Universal Semiorders

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May 5, 2016

Abstract

A $\mathbb{Z}$-product is a modified lexicographic product of three total preorders such that the middle factor is the chain of integers equipped with a shift operator. A $\mathbb{Z}$-line is a $\mathbb{Z}$-product having two linear orders as its extreme factors. We show that an arbitrary semiorder embeds into a $\mathbb{Z}$-product having the transitive closure as its first factor, and a sliced trace as its last factor. Sliced traces are modified forms of traces induced by suitable integer-valued maps, and their definition is reminiscent of constructions related to the Scott-Suppes representation of a semiorder. Further, we show that $\mathbb{Z}$-lines are universal semiorders, in the sense that they are semiorders, and each semiorder embeds into a $\mathbb{Z}$-line. As a corollary of this description, we derive the well known fact that the dimension of a strict semiorder is at most three.

Key words: Semiorder; interval order; trace; sliced trace; $\mathbb{Z}$-product; $\mathbb{Z}$-line; Scott-Suppes representation; order-dimension.

1 Introduction

Semiorders are among the most studied categories of binary relations in preference modeling. This is due to the vast range of scenarios which require the modelization of a preference structure to be more flexible and realistic than what a total preorder can provide. On this point, Chapter 2 of the monograph on semiorders by Pirlot and Vincke [41] gives a large account of possible applications of semiordered structures to various fields of research.

The concept of semiorder originally appeared in 1914 – albeit under a different name – in the work of Norbert Wiener [21, 50]. However, this notion is usually attributed to Duncan Luce [36], who formally defined a semiorder in 1956 as a pair (P, I) of binary relations satisfying suitable properties. The reason that motivated Luce to introduce such a structure was to study choice models in settings where economic agents exhibit preferences with an intransitive indifference. Luce's original definition takes into account the reciprocal behavior of the strict preference P (which is transitive) and the indifference I (which may fail to be transitive). Nowadays, a semiorder is equivalently defined as either a reflexive and complete relation that is Ferrers and

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semitransitive (sometimes called a *weak semiorder*), or an asymmetric relation that is Ferrers and semitransitive (sometimes called a *strict semiorder*).

Due to the universally acknowledged importance of semiordered structures, several contributions to this field of research have appeared since Luce's seminal work. Many papers on the topic deal with representations of semiorders by means of real-valued functions [5, 8, 11, 12, 22, 34, 35, 37, 38, 39], whereas others study the weaker notion of *interval order*, introduced by Fishburn [17, 18, 20]. On the topic of real-valued representations of interval orders and semiorders, a relevant issue is the connection among several notions of *separability*: Cantor, Debreu, Jaffray, strongly, weakly, topological, interval order, semiorder, etc.: on the point, see, e.g., [6, 9] and references therein. The most comprehensive reference on semiorders is the monograph of Pirlot and Vincke [41]. For the relation among utility representations, preferences, and individual choices, we refer the reader to the recent treatise of Aleskerov *et al.* [3].[1]

In 1958 Scott and Suppes [46] tried to identify a semiorder by means of the existence of a *shifted* real-valued utility function u, in the following sense: xPy (to be read as "alternative y is strictly preferred to alternative x") holds if and only if $u(x) + 1 < u(y)$. In this representation, the real number 1 is to be intended as a "threshold of perception or discrimination", which gives rise to the so-called *just noticeable difference* [37]. The shifted utility function u is classically referred to as a *Scott-Suppes representation* of the semiorder.

It is well known that not every semiorder admits a Scott-Suppes representation. In fact, as S̀wistak points out in [48], the existence of a Scott-Suppes utility function imposes strong restrictions of the structure of a semiorder. However, this type of representation has been given a lot of attention over time, due to its importance in several fields of research, such as extensive measurement in mathematical psychology [34, 35], choice theory under risk [16], decision-making under risk [45], modelization of choice with errors [2], etc.[2]

Scott and Suppes [46] showed that every finite semiorder always admits such a representation (see also [42]). In 1981 Manders [37] proved that – under a suitable condition related to the non-existence of monotone sequences with an upper bound in the set (a property later on called *regularity*) – countable semiorders have a Scott-Suppes representation as well. A similar result was obtained in 1992 by Beja and Gilboa [5], who introduced new types of representations – *GNR* and *GUR*, having an appealing geometric flavor – of both interval orders and semiorders.

Following a stream of research providing "external" characterizations of Scott-Suppes representable semiorders [12], in 2010 Candeal and Induráin [11] obtained what they call an "internal" characterization of the Scott-Suppes representability of an arbitrary semiorder. Their characterization uses both regularity and *s-separability*, the latter being a condition similar to the Debreu-separability of a total preorder but involving the trace of the semiorder.[3]

There are many additional studies on semiorders, most of which however restrict their attention to the finite case. As a matter of fact, the monograph on semiorders [41] is almost entirely dedicated to finite semiorders, due to the intrinsic difficulties connected to the analysis of the

[1] On individual choice theory and the associated theory of revealed preferences, see also [13] (and references therein), where the authors develop an axiomatic approach based on the satisfaction of the so-called *weak* (m, n)-*Ferrers properties*, recently introduced by Giarlotta and Watson [31] (which include semiorders as particular cases, that is, binary relations that are both weakly $(2, 2)$-Ferrers and weakly $(3, 1)$-Ferrers).

[2] See [1] for a very recent survey on the Scott-Supper representability of a semiorder.

[3] By *external* the authors mean that the characterization is based on the construction of suitable ordered structures that are related to the given semiorder. On the other hand, *internal* means that the characterization is entirely expressed in terms of structural features of the semiorder.

infinite case.[4] Among the studies that concern infinite semiorders, let us mention the work of Rabinovitch [43], who proved in 1978 that the *dimension* of a strict semiorder is at most three (that is, the asymmetric part of a semiorder can be always written as the intersection of three strict linear orders).

In this paper, we describe the structure of an arbitrary semiorder, regardless of its size. In fact, we obtain a universal type of semiorder, in which every semiorder embeds (Theorem 5.6). These universal semiorders are suitably modified forms of lexicographic products of three total preorders. The modification is determined by a shift operator, which typically creates intransitive indifferences. Since the middle factor of these products is always the standard linear ordering $(\mathbb{Z}, \leq)$, and the shift operator is applied to it, we call these modified lexicographic structures *$\mathbb{Z}$-products*. In particular, we prove that *$\mathbb{Z}$-lines*, which are the $\mathbb{Z}$-products having linear orders as their extreme factors, are universal semiorders as well (Corollary 5.7).

Our results on semiorders are related to a general stream of research that uses lexicographic products to represent preference relations. In this direction, the literature in mathematical economics has been mainly focused on lexicographic representations of well-structured preferences, which assume the form of total preorders or linear orders. Historically – following some order-theoretic results of Hausdorff [32] and Sierpiński [47] concerning representations by means of lexicographically ordered transfinite sequences – Chipman [14] and Thrall [49] were the first authors to develop a theory of lexicographic preferences. Among the several important contributions that followed, let us recall the structural result of Beardon *et al.* [4], which provides a subordering classification of all chains that are *non-representable* in $\mathbb{R}$ (that is, they cannot be order-embedded into the reals).[5] The (dated but always valuable) survey of Fishburn [19] provides a good source of references on lexicographic representations of preferences.[6]

The results on lexicographic structures mentioned in the previous paragraph describe linear orders in terms of universal linear orders. The main result of this paper has a similar flavor, since it describes semiorders in terms of universal semiorders, that is, $\mathbb{Z}$-products (and, in particular, $\mathbb{Z}$-lines). In the process of obtaining such a representation, we explicitly construct a special $\mathbb{Z}$-product in which a given semiorder embeds (Theorem 5.6(iv)). The procedure that allows us to differentiate the elements of a semiordered structure can be summarized as follows:

(I) first consider a "macro-ordering", given by the transitive closure of the semiorder;

(II) then partition each equivalence class of the macro-ordering into "vertical slices" indexed by the integers, allowing only certain relationships between pairs of slices;

(III) finally establish a "micro-ordering" to further refine the distinction among elements of the semiorder, and obtain an order-embedding into a $\mathbb{Z}$-product.

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[4] To further emphasize this point, note that the First Edition (2002) of the treatise of Aleskerov *et al.* [3] on utility maximization, choice and preference was almost entirely dedicated to covering the analysis of the finite case. This is the main reason why a Second Edition of the book appeared in 2007. In fact, Chapter 6 of [3] is now entirely dedicated to preference representation theory for the infinite case (in particular, infinite semiorders).

[5] The mentioned result directly involves a basic prototype of lexicographic product, namely, the lexicographically ordered real plane $\mathbb{R}^2_{\text{lex}}$. (Note that $\mathbb{R}^2_{\text{lex}}$ is the example used by Debreu [15] in his famous paper on the *Open Gap Lemma* to disprove the inveterate belief that ordered preferences admit a real-valued utility representation.) Beardon *et al.* [4] prove the following: A linear ordering is non-representable in $\mathbb{R}$ if and only if it is either (i) *long* (i.e., it contains a copy of the first uncountable ordinal ω_1 or its reverse ordering ω_1^*), or (ii) *large* (i.e., it contains a copy of a non-representable subset of $\mathbb{R}^2_{\text{lex}}$), or (iii) *wild* (i.e., it contains a copy of an *Aronszajn line*, which is an uncountable chain such that neither ω_1 nor ω_1^* nor an uncountable subchain of the reals embeds into it.)

[6] For recent contributions on the topic, the reader may consult [10, 23, 24, 28, 29, 31, 33] and references therein.

The binary relations used at each stage of the construction are total preorders. This fact is obvious for the macro-ordering employed at stage (I). The partition of each indifference class of the transitive closure – done at stage (II) – is obtained by using a so-called *locally monotonic integer slicer (LMIS)*, which is an integer-valued map having some desirable order-preserving properties (Theorem 3.6). The micro-ordering employed at stage (III) is a modified form of trace, called *sliced trace*, which allows "backward paths" with respect to an LMIS (Theorem 4.8).

The three-step procedure described above is an abstraction/generalization of the shifting process that is classically applied for Scott-Suppes representations by using a threshold of discrimination. Representing semiorders as subsets of $\mathbb{Z}$-products and, in particular, $\mathbb{Z}$-lines allows us to gain a better insight into their structure. In fact, we believe that many of the results on semiorders scattered in the literature (e.g., Beja-Gilboa's GNR and GUR representations, Candeal and Induráin's internal characterization of semiorders, etc.) are subsumed by this description, and can be suitably generalized. Here we start giving a direct application of our results, and show that Rabinovitch's theorem on the dimension of a strict semiorder is a consequence of the main structure theorem (Corollary 5.9).

The paper is organized as follows. In Section 2 we recall all basic notions on semiorders, with particular emphasis on the properties of the trace of a semiorder. In Section 3 we introduce the notion of locally monotonic integer slicer, and prove that such a map always exists for a semiorder. In Section 4 we define a modified type of trace, called sliced trace, which is induced by a locally monotonic integer slicer and is based on the notion of backward path. In particular, we show that a slice trace of a semiorder is always a total preorder. Section 5 contains the descriptive characterization of a semiorder, in the form of its embeddability into the $\mathbb{Z}$-product having the transitive closure as first factor and a slide trace as last factor. We also show that $\mathbb{Z}$-lines are universal semiorders, and derive as a corollary Rabinovitch's result on the dimension of a strict semiorder. Section 6 concludes our analysis by summarizing the findings of the paper and suggesting future directions of research.

