phi}_{t_i}$, where we are denoting with the “bar” the value of $\Phi_{\alpha_1},\Phi_{\alpha_n},\Phi_{t_i}$ at the point $(\bar{t},0,\dots,0)$. Set

$$e_j = \frac{\overline{\Phi}_{\alpha_j}}{\|\overline{\Phi}_{\alpha_j}\|} = \frac{1}{\sqrt{1+\overline{p}^2}}(0,\dots,0,\underbrace{1}_{2j},\underbrace{-\overline{p}}_{2j+1},0,\dots,0),\ j = 1,\dots n. \tag{7}$$

and, for fixed i,

$$e_{n+1}^i = \frac{1}{\overline{\mu}_{t_i}}(\overline{p}_{t_i}, \frac{\overline{p}\ \overline{w}_{1t_i}}{1+\overline{p}^2}, \frac{\overline{w}_{1t_i}}{1+\overline{p}^2}, \dots, \frac{\overline{p}\ \overline{w}_{nt_i}}{1+\overline{p}^2}, \frac{\overline{w}_{nt_i}}{1+\overline{p}^2}) = \overline{\Phi}_{t_i} + \sum_{j=1}^{n} \frac{\overline{p}\ \overline{w}_{jt_i}}{1+\overline{p}^2} \overline{\Phi}_{\alpha_j}, \tag{8}$$

where $\mu_{t_i}^2 = p_{t_i}^2 + \frac{1}{1+p^2}(w_{1t_i}^2 + \cdots + w_{nt_i}^2)$.

Then it is straightforward to verify that $e_1, \dots, e_n, e_{n+1}^i$ is a g-orthonormal basis of V_i with respect to the metric g induced by the Euclidean metric of $S \times \Omega(r)$. Let $e_{n+2}^i, \dots, e_{2n}^i$ be a g-orthonormal basis for $V_i^\perp$, the g-orthogonal complement of V_i (we do not need the explicit expression of this basis).

The entries of the $2n \times 2n$ symmetric matrix which represents dN_{x_0} with respect to the orthonomal basis $e_1, \dots, e_n, e_{n+1}^i, \dots e_{2n}^i$ are given by:

$$N_{\alpha\beta}^i := dN_{x_0}(e_\alpha^i) \cdot e_\beta^i = dN_{x_0}(e_\beta^i) \cdot e_\alpha^i, \quad \alpha, \beta = 1, \dots, 2n.$$

Notice that

$$dN_{x_0}(e_j) = dN_{x_0}(\frac{\overline{\Phi}_{\alpha_j}}{\|\overline{\Phi}_{\alpha_j}\|}) = \frac{dN_{x_0}(\overline{\Phi}_{\alpha_j})}{\|\overline{\Phi}_{\alpha_j}\|} = \frac{\overline{N}_{\alpha_j}}{\sqrt{1+\overline{p}^2}}$$

Hence after a straightforward computation one gets, for $j = 1, \dots n$,

$$dN_{x_0}(e_j) = -\frac{(-1)^j \overline{A}_j \overline{B}^{-\frac{3}{2}}}{\sqrt{1+\overline{p}^2}} \left(\overline{A}_0^2 - \overline{B}, -\overline{A}_0\overline{A}_1\overline{p}, -\overline{A}_0\overline{A}_1, \dots, (-1)^n \overline{A}_0\overline{A}_n\overline{p}, (-1)^n \overline{A}_0\overline{A}_n\right) \tag{9}$$

It follows by (7) and (9) that

$$N_{jk}^i = dN_{x_0}(e_j) \cdot e_k = 0, \ j, k = 1, \dots n. \tag{10}$$

(notice that, by contruction, N_{jk}^i does not depend on i).

On the other hand, by (8) and (9) we get

$$N_{j,n+1}^i = N_{n+1,j}^i = dN_{x_0}(e_j) \cdot e_{n+1}^i = \frac{(-1)^j \overline{A}_j \overline{B}^{-\frac{1}{2}} \overline{p}_{t_i}}{\overline{\mu}_{t_i}\sqrt{1+\overline{p}^2}} - \frac{(-1)^j \overline{A}_0 \overline{A}_j \overline{B}^{-\frac{3}{2}}}{\overline{\mu}_{t_i}\sqrt{1+\overline{p}^2}} [\overline{A}_0 \overline{p}_{t_i} + \sum_{k=1}^{n} (-1)^k \overline{A}_k \overline{w}_{kt_i}]$$

Observe that $\overline{A}_0 \overline{p}_{t_i} + \sum_{k=1}^n (-1)^k \overline{A}_k \overline{w}_{kt_i}$ vanishes for all $i = 1, \dots, n$ being the determinant of the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} \overline{p}_{t_1} & \dots & \overline{p}_{t_n} & \overline{p}_{t_i} \\ \overline{w}_{1t_1} & \dots & \overline{w}_{1t_n} & \overline{w}_{1,t_i} \\ \vdots & \dots & \vdots & \vdots \\ \overline{w}_{nt_1} & \dots & \overline{w}_{nt_n} & \overline{w}_{n,t_i} \end{pmatrix}.$$

Hence

$$N^i_{j,n+1} = \frac{(-1)^j \overline{A}_j \overline{B}^{-\frac{1}{2}} \overline{p}_{t_i}}{\overline{\mu}_{t_i}\sqrt{1+\overline{p}^2}}, i, j = 1, \dots n. \qquad (11)$$

Notice that if the sectional curvature of $E(r)$ (with respect to the metric g induced by the Euclidean metric of $S \times \Omega(r)$) vanishes then the rank of the matrix $\{N_{\alpha\beta}\}_{\alpha,\beta=1,\dots,2n}$ is less or equal to 1. If, in particular, the sectional curvature vanishes at x_0 then, in particular, for all $i, j = 1, \dots n$ the 2×2 matrix

$$\begin{pmatrix} N_{jj} & N^i_{j,n+1} \\ N^i_{n+1,j} & N^i_{n+1,n+1} \end{pmatrix}$$

has zero determinant, i.e., by (10) and (11),

$$\frac{\overline{A}^2_j \overline{p}^2_{t_i}}{\overline{\mu}^2_{t_i} \overline{B}(1+\overline{p}^2)} = 0, i, j = 1, \dots n.$$

Hence either $\overline{p}_{t_i} = 0$ for all $i = 1, \dots, n$ or $\overline{A}_j = 0$ for all $j = 1, \dots, n$. Since p_{t_i} and A_j do not depend on $\alpha_1, \dots \alpha_n$, it follows by (5) that in both cases the normal vector N of $E(r)$ at *any* point $x = \Phi(t, \alpha_1, \dots, \alpha_n)$ is constant, namely $N_x = (1, 0, \dots, 0)$, and hence $E(r)$ is an hyperplane in $S \times \Omega(r)$ parallel to $\{0\} \times \Omega(r)$. The proof concludes by combining this result with Theorem 2.2. □

Remark 4.2 Actually a more general result has been proved: namely, it is enough to have zero curvature at x_0 to deduce the uniqueness of equilibrium for every ω in $\Omega(r)$.

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