2 Preliminaries

In this paper, X denotes a nonempty – possibly infinite – set of alternatives, and $\precsim$ a reflexive[7] and complete[8] binary relation on X. We interpret "$x \precsim y$" as "alternative y is at least as good as alternative x". The pair $(X, \precsim)$ is called a *simple preference*; by a slight abuse of terminology, we also call the reflexive and complete relation $\precsim$ a simple preference (on X). As usual, the following two binary relations are associated to a simple preference $\precsim$ on X: its asymmetric[9] part $\prec$, called *strict preference*, and its symmetric[10] part $\sim$, called *indifference*. Thus, for each $x, y \in X$, we have by definition $x \prec y$ if $x \precsim y$ and $\neg(y \precsim x)$, and $x \sim y$ if $x \precsim y$ and $y \precsim x$. Note that a simple preference is the disjoint union of its strict preference and its indifference.

The process of passing from a simple preference to its asymmetric part is reversible. In fact, if $\prec$ is an asymmetric binary relation, then its *canonical completion* $\precsim$ is the simple preference defined by $x \precsim y$ if $\neg(y \prec x)$. Then the indifference $\sim$ associated to the primitive strict preference $\prec$ is defined exactly as in the previous case. As a consequence, whenever completeness is

[7] The relation $\precsim$ is *reflexive* if $x \precsim x$ for each $x \in X$.
[8] The relation $\precsim$ is *complete* (or *total*) if $x \precsim y$ or $y \precsim x$ for each distinct $x, y \in X$.
[9] The relation $\prec$ is *asymmetric* if $x \prec y$ implies $\neg(y \prec x)$ for each $x, y \in X$.
[10] The relation $\sim$ is *symmetric* if $x \sim y$ implies $y \sim x$ for each $x, y \in X$.

assumed, it is immaterial whether we take either a simple preference or an asymmetric preference as the primitive binary relation representing the preference structure of an economic agent.[11]

Recall that a reflexive (not necessarily complete) preference $\precsim$ on X is:

- *acyclic* if it contains no sequence of the type $x \prec x_1 \prec \ldots \prec x_n \prec x$, where $n \geq 1$;
- *quasi-transitive* if its strict preference $\prec$ is transitive;
- *Ferrers* if for all $x, x', y, y' \in X$, $\big(x \prec x' \wedge y \prec y'\big) \Longrightarrow \big(x \prec y' \vee y \prec x'\big)$ or, equivalently, $\big(x \precsim x' \wedge y \precsim y'\big) \Longrightarrow \big(x \precsim y' \vee y \precsim x'\big)$;
- *semitransitive* if for all $x, x', x'', y \in X$, $\big(x \prec x' \wedge x' \prec x''\big) \Longrightarrow \big(x \prec y \vee y \prec x''\big)$ or, equivalently, if $\big(x \precsim x' \wedge x' \precsim x''\big) \Longrightarrow \big(x \precsim y \vee y \precsim x''\big)$;
- an *interval order* if it is Ferrers, a *semiorder* if it is a semitransitive interval order, a *preorder* if it is transitive, and a *linear order* if it is an antisymmetric total preorder.[12]

Note that a total preorder is a transitive semiorder. It is immediate to check that the following implications hold for a reflexive preference $\precsim$:

$$\precsim \text{ Ferrers or semitransitive} \quad \Longrightarrow \quad \precsim \text{ quasi-transitive (and complete)} \quad \Longrightarrow \quad \precsim \text{ acyclic.}$$

In particular, an interval order (hence a semiorder) is a quasi-transitive simple preference. Sometimes, the strict part of an interval order (respectively, semiorder) is called a *strict interval order* (respectively, *strict semiorder*). The following well-known equivalences provide useful characterizations of quasi-transitive simple preferences, interval orders, and semiorders.

Lemma 2.1 *Let $(X, \precsim)$ be a simple*[13] *preference.*

(i) $\precsim$ *is quasi-transitive* $\iff$ $(\forall x, y, z \in X)\ \big((x \precsim y \prec z \ \vee\ x \prec y \precsim z) \Longrightarrow x \precsim z\big)$.

(ii) $\precsim$ *is Ferrers* $\iff$ $(\forall x, y, z, w \in X)\ \big(x \prec y \precsim z \prec w \Longrightarrow x \prec w\big)$.

(iii) $\precsim$ *is semitransitive* $\iff$ $(\forall x, y, z, w \in X)\ \big(x \precsim y \prec z \prec w \Longrightarrow x \prec w\big)$

$\iff$ $(\forall x, y, z, w \in X)\ \big(x \prec y \prec z \precsim w \Longrightarrow x \prec w\big)$.

Given a simple preference $\precsim$, its *transitive closure* $\precsim_{\rm tc}$ is defined as the smallest transitive relation containing $\precsim$. Observe that, due to the completeness of the simple preference $\precsim$, its transitive closure $\precsim_{\rm tc}$ is a total preorder, which simultaneously reduces strict preferences and augments indifferences: thus, the inclusions $\prec_{\rm tc} \subseteq \prec$ and $\sim \subseteq \sim_{\rm tc}$ hold.[14] In particular, a simple preference is a total preorder if and only if it is equal to its transitive closure.

Following Fishburn [17] (see also [3]), next we recall the notion of the trace of a simple preference, which is dual to that of transitive closure.

[11] On the point, see also the discussion about *injective* and *projective* families of binary relations in Section 2 of [26], in particular Example 1. Injectiveness and projectiveness are extensions (to a family of binary relations) of the properties of, respectively, antisymmetry and completeness of a single binary relation.

[12] Using the terminology of *weak/strict (m, n)-Ferrers properties* recently introduced in [31], a semiorder is a binary relation that is weakly (or strictly) $(2, 2)$-Ferrers and $(3, 1)$-Ferrers. See the last section of this paper for further details on (m, n)-Ferrers properties.

[13] Recall that a simple preference is reflexive and complete. Completeness is a necessary hypothesis.

[14] Here $\sim_{\rm tc}$ denotes the symmetric part of $\precsim_{\rm tc}$, which may be larger than the transitive closure of $\sim$.

Definition 2.2 Let $(X, \precsim)$ be a simple preference. For each $x, y \in X$, let

$$
\begin{array}{lcl}
x \prec^{*} y & \overset{\text{def}}{\Longleftrightarrow} & (\exists z \in X)\,(x \prec z \precsim y) \\
x \prec^{**} y & \overset{\text{def}}{\Longleftrightarrow} & (\exists z \in X)\,(x \precsim z \prec y) \\
x \prec_0 y & \overset{\text{def}}{\Longleftrightarrow} & (x \prec^{*} y) \vee (x \prec^{**} y) \\
x \precsim^{*} y & \overset{\text{def}}{\Longleftrightarrow} & \neg(y \prec^{*} x) \\
x \precsim^{**} y & \overset{\text{def}}{\Longleftrightarrow} & \neg(y \prec^{**} x) \\
x \precsim_0 y & \overset{\text{def}}{\Longleftrightarrow} & \neg(y \prec_0 x).
\end{array}
$$

The relations $\precsim^{*}$ and $\precsim^{**}$ are called, respectively, the *left trace* and the *right trace* of $\precsim$, whereas $\precsim_0$ is the *(global) trace* of $\precsim$. Further, for each $x \in X$, define

- *(weak lower section)* $x^{\downarrow,\precsim} := \{y \in X : y \precsim x\}$
- *(weak upper section)* $x^{\uparrow,\precsim} := \{y \in X : x \precsim y\}$
- *(strict lower section)* $x^{\downarrow,\prec} := \{y \in X : y \prec x\}$
- *(strict upper section)* $x^{\uparrow,\prec} := \{y \in X : x \prec y\}$.

The next result connects the trace of an interval order with upper and lower sections, and characterizes stronger types of preferences in terms of their traces: see, e.g., [3] (Sections 3.3–3.4), [7] (p. 105), and [38].

Lemma 2.3 *Let $\precsim$ be an interval order on X. For each $x, y \in X$, the following holds:*

$$
\begin{array}{lcccc}
x \precsim^{*} y & \Longleftrightarrow & x^{\downarrow,\precsim} \subseteq y^{\downarrow,\precsim} & \Longleftrightarrow & y^{\uparrow,\prec} \subseteq x^{\uparrow,\prec} \\
x \precsim^{**} y & \Longleftrightarrow & y^{\uparrow,\precsim} \subseteq x^{\uparrow,\precsim} & \Longleftrightarrow & x^{\downarrow,\prec} \subseteq y^{\downarrow,\prec} \\
x \precsim_0 y & \Longleftrightarrow & x^{\downarrow,\precsim} \subseteq y^{\downarrow,\precsim} \;\wedge\; y^{\uparrow,\precsim} \subseteq x^{\uparrow,\precsim} & \Longleftrightarrow & x^{\downarrow,\prec} \subseteq y^{\downarrow,\prec} \;\wedge\; y^{\uparrow,\prec} \subseteq x^{\uparrow,\prec}.
\end{array}
$$

Furthermore, we have:

(i) $\precsim^{*}$ *and* $\precsim^{**}$ *are total preorders contained in* $\precsim$*;*

(ii) $\precsim_0 = \precsim^{*} \cap \precsim^{**}$ *is a preorder contained in* $\precsim$ *such that for all* $x, y, z \in X$, $x \precsim y \precsim_0 z$ *implies* $x \precsim z$, *and* $x \precsim_0 y \precsim z$ *implies* $x \precsim z$*;*

(iii) $\prec_0$ *is asymmetric* $\Longleftrightarrow$ $\precsim_0$ *is a total preorder* $\Longleftrightarrow$ $\precsim$ *is a semiorder;*

(iv) *the equalities* $\precsim_{\mathrm{tc}} = \precsim^{*} = \precsim^{**} = \precsim_0 = \precsim$ *hold* $\Longleftrightarrow$ $\precsim$ *is a total preorder.*

Note that (i) says that the left and right traces have properties that are dual to those of the transitive closure; in particular, the inclusions $\prec \subseteq \prec^{*}$, $\prec \subseteq \prec^{**}$, $\sim^{*} \subseteq \sim$, and $\sim^{**} \subseteq \sim$ hold. Further, by (ii) and (iii), the trace of an interval order $\precsim$ is always reflexive and transitive, but completeness holds if and only if $\precsim$ is a semiorder.

We end this section by recalling the notions of embedding and isomorphism.

Definition 2.4 Let $(X, \precsim_X)$ and $(Y, \precsim_Y)$ be simple preferences. An injective map $f\colon X \to Y$ is an *order-embedding* (for short, *embedding*) if for each $x, x' \in X$, the equivalence

$$x \precsim_X x' \iff f(x) \precsim_Y f(x') \tag{1}$$

holds. Note that, since simple preferences are complete, (1) can be equivalently stated as

$$x \prec_X x' \iff f(x) \prec_Y f(x')\,. \tag{2}$$

A surjective embedding is called an *isomorphism*. We denote by $X \cong Y$ the fact that $(X, \precsim_X)$ and $(X, \precsim_Y)$ are *isomorphic* (i.e., there exists an isomorphism between them).

3 Locally monotonic integer slicers

In this section we show that a semiorder can be mapped to the integers in a "locally monotonic" fashion: in fact, we prove that each semiorder possesses a *locally monotonic integer slicer* (LMIS). Roughly speaking, an LMIS is obtained as the pasting of various integer-valued maps, whose domains are the equivalence classes of the transitive closure of the semiorder. These local maps satisfy suitable properties, which involve both the original semiorder and its trace.

To begin, we introduce a discrete measure of the strict domination of an alternative over a different one.

Definition 3.1 Let $(X, \precsim)$ be a simple preference and $x, y \in X$. A *strict chain* C from x to y is a finite sequence

$$x = w_0 \prec \ldots \prec w_n = y$$

of $n \geq 1$ strict relationships; in this case, $l(C) = n$ is the *length* of C. Denoted by $\mathsf{Ch}(x, y)$ the set of all strict chains from x to y (where x and y are not necessarily distinct), define

$$n(x, y) \ := \ \begin{cases} \sup\{l(C) : C \in \mathsf{Ch}(x, y)\} & \text{if } x \prec y \\ 0 & \text{otherwise.} \end{cases}$$

Note that $(X, \precsim)$ is acyclic if and only if each set $\mathsf{Ch}(x, x)$ is empty. Intuitively, $n(x, y)$ provides a rough evaluation of how strong the strict preference of y over x is. Some immediate consequences of the definition of $n(x, y)$ are listed below; their simple proof is left to the reader.[15]

Lemma 3.2 *Let $(X, \precsim)$ be a quasi-transitive simple preference. For each $x, y, z \in X$, we have:*

(i) $x \prec y \implies 1 \leq n(x, y) \leq \infty$;

(ii) $y \precsim x \implies n(x, y) = 0$;

(iii) $n(x, y) + n(y, x) \ = \ \max\{n(x, y), n(y, x)\}$;

(iv) $n(x, y) = n(y, x) = 0 \iff x \sim y$;

(v) *if $x \prec y \prec z$, then $n(x, y) + n(y, z) \leq n(x, z)$.*

[15] As usual, we assume that for each positive integer n, we have $n < \infty$, $n + \infty = \infty + n = \infty$, etc.

Observe that the hypothesis of quasi-transitivity is needed in Lemma 3.2. Under semitransitivity (but not necessarily Ferrers), $n(x,y)$ satisfies some additional properties.

Lemma 3.3 *Let $(X,\preceq)$ be a semitransitive simple preference. For each $x,y,z \in X$, we have:*

(i) *if $n(x,z) \geq 2$ and $y \precsim x$, then $n(y,z) \geq n(x,z)-1$;*

(ii) *if $n(x,z) \geq 2$ and $z \precsim y$, then $n(x,y) \geq n(x,z)-1$;*

(iii) *if $z \precsim y \precsim x$, then $n(x,z) \leq 1$;*

(iv) *if $x \sim_{\mathrm{tc}} y$, then $n(x,y) < \infty$.*

Proof. Let $x,y,z \in X$.

(i): Assume that $y \precsim x$. If $n(x,z)=2$, then there exists $w_1 \in X$ such that $y \precsim x \prec w_1 \prec z$. Since $\precsim$ is semitransitive, by Lemma 2.1(iii) we obtain $y \prec z$, and so $n(y,z) \geq 1 = n(x,z)-1$. Similarly, for $n(x,z) > 2$, there is a strict chain $x \prec w_1 \prec w_2 \prec \ldots \prec z$ of length ≥ 3, and so Lemma 2.1(iii) yields that $y \prec w_2 \prec \ldots \prec z$ is a strict chain from y to z.

(ii): This is dual to (i).

(iii): Assume by contradiction that $z \precsim y \precsim x$ and $n(x,z) \geq 2$. Thus, there exists $w \in X$ such that $y \precsim x \prec w \prec z$. However, this implies $y \prec z$ by Lemma 2.1(iii), which is impossible.

(iv): Assume that $x \sim_{\mathrm{tc}} y$. (Recall that $\sim_{\mathrm{tc}}$ is the symmetric part of $\precsim_{\mathrm{tc}}$.) By the definition of transitive closure, there are $n \geq 1$ and $w_1,\ldots,w_n \in X$ such that $y = w_1 \precsim \ldots \precsim w_n = x$. If $n \leq 3$, then we are immediately done by part (iii). Let $n \geq 4$. First, note that $n(x,w_{n-2}) \leq 1$ by part (iii). Further, since $w_j \precsim w_{j+1}$ for each j, part (ii) yields that if $n(x,w_{j+1})$ is finite, then so is $n(x,w_j)$. Thus the claim $n(x,w_1)=n(x,y)<\infty$ follows by induction. □

We shall use the family of integers $\{n(x,y) : x,y \in X\}$ to associate a locally monotonic map to a semiorder.[16] By "locally monotonic" we mean that this map behaves well when restricted to each equivalence class of the transitive closure. The next definition makes this notion precise for the general case of a simple preference.

Definition 3.4 Let $(X,\precsim)$ be a simple preference. A *locally monotonic integer slicer* (*LMIS*) for $(X,\precsim)$ is a function $\zeta\colon X \to \mathbb{Z}$ satisfying the following properties for each $x,y \in X$ belonging to the same indifference class of the transitive closure $\precsim_c$ of $\precsim$ (i.e., $x \sim_{\mathrm{tc}} y$):

(S1) $x \prec y \quad \Longrightarrow \quad \zeta(x) < \zeta(y)\,;$

(S2) $\zeta(x)+1 < \zeta(y) \quad \Longrightarrow \quad x \prec y\,;$

(S3) $\zeta(x) < \zeta(y) \quad \Longrightarrow \quad x \prec_0 y\,.$

If ζ only satisfies property (S1), then we call it a *weak LMIS*.

Before discussing the semantics of Definition 3.4, let us see what happens in the limit case of a "well-behaved" simple preference, that is, for a total preorder. The next proposition shows that if a simple preference is transitive, then the action of a locally monotonic integer slicer is limited to collecting indifferent elements together. In particular, the semantics of locally monotonic

[16]See the proof of Theorem 3.6.

integer slicers totally vanishes in the very special case of a linear order. Indeed, in this limit case, the transitive closure is equal to the linear order, and the latter already distinguishes all alternatives; therefore, an LMIS gives no contribution in the differentiation process.

Proposition 3.5 *Let $(X, \precsim)$ be a total preorder.*[17] *For any map $\zeta\colon X \to \mathbb{Z}$, the following statements are equivalent:*

(i) *ζ is an LMIS;*

(ii) *each preimage $\zeta^{-1}(n)$ is either empty or a union of $\sim$-equivalence classes;*

(iii) *ζ factors through the canonical projection $\pi\colon X \to X/\!\sim$, defined by $x \mapsto [x]$.*

In particular, any map from a linear order to the integers is an LMIS.

Proof. It suffices to prove that (i) is equivalent to (ii). To start, observe that since $(X, \precsim)$ is a total preorder, the equalities $\precsim \,=\, \precsim_{\rm tc} \,=\, \precsim_0$ hold by Lemma 2.3(iv); in particular, we have $\sim \,=\, \sim_{\rm tc} \,=\, \sim_0$. In this setting, any map $\zeta\colon X \to \mathbb{Z}$ satisfying property (S3) in Definition 3.4 automatically satisfies property (S2). Further, the two properties (S1) and (S3) hold for ζ if and only if so does the logical equivalence "$x \prec y \iff \zeta(x) < \zeta(y)$" for each $x, y \in X$ such that $x \sim y$. Thus, denoted by $[x']$ the class of elements in X that are $\sim$-indifferent to x', we obtain:

$$\begin{array}{lll} \zeta \text{ is an LMIS} & \iff & (\forall x, y \in X)\ \big(x \sim y \implies (x \prec y \iff \zeta(x) < \zeta(y))\big) \\ & \iff & (\forall x, y \in X)\ \big(x \sim y \implies \zeta(x) = \zeta(y)\big) \\ & \iff & (\forall n \in \mathbb{Z})\quad (\exists X' \subseteq X)\ \big(\zeta^{-1}(n) = \bigcup_{x' \in X'} [x']\big) \end{array}$$

where the set X' can possibly be empty (which happens if and only if $\zeta^{-1}(n) = \emptyset$). This proves the claim. □

Although we shall only use the notion of an LMIS for the case of semiorders, Definition 3.4 applies to an arbitrary simple preference. To get a better insight in the semantics of an LMIS, below we discuss the three properties that it must satisfy. Later on we shall prove that an LIMS of a semiorder can be obtained by using its trace (see the proof of Theorem 3.6).

As a preliminary remark, observe that an LIMS for a simple preference $\precsim$ can be seen as the union of mappings defined on the various equivalence classes of $\sim_{\rm tc}$, with no compatibility condition whatsoever between any two such maps. Condition (S1) is a natural monotonicity condition, which implies that any two elements mapped by an LMIS on the same integer must be indifferent according to the original preference relation. Condition (S2) in Definition 3.4 can be equivalently written in the following way:

(S2)$'$ $x \precsim y \implies \zeta(x) \le \zeta(y) + 1.$

Property (S2)$'$ is reminiscent of (a local version of) Scott-Suppes representability [46]. Further, observe that the combination of properties (S1) and (S2)$'$ is somehow analogous to the *Richter-Peleg representation* [40, 44] of a preorder. Finally, property (S3), which relates an LMIS to the trace of the simple preference, is more typical of semiorders (whose properties shall be needed to guarantee that the trace is a total preorder: see Lemma 2.3(iii)).

[17] In what follows $[x]$ denotes the $\sim$-equivalence class of $x \in X$. Similarly, $[x]_{\rm tc}$ is the $\sim_{\rm tc}$-equivalence class of x.

Intuitively, an LMIS arranges all elements belonging to one equivalence class of the transitive closure into "vertical slices" labeled by the integers, and behaves as a sort of strict embedding on each equivalence class. In fact, property (S1) says that whenever x and y are indifferent in the transitive closure, if y is strictly preferred to x in the original simple preference, then y is on a vertical slice that is located to the right of the vertical slice of x. Despite the converse of (S1) need not hold in general, properties (S2) and (S3) do guarantee partial forms of it (whence the appellative of "locally monotonic").

Specifically, property (S2) says that if y is on a vertical slice that is located to the right of the vertical slice of x but not immediately adjacent to it, then y is strictly preferred to x. Thus, in particular, if the vertical slice of x is immediately to the left of the vertical slice of y (i.e., $\zeta(x) + 1 = \zeta(y)$), then we might have either a strict preference of y over x or an indifference between x and y. Finally, property (S3) further contributes to a partial converse of (S1) by requiring that whenever y is on a slice that is located immediately to the right of that of x, at least the trace must record a strict preference of y over x. A graphical representation of the properties of an LMIS (on a single equivalence class of $\sim_{\mathrm{tc}}$) is given in Figure 1.

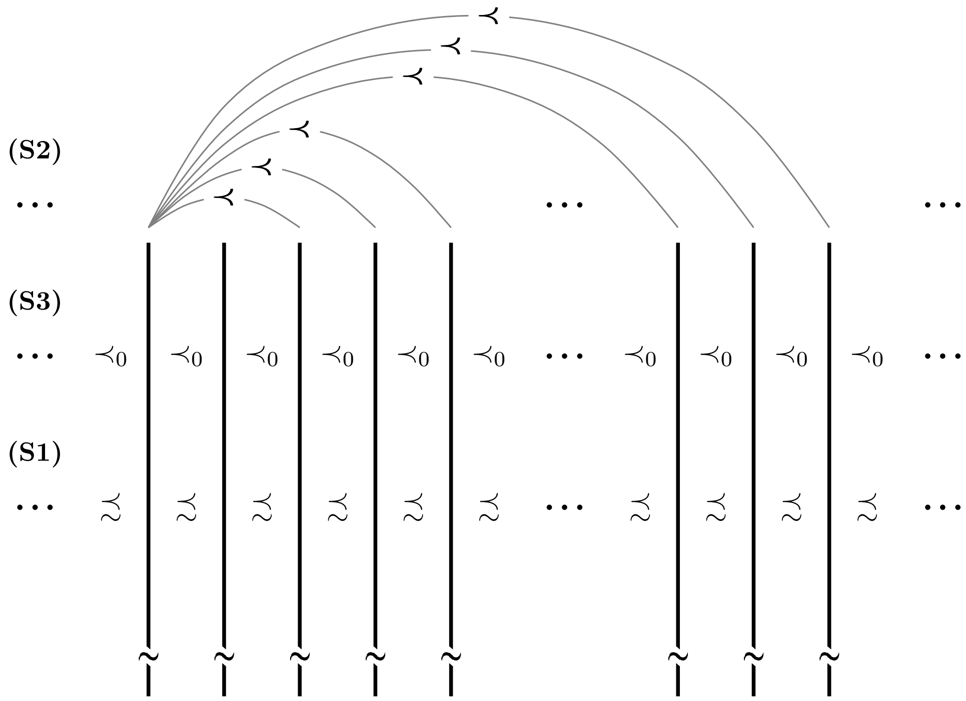

Figure 1: How an LMIS arranges a single equivalence class of $\sim_{\mathrm{tc}}$ in vertical slices.

Summarizing, if a simple preference $\precsim$ admits an LMIS, then the following holds for any two elements that are in the same $\sim_{\mathrm{tc}}$-equivalence class:

- if they are located on the same vertical slice, then they are indifferent (w.r.t. $\sim$);

- if they are located in adjacent vertical slices, then either the right element is strictly preferred to the left element, or they are indifferent; at any rate, the right element is always strictly trace-preferred to the left element;

- if they are located in vertical slices that are more than one slice far apart, then there is always a strict preference of the right element over the left element.

The next result shows that, in the special case of a semiorder, we can use the trace and the integers $n(x,y)$ (see Definition 3.1) to obtain an LMIS.

Theorem 3.6 *Each semiorder admits a locally monotonic integer slicer.*

PROOF. Let $\precsim$ be a semiorder on X. Consider the partition of X induced by the indifference $\sim_{\rm tc}$ associated to the transitive closure $\precsim_{\rm tc}$ of $\precsim$, and let T be a *transversal* of this partition (i.e., T intersects each equivalence class in a singleton). For each $t \in T$, denote by $[t]_{\rm tc}$ the $\sim_{\rm tc}$-equivalence class of t. In what follows, we define a map $\zeta_t \colon [t]_{\rm tc} \to \mathbb{Z}$ such that the three properties (S1)-(S3) in Definition 3.4 hold for all $x, y \in [t]_{\rm tc}$. Then $\zeta := \bigcup_{t\in T} \zeta_t$ is an LMIS for $(X, \precsim)$, as claimed. For each $x \in [t]_{\rm tc}$, let

$$\zeta_t(x) \;:=\; \begin{cases} 0 & \text{if } \; t \sim x \; \text{ and } \; x \precsim_0 t \\ 1 & \text{if } \; t \sim x \; \text{ and } \; t \prec_0 x \\ -n(x,t) & \text{if } \; x \prec t \\ n(t,x)+1 & \text{if } \; t \prec x \,. \end{cases}$$

The function ζ_t is well-defined by Lemma 2.1(iii) (which ensures that the trace $\precsim_0$ is complete) and Lemma 3.3(iv) (which ensures that $n(x,t)$ and $n(t,x)$ are integers). To complete the proof, we show that properties (S1), (S2) and (S3) hold for ζ_t.

(S1): Assume that $x, y \in [t]_{\rm tc}$ are such that $x \prec y$. We split the analysis in the following exhaustive list of cases:

(1) $x \sim t$ or $y \sim t$;

(2) $x, y \prec t$ or $t \prec x, y$;

(3) $x \prec t \prec y$ or $y \prec t \prec x$.

For case (1), first let $x \sim t$. If $t \prec y$, then we are done, since $\zeta_t(x) \leq 1 < n(t,y)+1 = \zeta(y)$. Thus, assume that $y \precsim t$, in fact $y \sim t$. (Indeed, if $y \prec t$, then $x \prec y \prec t$, which is impossible.) It follows that both $x \prec y \precsim t$ and $t \precsim x \prec y$ hold, whence $x \prec_0 t \prec_0 y$. Now the definition of ζ_t gives $\zeta_t(x) = 0 < 1 = \zeta_t(y)$, and the claim holds. The subcase $y \sim t$ is dual to the previous one: if $x \prec t$, then $\zeta_t(x) = -n(x,t) < 0 \leq \zeta_t(y)$; on the other hand, if $t \precsim x$ (in fact, $t \sim x$), then $x \prec_0 t \prec_0 y$, and $\zeta_t(x) = 0 < 1 = \zeta_t(y)$.

For case (2), assume that $x \prec t$ and $y \prec t$, hence $\zeta_t(x) = -n(x,t)$ and $\zeta_t(y) = -n(y,t)$. The hypothesis $x \prec y$ implies that any strict chain from y to t can be elongated toward the left by appending x, and so $n(y,t) < n(x,t)$, which implies $\zeta_t(x) < \zeta_t(y)$. The subcase $t \prec x, y$ is dual to the previous one.

Finally, $x \prec t \prec y$ readily yields $\zeta_t(x) < 0 < \zeta_t(y)$, whereas $y \prec t \prec x$ contradicts the quasi-transitivity of $\precsim$. This completes the proof that ζ_t satisfies (i).

(S2): We prove the contrapositive, i.e., we assume that $y \precsim x$ and show that $\zeta_t(y) - 1 \leq \zeta_t(x)$. As in part (i), we analyze separately cases (1)-(3).

For case (1), first assume that $x \sim t$, hence $0 \leq \zeta_t(x) \leq 1$. Since $y \precsim x \precsim t$, Lemma 3.3(iii) yields $n(t,y) \leq 1$. Thus, the inequality $\zeta_t(y) - 1 \leq \zeta_t(x)$ holds in any circumstance unless $\zeta_t(x) = 0$ and $\zeta_t(y) = n(t,y) + 1 = 2$. However, the latter situation may happen only if $x \precsim_0 t \prec y$, which is impossible since $y \in x^{\downarrow,\precsim} \setminus t^{\downarrow,\precsim}$ contradicts the characterization of the trace $\precsim_0$ given by Lemma 2.3. To complete the analysis of case (1), assume that $y \sim t$ and $\neg(x \sim t)$. The claim holds trivially whenever $t \prec x$, thus let $x \prec t$. It follows that $y \prec_0 t$, and so $\zeta_t(y) = 0$. On the other hand, $t \precsim y \precsim x$ implies $n(x,t) \leq 1$ by Lemma 3.3(iii). Thus $\zeta_t(x) = -n(x,t) = -1$, and the claim is verified also in this circumstance.

In case (2), let $x \prec t$ and $y \prec t$, hence $\zeta_t(x) = -n(x,t)$ and $\zeta_t(y) = -n(y,t)$. If $n(x,t) = 1$, then the claim holds trivially. Otherwise, $n(x,t) \geq 2$, and we can apply Lemma 3.3(i) to obtain $\zeta_t(y) - 1 \leq \zeta_t(x)$. The subcase $t \prec x, y$ can be handled similarly, using Lemma 3.3(ii).

For case (3), $x \prec t \prec y$ contradicts the hypothesis $y \precsim x$, whereas the claim holds trivially whenever $y \prec t \prec x$.

(S3): Suppose $\zeta_t(x) < \zeta_t(y)$. If $\zeta_t(x) < 0 \leq \zeta_t(y)$, then $x \prec t \precsim y$, hence $x \prec^* y$, which implies $x \prec_0 y$ (see Definition 2.2). Dually, if $\zeta_t(x) \leq 1 < \zeta_t(y)$, then $x \precsim t \prec y$, hence $x \prec^{**} y$, and so $x \prec_0 y$. If $\zeta_t(x) < \zeta_t(y) < 0$, then let $x \prec w \prec \ldots \prec t$ be a strict chain from x to t of length $n(x,t) \geq 2$. Since $n(y,t) < n(x,t)$, we have $\neg(y \prec w)$, hence $x \prec w \precsim y$, which implies $x \prec_0 y$. Dually, if $1 < \zeta_t(x) < \zeta_t(y)$, let $t \prec \ldots \prec w \prec y$ be a strict chain from t to y of length $n(t,y) \geq 2$. Since $n(t,x) < n(t,y)$, we have $\neg(w \prec x)$, hence $x \precsim w \prec y$, which again implies $x \prec_0 y$. Finally, if $\zeta_t(x) = 0 < \zeta_t(y) = 1$, then $x \prec_0 t \prec_0 y$. □

The following example exhibits an LMIS for a classical type of (Scott-Suppes representable) semiorder on the reals, according to the construction described in the proof of Theorem 3.6.[18]

Example 3.7 Let $\precsim$ be the typical Scott-Suppes representable semiorder on $\mathbb{R}$, defined as follows for each $x, y \in \mathbb{R}$:

$$x \precsim y \quad \overset{\text{def}}{\iff} \quad x \leq y + 1.$$

Note that the trace $\precsim_0$ is the usual linear order $\leq$ of the reals, and the transitive closure $\precsim_{\text{tc}}$ is the whole $\mathbb{R}^2$ (that is, all reals are in a single $\sim_{\text{tc}}$-equivalence class). Thus, using the notation in the proof of Theorem 3.6, we may take $T = \{0\}$ as a transversal, whence $[0]_{\text{tc}} = \mathbb{R}$ and $\zeta = \zeta_0$. An easy computation shows that, for all $x \in \mathbb{R}$, we have

$$n(x,0) = \begin{cases} 0 & \text{if } x \in [-1, +\infty) \\ k & \text{if } x \in [-k-1, -k) \end{cases} \qquad \text{and} \qquad n(0,x) = \begin{cases} 0 & \text{if } x \in (-\infty, 1] \\ k & \text{if } x \in (k, k+1] \end{cases}$$

where k ranges over $\mathbb{N} \setminus \{0\}$. Consequently, the LMIS $\zeta = \zeta_0 \colon \mathbb{R} \to \mathbb{Z}$ is defined as follows for each $x \in \mathbb{R}$:

$$\zeta(x) = \begin{cases} 0 & \text{if } x \sim 0 \wedge x \precsim_0 0 & (\iff x \in [-1, 0]) \\ 1 & \text{if } x \sim 0 \wedge 0 \prec_0 x & (\iff x \in (0, 1]) \\ -n(x,0) & \text{if } x \prec 0 & (\iff x \in (-\infty, -1)) \\ n(0,x) & \text{if } 0 \prec x & (\iff x \in (1, +\infty)) \end{cases}$$

[18] We thank the two referees for suggesting this classical example of semiorder as an illustration of our approach.

that is,

$$\zeta(x) = \begin{cases} 0 & \text{if } x \in [-1, 0] \\ 1 & \text{if } x \in (0, 1] \\ -k & \text{if } x \in [-k-1, -k) \\ k+1 & \text{if } x \in (k, k+1] \end{cases}$$

where $k \in \mathbb{N} \setminus \{0\}$. In conclusion, if we select the singleton $T = \{0\}$ as transversal, then the LMIS $\zeta = \zeta_0$ is defined by

$$\zeta(x) = \begin{cases} \lfloor x \rfloor + 1 & \text{if } x < -1 \\ 0 & \text{if } -1 \leq x \leq 0 \\ \lceil x \rceil & \text{if } x > 0 \end{cases}$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ are the floor and the ceiling of $x \in \mathbb{R}$, respectively: see Figure 2. Note that if we choose another representative (different from zero) of the unique equivalence class of the transitive closure, then we get an LMIS that is a translation of ζ_0. As we shall see below, LMIS's are not unique (up to translations): see Example 4.4 for a simpler instance of an LMIS for the same semiorder $(\mathbb{R}, \precsim)$.

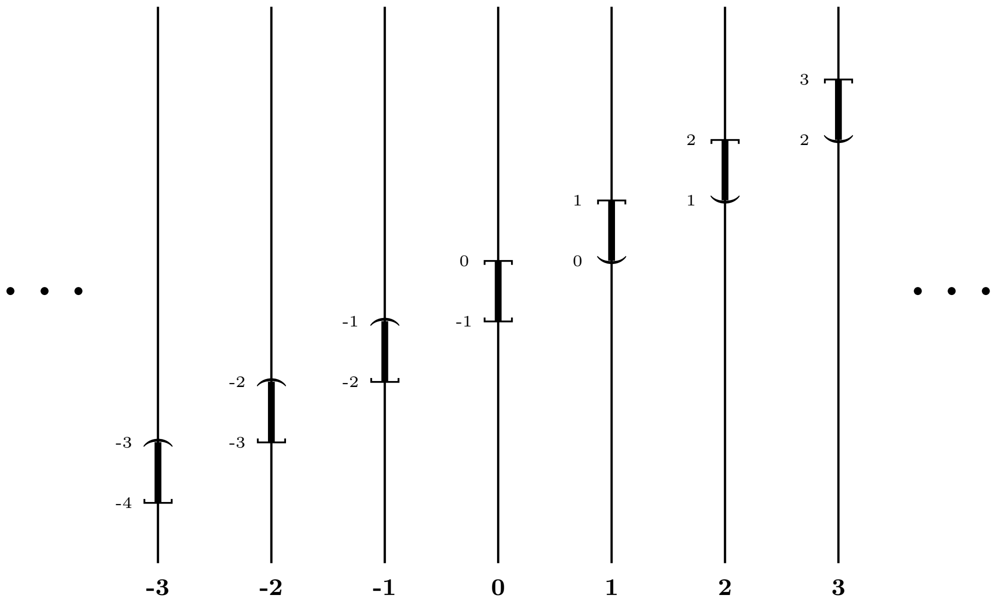

Figure 2: The LMIS ζ_0 defined in the proof of Theorem 3.6 for the classical semiorder on $\mathbb{R}$. (In accordance with Figure 1, elements of $\mathbb{R}$ are arranged vertically, and values assumed by ζ_0 run horizontally.)

In the proof of Theorem 3.6, the definition of each function $\zeta_t \colon [t]_{\rm tc} \to \mathbb{Z}$ that is used to obtain the LMIS $\zeta = \bigcup_{t \in T} \zeta_t$ is asymmetric, in the sense that ζ_t maps the subset $[t] \subseteq [t]_{\rm tc}$ to 0 or 1. (Recall that $[t]$ denotes the set of elements of X that are $\sim$-indifferent to t.) One may wonder whether it is possible to make the definition of ζ_t symmetric, by mapping $[t] \mapsto 0$, regardless of the trace $\precsim_0$. This question is relevant in view of the fact that the only place in the proof where we use the full power of a semiorder (that is, the Ferrers property, since semitransitivity is needed throughout) is to show that ζ_t is well-defined. However, the answer to the above question is negative, as the next example shows.

Example 3.8 Let $\prec$ be the asymmetric relation on $X = \{x_1, x_2, x_3, x_4, x_5\}$ defined as follows:

$$x_i \prec x_j \quad \overset{\text{def}}{\Longleftrightarrow} \quad i + 2 \leq j.$$

Let $\precsim$ be the canonical completion of $\prec$, that is, $x_i \precsim x_j \overset{\text{def}}{\Longleftrightarrow} \neg(x_j \prec x_i)$. It is immediate to check that $\precsim$ is a semiorder on X such that $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_5$. In particular, we have $[x_3] = \{x_2, x_3, x_4\}$ and $x_2 \prec_0 x_3 \prec_0 x_4$. However, since $x_2 \prec x_4$, a function that maps the whole $\sim$-indifference class $[x_3]$ to a single element is not even a weak LMIS, because property (S1) in Definition 3.4 fails to hold.

4 The sliced trace of a semiorder

As summarized in the Introduction, the idea of our approach is to successively distinguish alternatives of a semiordered structure in three stages: (I) take the transitive closure of the semiorder as a "macro-ordering", (II) suitably partition each equivalence class of the macro-ordering into "vertical slices", and (III) establish a "micro-ordering" to further refine the distinction among elements of the semiorder. To make the procedure effective, we have to make sure that the ordering obtained at every stage is indeed a total preorder.

In this section we describe the micro-ordering used in stage (III), hereafter referred to as a "sliced trace" of the primitive semiorder. Its name comes from the fact that a sliced trace is a suitable modification of the (global) trace of a semiorder, obtained by means of an integer-valued map that locally preserves the asymmetric ordering but not necessarily the associated indifference. In fact, we shall construct a sliced trace of a semiorder by using a weak LMIS, since the properties (S2) and (S3) of a (general) LMIS (see Definition 3.4) are not needed at this stage. The main result of this section is that a sliced trace is indeed a total preorder.

To formally define what a sliced trace is, first we introduce the preliminary notion of a "backward path" with respect to a weak LMIS. This notion only applies to elements that are indiscernible by the "macro-ordering", that is, elements that are indifferent with respect to the transitive closure of the semiorder.

Definition 4.1 Let $\precsim$ be a semiorder on X, and $\zeta \colon (X, \precsim) \to (\mathbb{Z}, \leq)$ a weak LMIS (i.e., if $x \sim_{\text{tc}} y$ and $x \prec y$, then $\zeta(x) < \zeta(y)$). For each $x, y \in X$ such that $x \sim_{\text{tc}} y$, a *ζ-backward path* from x to y is a sequence $y = w_n \succsim \ldots \succsim w_0 = x$ of positive length $n \geq 1$ such that $\zeta(w_{i+1}) < \zeta(w_i)$ for each $0 \leq i \leq n-1$. (Observe that the notion of backward path is not defined for elements belonging to distinct equivalence classes of the transitive closure.) We denote the existence of a ζ-backward path from x to y by $y \curvearrowleft_\zeta x$.

The symbol "$\curvearrowleft_\zeta$", here employed for the existence of a ζ-backward path, is suggestive of its semantics. In fact, since a weak LMIS ζ arranges $\sim_{\text{tc}}$-equivalent elements of the semiorder $(X, \precsim)$ into slices indexed by the integers, $y \curvearrowleft_\zeta x$ means that x and y are connected by a finite sequence of indifference relationships $\sim$ (being $y \sim_{\text{tc}} x$), but y is on a slice located to the left of the slice where x is. For the same reason, in Definition 4.1 we use the reverse of the semiorder (that is, $\succsim$ in place of $\precsim$) to describe a ζ-backward path. The following example describes some instances of backward paths in the classical semiorder on the reals.

Example 4.2 Let $\zeta = \zeta_0 \colon \mathbb{R} \to \mathbb{Z}$ be the LMIS for the semiorder $(\mathbb{R}, \precsim)$ defined in Example 3.7 (and represented in Figure 2). In what follows, we construct a ζ-backward path of arbitrary positive length. Let $\{\varepsilon_n : n \in \mathbb{N}\}$ be a strictly increasing sequence in the open interval $(0, 1/4)$. For each $i \in \mathbb{Z}$, set

$$x_i \ := \begin{cases} i + \frac{1}{2} - \varepsilon_i & \text{if } i \geq 0 \\ \\ i + \frac{1}{2} + \varepsilon_{-i} & \text{if } i < 0\,. \end{cases}$$

The elements of the $\mathbb{Z}$-sequence $\{x_i : i \in \mathbb{Z}\}$ are obviously located in distinct slices of the given representation. Since the sequence $\{\varepsilon_n : n \in \mathbb{N}\}$ is strictly increasing, it follows that two elements of the $\mathbb{Z}$-sequence are indifferent if and only if they are consecutive. Moreover, we have:

$$\begin{array}{ccccccccccccc} \ldots & \succsim & x_{-2} & \succsim & x_{-1} & \succsim & x_0 & \succsim & x_1 & \succsim & x_2 & \succsim & \ldots \\ \ldots & < & \zeta(x_{-2}) & < & \zeta(x_{-1}) & < & \zeta(x_0) & < & \zeta(x_1) & < & \zeta(x_2) & < & \ldots \end{array}$$

that is, $x_r \curvearrowleft_\zeta x_s$ for each $r, s \in \mathbb{Z}$ with $r < s$. For instance, the sequence $x_{-1} \succsim x_0 \succsim x_1 \succsim x_2$ is a ζ-backward path of length 3 from $x_2 = 5/2 - \varepsilon_2$ to $x_{-1} = 1/2 + \varepsilon_1$. In fact, the set $\{x_i : i \in \mathbb{Z}\}$ can be though as a limit type of ζ-backward paths (in order-type $\mathbb{Z}$) going from $+\infty$ to $-\infty$. (Note that all elements of $\mathbb{R}$ are indifferent in the transitive closure of the semiorder $\precsim$.)

Observe that since a weak LMIS ζ only preserves the strict ordering (on each indifference class of the transitive closure), it may well happen that $x \sim y$ and $\zeta(x) \neq \zeta(y)$. In particular, ζ may fail to be a homomorphism[19] even locally, that is, when restricted to each equivalence class of the transitive closure.

In the next definition we describe the "micro-ordering" of stage (III) on the basis of the trace and the possible existence of backward paths with respect to a weak LMIS.

Definition 4.3 Let $\precsim$ be a semiorder on X, and $\zeta \colon (X, \precsim) \to (\mathbb{Z}, \leq)$ a weak LMIS. The *sliced trace* of $\precsim$ *induced by* ζ (or, simply, the ζ*-trace* of $\precsim$) is the binary relation $\precsim_\zeta$ on X defined as follows for each $x, y \in X$:

$$x \precsim_\zeta y \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (x \precsim_0 y \ \wedge \ x \not\curvearrowleft_\zeta y) \ \vee \ y \curvearrowleft_\zeta x.$$

The following example exhibits an instance of a sliced trace for the classical semiorder on the reals. In order to provide the reader with a different perspective, the ζ-trace considered below is induced by an LMIS ζ that is a slight modification (in fact, a simplification) of the one given in Example 3.7.[20]

Example 4.4 Consider the semiorder $\precsim$ on $\mathbb{R}$ defined in Example 3.7. The floor function $\zeta \colon \mathbb{R} \to \mathbb{Z}$, defined by $\zeta(x) := \lfloor x \rfloor$ for each $x \in \mathbb{R}$, satisfies properties (S1)-(S3) in Definition 3.4, hence it is an LMIS for $(\mathbb{R}, \precsim)$. In order to describe the sliced trace induced by ζ, we start by determining ζ-backward paths. Since $\precsim_{\text{tc}} = \mathbb{R}^2$, for each $x, y \in \mathbb{R}$, we have:

$$y \curvearrowleft_\zeta x \quad \Longleftrightarrow \quad \lfloor y \rfloor < \lfloor x \rfloor \ \wedge \ y - \lfloor y \rfloor \geq x - \lfloor x \rfloor\,. \tag{3}$$

[19] Given two simple preferences $(X, \precsim_X)$ and $(Y, \precsim_Y)$, a *homomorphism* is a map $f \colon X \to Y$ such that $x \precsim_X x'$ implies $f(x) \precsim_Y f(x')$ for each $x, x' \in X$.

[20] See also Example 5.5.

To prove (3), let $x, y \in \mathbb{R}$. Assume that both $\lfloor y \rfloor < \lfloor x \rfloor$ and $y - \lfloor y \rfloor \geq x - \lfloor x \rfloor$ hold. Let r_n be a non-increasing finite sequence of real numbers, with $\lfloor y \rfloor = \zeta(y) \leq n \leq \zeta(x) = \lfloor x \rfloor$, such that $r_{\zeta(y)} = y - \lfloor y \rfloor$ and $r_{\zeta(x)} = x - \lfloor x \rfloor$. (The sequence is constant in case $x - \lfloor x \rfloor = y - \lfloor y \rfloor$.) Then, since all numbers in the sequence

$$y \succsim \zeta(y) + 1 + r_{\zeta(y)+1} \succsim \zeta(y) + 2 + r_{\zeta(y)+2} \succsim \ldots \succsim \zeta(x) - 2 + r_{\zeta(x)-2} \succsim \zeta(x) - 1 + r_{\zeta(x)-1} \succsim x$$

are located on distinct slices by construction, it follows that $y \curvearrowleft_\zeta x$. Conversely, if either $\lfloor y \rfloor \geq \lfloor x \rfloor$ or $y - \lfloor y \rfloor < x - \lfloor x \rfloor$ holds, then it is easy to check that there are no backward paths from x to y. This proves (3). Intuitively, there is a backward path from x to y if and only if (i) the slice Z_x of x is on the right of the slice Z_y of y, and (ii) the height of y in Z_y is greater or equal than the height of x in Z_x. (Here the "height" of an element x is given by $x - \lfloor x \rfloor$.)

We are now ready to describe the sliced trace induced by ζ. According to the definition of ζ-trace, the following chain of strict inequalities holds:

$$\ldots \succ_\zeta 1.2 \succ_\zeta 2.2 \succ_\zeta 3.2 \succ_\zeta 4.2 \succ_\zeta \ldots$$

This is true because there are backward paths from 4.2 to 3.2, from 3.2 to 2.2, etc., but no backward paths in the opposite direction. In other words, the ζ-trace $\precsim_\zeta$ reverses the order of the standard trace $\precsim_0$ (which is equal to the linear order of the reals) on each horizontal slice. ("Horizontal" slices are formed by elements at the same height.) Furthermore, we have

$$\ldots,\ 1.5\,,\ 2.5\,,\ 3.5 \prec_\zeta 3.6 \qquad \text{and} \qquad 3.6 \succ_\zeta 4.5\,,\ 5.5\,,\ 6.5\,, \ldots$$

The inequalities on the left hand side hold by the first part of definition of ζ-trace: in fact, the number 3.6 is strictly bigger than all elements of S in the trace, and there are no backward paths of any kind between 3.6 and the elements of $S := \{i + 0.5 : i \in \mathbb{Z}, i \leq 3\}$. On the other hand, the inequalities on the right hand side hold by the second part of the definition of ζ-trace, since there exist backward paths from each of the elements of $T := \{i + 0.5 : i \in \mathbb{N}, i \geq 4\}$ to the number 3.6. In other words, in the ζ-trace, any point at a certain height is strictly bigger that any point located at a strictly smaller height. From the above discussion it follows that

$$x \prec_\zeta y \qquad \Longleftrightarrow \qquad \big(x - \lfloor x \rfloor < y - \lfloor y \rfloor\big) \ \vee\ \Big(\big(x - \lfloor x \rfloor = y - \lfloor y \rfloor\big) \ \wedge\ \big(x > y\big)\Big)$$

for each $x, y \in \mathbb{R}$. Therefore, we can conclude that the ζ-trace is isomorphic to the linear order (lexicographic product) $[0,1) \times_{\mathrm{lex}} \mathbb{Z}^*$, where $\mathbb{Z}^* = (\mathbb{Z}, \geq)$ is the reverse ordering of the integers.

The next result lists some simple properties of each sliced trace of a semiorder.

Lemma 4.5 *Let $\precsim$ be a semiorder on X, and $\zeta\colon (X, \precsim) \to (\mathbb{Z}, \leq)$ a weak LMIS.*

(i) *The relation $\curvearrowleft_\zeta$ is a strict partial order.*[21]

(ii) *For each $x, y \in X$, if $y \curvearrowleft_\zeta x$ then $x \prec_\zeta y$.*

(iii) *For each $x, y \in X$ such that $\neg(x \sim_{\mathrm{tc}} y)$, we have $x \precsim_\zeta y \iff x \precsim_0 y$.*

(iv) *For each $x, y \in X$ such that $x \sim_{\mathrm{tc}} y$ and $\zeta(x) + 1 = \zeta(y)$, we have $x \prec_\zeta y \iff x \prec y$.*

[21] A *strict partial order* is an asymmetric and transitive binary relation.

(v) *For each $x, y \in X$ such that $x \sim_{\mathrm{tc}} y$ and $\zeta(x) = \zeta(y)$, we have $x \prec_\zeta y \iff x \prec_0 y$.*

(vi) *The relation $\precsim_\zeta$ is a simple preference.*

Proof. (i): We prove asymmetry first. Toward a contradiction, let $x, y \in X$ be such that $y \curvearrowleft_\zeta x$ and $x \curvearrowleft_\zeta y$. Thus, we have $x \sim_{\mathrm{tc}} y$ and $x = w_n \succsim \ldots \succsim w_0 = y = v_m \succsim \ldots \succsim v_0 = x$, with $\zeta(w_{i+1}) < \zeta(w_i)$ and $\zeta(v_{j+1}) < \zeta(v_j)$ for all i's and j's, where $m, n \geq 1$. In particular, $\zeta(x) < \zeta(y) < \zeta(x)$. This proves that $\curvearrowleft_\zeta$ is asymmetric. The proof of transitivity is similar, considering the union of two backward paths.

(ii): This is an immediate consequence of the definition of ζ-trace, using the asymmetry of $\curvearrowleft_\zeta$ established in (i).

(iii): This follows from the fact that $\neg(x \sim_{\mathrm{tc}} y)$ implies $x \not\curvearrowleft_\zeta y$ and $y \not\curvearrowleft_\zeta x$.

(iv): Let $x, y \in X$ be such that $x \sim_{\mathrm{tc}} y$ and $\zeta(x) + 1 = \zeta(y)$. For necessity, assume that $x \prec y$, in particular $x \prec_0 y$. Note that the hypothesis yields that there is no ζ-backward path from y to x (neither of length 1 because $\neg(x \succsim y)$, nor of length $n \geq 2$ because $\zeta(x) = \zeta(y) - 1$). Thus we have $x \not\curvearrowleft_\zeta y$, and so $x \precsim_\zeta y$ holds. Further, since $y \not\curvearrowleft_\zeta x$, we have $\neg(y \precsim_\zeta x)$. It follows that $x \prec_\zeta y$. Conversely, if $x \succsim y$, then there exists a ζ-backward path of length 1 from y to x, and so $x \curvearrowleft_\zeta y$. It follows that $y \precsim_\zeta x$, thus proving (v).

(v): This is an immediate consequence of the definition of ζ-trace.

(vi): We show that $\precsim_\zeta$ is reflexive and complete. By part (i), $\curvearrowleft_\zeta$ is irreflexive.[22] Since the trace is reflexive, it follows that so is the ζ-trace. The completeness of the ζ-trace is a consequence of that of the trace. To see this, assume that $x, y \in X$ are such that $x \neq y$. By part (ii), $y \curvearrowleft_\zeta x$ implies $x \prec_\zeta y$, and $x \curvearrowleft_\zeta y$ implies $y \prec_\zeta x$. On the other hand, if $y \not\curvearrowleft_\zeta x$ and $x \not\curvearrowleft_\zeta y$, then either $x \precsim_0 y$ and so $x \precsim_\zeta y$, or $y \precsim_0 x$ and so $y \precsim_\zeta x$. Thus (vii) holds as well, and the proof is complete. □

Whenever $\curvearrowleft_\zeta$ is empty, obviously the ζ-trace and the (standard) trace of a semiorder coincide. However, even when $\curvearrowleft_\zeta$ is nonempty, it turns out that the ζ-trace and the trace of a semiorder do have a similar structure, both of them being total preorders. The remainder of this section is devoted to prove that each sliced trace of a semiorder is indeed transitive. To that end, we need two technical results, which we prove first.

Lemma 4.6 *Let $\precsim_\zeta$ be the ζ-trace of a semiorder $\precsim$ on X. For each $x, y, z \in X$ such that $x \sim_{\mathrm{tc}} y \sim_{\mathrm{tc}} z$ and $x \precsim_\zeta y \precsim_\zeta z \prec_\zeta x$, we have:*

(i) *$x \not\curvearrowleft_\zeta y$, $y \not\curvearrowleft_\zeta z$, $z \not\curvearrowleft_\zeta x$, and*

(ii) *one of the following:*

(1) *$y \curvearrowleft_\zeta x$, $z \not\curvearrowleft_\zeta y$, $x \not\curvearrowleft_\zeta z$, and $y \precsim_0 z \precsim_0 x$, or*

(2) *$z \curvearrowleft_\zeta y$, $x \not\curvearrowleft_\zeta z$, $y \not\curvearrowleft_\zeta x$, and $z \precsim_0 x \precsim_0 y$, or*

(3) *$x \curvearrowleft_\zeta z$, $y \not\curvearrowleft_\zeta x$, $z \not\curvearrowleft_\zeta y$, and $x \precsim_0 y \precsim_0 z$.*

[22] A binary relation R on X is *irreflexive* if $\neg(xRx)$ for all $x \in X$. Asymmetry implies irreflexivity.

Proof. Let $x, y, z \in X$ be such that $x \sim_{\rm tc} y \sim_{\rm tc} z$ and $x \precsim_\zeta y \precsim_\zeta z \prec_\zeta x$. Part (i) readily follows from the definition of $\precsim_\zeta$, using Lemma 4.5(i). Next we show that one among (1), (2) and (3) holds. If $x \curvearrowleft_\zeta z$ and $z \curvearrowleft_\zeta y$, then $x \curvearrowleft_\zeta y$ by Lemma 4.5(i), and so $y \prec_\zeta x$ Lemma 4.5(ii), which contradicts the hypothesis $x \precsim_\zeta y$. If $y \curvearrowleft_\zeta x$ and $x \curvearrowleft_\zeta z$, then $y \curvearrowleft_\zeta z$ by Lemma 4.5(i), and so $z \prec_\zeta y$ by Lemma 4.5(ii), which contradicts the hypothesis $y \precsim_\zeta z$. If $z \curvearrowleft_\zeta y$ and $y \curvearrowleft_\zeta x$, then $z \curvearrowleft_\zeta x$, and so $x \prec_\zeta z$, which is false. Now a simple case analysis shows that one of the following cases must happen:

(4) $y \curvearrowleft_\zeta x$, $z \not\curvearrowleft_\zeta y$, $x \not\curvearrowleft_\zeta z$, or

(5) $z \curvearrowleft_\zeta y$, $x \not\curvearrowleft_\zeta z$, $y \not\curvearrowleft_\zeta x$, or

(6) $x \curvearrowleft_\zeta z$, $y \not\curvearrowleft_\zeta x$, $z \not\curvearrowleft_\zeta y$, or

(7) $x \not\curvearrowleft_\zeta z$, $y \not\curvearrowleft_\zeta x$, $z \not\curvearrowleft_\zeta y$.

In case (4), since $y \precsim_\zeta z$, we have $y \precsim_0 z$, and since $z \precsim_\zeta x$, we have $z \precsim_0 x$, and so (1) holds. In case (5), since $z \precsim_\zeta x$, we have $z \precsim_0 x$, and since $x \precsim_\zeta y$, we have $x \precsim_0 y$, and so (2) holds. In case (6), since $x \precsim_\zeta y$, we have $x \precsim_0 y$, a since $y \precsim_\zeta z$, we have $y \precsim_0 z$, and so (3) holds. To complete the proof, we show that case (7) cannot happen. Indeed, since $x \precsim_\zeta y$ implies $x \precsim_0 y$, and $y \precsim_\zeta z$ implies $y \precsim_0 z$, the transitivity of $\precsim_0$ yields $x \precsim_0 z$. The hypothesis $z \prec_\zeta x$ entails $\neg(x \precsim_\zeta z)$, hence $(x \precsim_0 z \wedge x \not\curvearrowleft_\zeta z)$ fails. However, this contradicts $x \precsim_0 z$. □

Lemma 4.7 *Let $\precsim_\zeta$ be the ζ-trace of a semiorder $\precsim$ on X. There are no $x, y, z \in X$, with $x \sim_{\rm tc} y \sim_{\rm tc} z$, which simultaneously satisfy the following properties:*

(a) $y \precsim_0 z \precsim_0 x$;

(b) $y \not\curvearrowleft_\zeta z$, $z \not\curvearrowleft_\zeta x$, $x \not\curvearrowleft_\zeta y$, $z \not\curvearrowleft_\zeta y$, *and* $x \not\curvearrowleft_\zeta z$;

(c) $y \curvearrowleft_\zeta x$.

Proof. For each $n \geq 1$, let A_n denote the set of all triples $(x, y, z) \in X^3$ belonging to the same equivalence class of the transitive closure, and satisfying properties (a), (b), and

(c)$_n$ there is a ζ-backward path from x to y of length n.

To prove the lemma, it suffices to show that each A_n is empty. Toward a contradiction, assume that some A_n is nonempty. Fix $n \geq 1$ minimal such that $A_n \neq \emptyset$, in particular $A_k = \emptyset$ for $1 \leq k < n$. Choose (x, y, z) arbitrary in A_n. By (c)$_n$, we have

$$y = w_n \succsim \ldots \succsim w_0 = x$$

with $\zeta(w_n) < \ldots < \zeta(w_0)$. In what follows we argue according to the length n of the ζ-backward path from x to y, and obtain a contradiction in each case.

Case 1: $n = 1$. In this case, we have $y \succsim x$, with $\zeta(y) < \zeta(x)$. By the properties of the trace (Lemma 2.3(ii)), $x \precsim y \precsim_0 z$ implies $x \precsim z$, and $z \precsim_0 x \precsim y$ implies $z \precsim y$. Since $\zeta(y) < \zeta(x)$ holds by hypothesis, we have either $\zeta(z) < \zeta(x)$ or $\zeta(z) > \zeta(y)$. Now the first case yields $z \curvearrowleft_\zeta x$, and the second $y \curvearrowleft_\zeta z$. However, both conclusions contradict (b).

Case 2: $n = 2$. In this case, there is w such that $y \succsim w \succsim x$ and $\zeta(y) < \zeta(w) < \zeta(x)$. Now property (a) yields $w \sim z$, using Lemma 2.3(ii). We claim that $\zeta(w) = \zeta(z)$. Indeed, if $\zeta(w) < \zeta(z)$, then $y \curvearrowleft_\zeta z$ (since $y \succsim w \succsim z$ and $\zeta(y) < \zeta(w) < \zeta(z)$), and if $\zeta(z) < \zeta(w)$, then $z \curvearrowleft_\zeta x$ (since $z \succsim w \succsim x$ and $\zeta(z) < \zeta(w) < \zeta(x)$. However, both conclusions contradict property (b), and so the equality $\zeta(w) = \zeta(z)$ holds. Now $z \succsim x$ implies $z \curvearrowleft_\zeta x$, and $y \succsim z$ implies $y \curvearrowleft_\zeta z$, both of which are impossible. It follows that $y \prec z \prec x$ holds. However, the latter contradicts semitransitivity, since we have $y \succsim w \succsim x$ by hypothesis.

Case 3: $n \geq 3$. In this case, there are distinct $x', y' \in X$ such that $y \succsim y' \succsim \ldots \succsim x' \succsim x$ and $\zeta(y) < \zeta(y') < \ldots < \zeta(x') < \zeta(x)$. We claim that $y' \succsim z \succsim x'$.

To prove the claim, we argue by contradiction. Assume that $z \prec x'$ holds. The definition of trace gives $z \precsim_0 x'$. Since $\zeta(y) < \zeta(x')$, we obtain $x' \not\curvearrowleft_\zeta y$. Since $z \prec x'$, the property (S1) of a weak LMIS yields $\zeta(z) < \zeta(x')$, and so $x' \not\curvearrowleft_\zeta z$. If $z \curvearrowleft_\zeta x'$, then since $x' \curvearrowleft_\zeta x$, Lemma 4.5(i) yields $z \curvearrowleft_\zeta x$, which is against (b). It follows that $(x', y, z) \in A_{n-1} = \emptyset$, a contradiction. In a similar way, one can show that $y' \prec z$ implies $(x, y', z) \in A_{n-1} = \emptyset$, which is again impossible. Thus the claim holds.

Now since $\zeta(y') < \zeta(x')$ by hypothesis, we have either $\zeta(z) < \zeta(x')$ or $\zeta(z) > \zeta(y')$. In the first case, $z \curvearrowleft_\zeta x$ (since $z \succsim x' \succsim x$ and $\zeta(z) < \zeta(x') < \zeta(x)$), which is false. In the second case, $y \curvearrowleft_\zeta z$ (since $y \succsim y' \succsim z$ and $\zeta(y) < \zeta(y') < \zeta(z)$), which is false. This completes the proof. □

Next we use Lemmas 4.5, 4.6 and 4.7 to prove the main result of this section.

Theorem 4.8 *Any sliced trace of a semiorder is a total preorder.*

PROOF. Let $\precsim$ be a semiorder on X, and $\precsim_\zeta$ its ζ-trace for some weak LMIS $\zeta \colon X \to \mathbb{Z}$. By Lemma 4.5(vi), $\precsim_\zeta$ is reflexive and complete. To complete the proof of the theorem, we show that $\precsim_\zeta$ is transitive.

Toward a contradiction, assume that there exist $x, y, z \in X$ such that $x \precsim_\zeta y \precsim_\zeta z$ but $\neg(x \precsim_\zeta z)$. If $x, y, z \in X$ do not belong to the same equivalence class of the transitive closure, then we immediately get a contradiction, due to the definition of $\precsim_\zeta$. Therefore, we can assume that $x \sim_{\mathrm{tc}} y \sim_{\mathrm{tc}} z$. By the completeness of the ζ-trace, it follows that $x \precsim_\zeta y \precsim_\zeta z \prec_\zeta x$ holds, so we can apply Lemma 4.6. Below we consider an exhaustive list of all possible cases, and get a contradiction in each of them.

Case 1: $x \not\curvearrowleft_\zeta y$, $y \not\curvearrowleft_\zeta z$, $z \not\curvearrowleft_\zeta x$, $y \curvearrowleft_\zeta x$, $z \not\curvearrowleft_\zeta y$, $x \not\curvearrowleft_\zeta z$, and $y \precsim_0 z \precsim_0 x$. Apply Lemma 4.7 to get a contradiction.

Case 2: $x \not\curvearrowleft_\zeta y$, $y \not\curvearrowleft_\zeta z$, $z \not\curvearrowleft_\zeta x$, $z \curvearrowleft_\zeta y$, $x \not\curvearrowleft_\zeta z$, $y \not\curvearrowleft_\zeta x$, and $z \precsim_0 x \precsim_0 y$. Apply Lemma 4.7 with y, z, x in place of x, y, z to get a contradiction.

Case 3: $x \not\curvearrowleft_\zeta y$, $y \not\curvearrowleft_\zeta z$, $z \not\curvearrowleft_\zeta x$, $x \curvearrowleft_\zeta z$, $y \not\curvearrowleft_\zeta x$, $z \not\curvearrowleft_\zeta y$, and $x \precsim_0 y \precsim_0 z$. Apply Lemma 4.7 with z, x, y in place of x, y, z to get a contradiction. □

5 Universal semiorders: $\mathbb{Z}$-products and $\mathbb{Z}$-lines

We are finally able to provide a full description of the internal structure of an arbitrary semiorder. To start, we characterize the asymmetric part of an arbitrary semiorder by means of the notions of transitive closure, locally monotonic integer slicer, and ζ-trace.

Theorem 5.1 *Let $(X, \precsim)$ be a semiorder. There is a function $\zeta\colon X \to \mathbb{Z}$ such that for each $x, y \in X$, we have:*

$$x \prec y \iff \begin{array}{ll} (1) & x \prec_{\mathrm{tc}} y\,, \text{ or} \\ (2) & x \sim_{\mathrm{tc}} y \text{ and } \zeta(x)+1 < \zeta(y)\,, \text{ or} \\ (3) & x \sim_{\mathrm{tc}} y \text{ and } \zeta(x)+1 = \zeta(y) \text{ and } x \prec_{\zeta} y\,. \end{array}$$

Moreover, ζ satisfies the following additional properties for each $x, y \in X$ such that $x \sim_{\mathrm{tc}} y$:

(i) *if $\zeta(x) < \zeta(y)$, then $x \prec_0 y$;*

(ii) *if $\zeta(x) = \zeta(y)$, then $x \prec_0 y \iff x \prec_\zeta y$.*

Proof. Let $x, y \in X$ be arbitrary. Then either (I) $x \prec_{\mathrm{tc}} y$, or (I)$'$ $y \prec_{\mathrm{tc}} x$, or (II) $x \sim_{\mathrm{tc}} y$. Since $\precsim$ is complete and $\precsim_{\mathrm{tc}}$ is an extension of $\precsim$, we obtain $x \prec y \iff x \prec_{\mathrm{tc}} y$ in case (I), and $y \prec x \iff y \prec_{\mathrm{tc}} x$ in case (I)$'$. It follows that the equivalence in the statement of the theorem holds in cases (I) and (I)$'$.

In what follows, we show that the claimed equivalence also holds in case (II). Since $(X, \precsim)$ is a semiorder, it admits an LMIS by Theorem 3.6, say, $\zeta\colon (X, \precsim) \to (\mathbb{Z}, \leq)$. In particular, ζ is a weak LMIS, and so Lemma 4.5(v) applies as well. Now assume that $x \prec y$. The property (S1) of an LMIS (see Definition 3.4) yields $\zeta(x) < \zeta(y)$. Thus, either $\zeta(x)+1 < \zeta(y)$ or $\zeta(x)+1 = \zeta(y)$. In the first case, (2) holds. In the second case, (3) holds by Lemma 4.5(iv). Conversely, assume that either (2) or (3) holds. In case (2), we obtain $x \prec y$ by the property (S2) of an LMIS. In case (3), we obtain $x \prec y$ by Lemma 4.5(iv).

Finally, let $x, y \in X$ be such that $x \sim_{\mathrm{tc}} y$. Then the implication (i) is property (S3) in the definition of an LMIS, whereas the implication (ii) follows from Lemma 4.5(v). □

Next, we introduce a modified notion of the lexicographic product of three total preorders, having the chain of integer numbers as middle factor, equipped with a shifting operator.

Definition 5.2 Let $(A, \precsim_A)$ and $(B, \precsim_B)$ be total preorders. The $\mathbb{Z}$*-product* of A and B is the triple $\big(R, \oplus 1, \precsim^{\oplus 1}_{\mathrm{lex}}\big)$, where:

- R is the Cartesian product $A \times \mathbb{Z} \times B$;
- $\oplus 1$ is the unary operation on R defined by $(a,n,b) \oplus 1 := (a, n+1, b)$ for each $(a,n,b) \in R$;
- $\precsim^{\oplus 1}_{\mathrm{lex}}$ is the canonical completion of the $\mathbb{Z}$*-shifted lexicographic order* $\prec^{\oplus 1}_{\mathrm{lex}}$ on R, defined by

$$(a,n,b) \prec^{\oplus 1}_{\mathrm{lex}} (a',n',b') \quad \overset{\mathrm{def}}{\iff} \quad (a,n,b) \oplus 1 \prec_{\mathrm{lex}} (a',n',b')$$

for each $(a,n,b), (a',n',b') \in R$, with $\prec_{\mathrm{lex}}$ being the standard lexicographic order on R.

We denote by $A \otimes_{\mathbb{Z}} B$ the $\mathbb{Z}$-product of the total preorders $(A, \precsim_A)$ and $(B, \precsim_B)$. The $\mathbb{Z}$-product of two linear orders is called a $\mathbb{Z}$*-line*.

Remark 5.3 For each $(a,n,b), (a',n',b') \in A \otimes_{\mathbb{Z}} B$, we have:

$$(a,n,b) \prec^{\oplus 1}_{\mathrm{lex}} (a',n',b') \iff \begin{array}{ll} (1) & a \prec_A a', \text{ or} \\ (2) & a \sim_A a' \text{ and } n+1 < n', \text{ or} \\ (3) & a \sim_A a', n+1 = n' \text{ and } b \prec_B b'. \end{array}$$

Semiorders with special properties can be embedded into particular $\mathbb{Z}$-lines. The next example presents a few instances of this kind.

Example 5.4 Recall that a semiorder $(X, \precsim)$ is:

- *regular* if there is no strictly increasing (respectively, strictly decreasing) sequence $\{x_n : n \geq 0\} \subseteq X$ and an element $x_\infty \in X$ such that $x_n \prec x_\infty$ (respectively, $x_\infty \prec x_n$) for all $n \geq 0$;[23]
- *s-separable* if there exists a countable set $D \subseteq X$ such that for each $x, y \in X$ with $x \prec y$, there are $d_1, d_2 \in D$ such that $x \prec d_1 \precsim_0 y$ and $x \precsim_0 d_2 \prec y$;
- *Scott-Suppes representable* if there is a function $f \colon X \to \mathbb{R}$ such that the equivalence $x \prec y \iff f(x) + 1 < f(y)$ holds for all $x, y \in X$.

Denoted by $\mathbf{1} := \{0\}$ the chain with exactly one element, the following facts hold:[24]

(i) $(X, \precsim)$ is regular if and only if it embeds into some $\mathbb{Z}$-line $\mathbf{1} \otimes_{\mathbb{Z}} B$;

(ii) $(X, \precsim)$ is s-separable if and only if there is an embedding $f \colon X \to \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}$ such that $|f(X) \cap \{r\} \times \mathbb{Z} \times \mathbb{R}| \leq 1$ for all but countably many $r \in \mathbb{R}$;

(iii) $(X, \precsim)$ is Scott-Suppes representable if and only if it embeds into $\mathbf{1} \otimes_{\mathbb{Z}} \mathbb{R}$.

As a corollary, we obtain the internal characterization of the Scott-Suppes representability of a semiorder recently proved in [11], namely:

Theorem (Candeal and Induráin, 2010) *A semiorder is Scott-Suppes representable if and only if it is regular and s-separable.*[25]

The next example is complementary to Example 5.4. In fact, it explicitly exhibits an embedding of the Scott-Suppes representable semiorder $(\mathbb{R}, \precsim)$ (examined in Example 3.7) into a suitable $\mathbb{Z}$-line. It turns out that this embedding is in fact an isomorphism.[26]

Example 5.5 Let $(\mathbb{R}, \precsim)$ the semiorder defined in Example 3.7 (see also Examples 3.7 and 4.4). We claim that

$$(\mathbb{R}, \precsim) \;\cong\; \mathbf{1} \otimes_{\mathbb{Z}} [0, 1)$$

where $\mathbf{1} = \{0\}$ is the chain with a unique element, and the interval $[0, 1) \subseteq \mathbb{R}$ is equipped with the usual order. To prove the claim, we show that the function

$$f \colon (\mathbb{R}, \precsim) \to \mathbf{1} \otimes_{\mathbb{Z}} [0, 1)\,, \quad f(x) \;:=\; \big(0, \lfloor x \rfloor, x - \lfloor x \rfloor\big) \quad \forall x \in \mathbb{R}$$

[23] A sequence $\{x_n : n \geq 0\}$ in $(X, \precsim)$ is *strictly increasing* if $x_n \prec x_{n+1}$ for each $n \geq 0$; the notion of *strictly decreasing* is defined dually. Note that regularity can be formulated in a neater way by using ordinal numbers: in fact, X is regular if and only if neither the ordinal $\omega + 1$ nor its reverse ordering $(\omega + 1)^*$ embeds into X (where, as usual, ω denotes the first infinite ordinal, and $\omega + 1$ is its immediate successor in the ordinal hierarchy).

[24] Details are available upon request. In fact, these results are included here only for the sake of illustration, and they will be extensively discussed in a forthcoming paper.

[25] The very definition of regularity already suggests that regular semiorders are far from being general. The result proved by Candeal and Induráin in [11] (as well as the characterization (iii) stated in Example 5.4) provides formal arguments that confirm the specialty of Scott-Suppes representable semiorders, a fact that was already pointed out by Ŝwistak in [48].

[26] We thank one of the referees for suggesting this interesting example.

(where $\lfloor x \rfloor$ denotes the floor of x) is an isomorphism. Indeed, f is a bijection, and the following chain of equivalences holds for each $x, y \in \mathbb{R}$:

$$\begin{aligned}
f(x) \prec_{\text{lex}}^{\oplus 1} f(y) &\iff \big(0, \lfloor x \rfloor, x - \lfloor x \rfloor\big) \prec_{\text{lex}}^{\oplus 1} \big(0, \lfloor y \rfloor, y - \lfloor y \rfloor\big) \\
&\iff \big(0, \lfloor x \rfloor, x - \lfloor x \rfloor\big) \oplus 1 \prec_{\text{lex}} \big(0, \lfloor y \rfloor, y - \lfloor y \rfloor\big) \\
&\iff \big(0, \lfloor x \rfloor + 1, x - \lfloor x \rfloor\big) \prec_{\text{lex}} \big(0, \lfloor y \rfloor, y - \lfloor y \rfloor\big) \\
&\iff \big(\lfloor x \rfloor + 1, x - \lfloor x \rfloor\big) \prec_{\text{lex}} \big(\lfloor y \rfloor, y - \lfloor y \rfloor\big) \\
&\iff \big(\lfloor x \rfloor + 1 < \lfloor y \rfloor\big) \quad \vee \quad \big(\lfloor x \rfloor + 1 = \lfloor y \rfloor \ \wedge \ x - \lfloor x \rfloor < y - \lfloor y \rfloor\big) \\
&\iff x + 1 < y \quad \iff \quad x \prec y.
\end{aligned}$$

This proves that the equivalence (2) in Definition 2.4 holds, and so f is an isomorphism. The isomorphism f between the classical semiorder $(\mathbb{R}, \precsim)$ and the $\mathbb{Z}$-line $\mathbf{1} \otimes_{\mathbb{Z}} [0,1)$ is interesting in view of the study of the Scott-Suppes representability of a semiorder: in fact, a semiorder is Scott-Suppes representable if and only if it embeds into $(\mathbb{R}, \precsim)$. In this direction, it is worth mentioning the following structural result obtained in [12], which has a similar flavour:

Theorem (Candeal *et al.*, 2002) *Every representable (strict) semiorder is isotonic to a subset of the cartesian product $\mathbb{Z} \times [0,1)$ endowed with the following ordering $\lhd$:*

$$(a,b) \lhd (c,d) \quad \stackrel{\text{def}}{\iff} \quad \big(c - a \geq 2\big) \ \text{ or } \ \big(c - a = 1 \quad \text{and} \quad d - b > 0\big).$$

Note that $\mathbb{Z} \times [0,1)$ endowed with the canonical completion of $\lhd$ is isomorphic to $\mathbf{1} \otimes_{\mathbb{Z}} [0,1)$ (and therefore to $(\mathbb{R}, \precsim)$).

We are ready to state the main result of this paper.

Theorem 5.6 *The following statements are equivalent for a simple preference $(X, \precsim)$:*

(i) $(X, \precsim)$ *is a semiorder;*

(ii) $(X, \precsim)$ *embeds into a $\mathbb{Z}$-product;*

(iii) $(X, \precsim)$ *embeds into a $\mathbb{Z}$-line;*

(iv) $(X, \precsim)$ *embeds into $(X, \precsim_{\text{tc}}) \otimes_{\mathbb{Z}} (X, \precsim_{\zeta})$ for some ζ-trace of $\precsim$.*

Proof. We prove (i) $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i), and (ii) $\Leftrightarrow$ (iii).

(i) $\Rightarrow$ (iv): Apply Theorem 5.1.

(iv) $\Rightarrow$ (ii): Obvious by Theorem 4.8.

(ii) $\Rightarrow$ (i): Assume that $(W, \precsim)$ embeds into $A \otimes_{\mathbb{Z}} B$ for some total preorders $(A, \precsim_A)$ and $(B, \precsim_B)$. To show that $\precsim$ is semitransitive, assume by contradiction that (using Lemma 2.1(iii))

$$(a_1, n_1, b_1) \prec_{\text{lex}}^{\oplus 1} (a_2, n_2, b_2) \prec_{\text{lex}}^{\oplus 1} (a_3, n_3, b_3) \precsim_{\text{lex}}^{\oplus 1} (a_4, n_4, b_4) \precsim_{\text{lex}}^{\oplus 1} (a_1, n_1, b_1)$$

holds. It follows that $a_1 \sim_A a_2 \sim_A a_3 \sim_A a_4$. Since $(a_1, n_1, b_1) \prec^{\oplus 1}_{\text{lex}} (a_2, n_2, b_2) \prec^{\oplus 1}_{\text{lex}} (a_3, n_3, b_3)$, we get (1) $n_1 + 1 \leq n_2$, and (2) $n_2 + 1 \leq n_3$. Further, since $(a_3, n_3, b_3) \precsim^{\oplus 1}_{\text{lex}} (a_4, n_4, b_4) \precsim^{\oplus 1}_{\text{lex}} (a_1, n_1, b_1)$, we get (3) $n_3 \leq n_4 + 1$, and (4) $n_4 \leq n_1 + 1$. Now if either one of the inequalities (1) and (2) is strict, then we obtain

$$n_1 + 2 < n_3 \leq n_4 + 1 \leq n_1 + 2$$

which is impossible. A similar contradiction arises in case either (3) or (4) holds with strict inequality. It follows that all of (1), (2), (3) and (4) are equalities, which implies $n_2 = n_4 = n_1 + 1$ and $n_3 = n_1 + 2$. However, now the hypothesis yields $b_1 \prec_B b_2 \prec_B b_3 \precsim_B b_4 \precsim_B b_1$, which is impossible because $\precsim_B$ is transitive. This shows that $\precsim$ is semitransitive.

To prove that $\precsim$ is also Ferrers, assume by contradiction that (using Lemma 2.1(ii))

$$(a_1, n_1, b_1) \prec^{\oplus 1}_{\text{lex}} (a_2, n_2, b_2) \precsim^{\oplus 1}_{\text{lex}} (a_3, n_3, b_3) \prec^{\oplus 1}_{\text{lex}} (a_4, n_4, b_4) \precsim^{\oplus 1}_{\text{lex}} (a_1, n_1, b_1).$$

By an argument similar to the one of the previous paragraph, we obtain $a_1 \sim_A a_2 \sim_A a_3 \sim_A a_4$, $n_1 = n_3$ and $n_2 = n_4 = n_1 + 1$. It follows that $b_1 \prec_B b_2 \precsim_B b_3 \prec_B b_4 \precsim_B b_1$, which is impossible. Thus $\precsim$ is a semiorder.

(ii) $\Leftrightarrow$ (iii): It suffices to show that (ii) implies (iii). Let $f\colon (X, \precsim) \hookrightarrow (A, \precsim_A) \otimes_{\mathbb{Z}} (B, \precsim_B)$ be an embedding into a $\mathbb{Z}$-product of total preorders. Fix an arbitrary linear order $\leq_X$ on X. Let $(\overline{A}, \leq_{\overline{A}})$ be the quotient linear ordering $(A, \precsim_A)/\sim_A$, obtained by collapsing each $\sim_A$-equivalence class to a point. Likewise, let $(\overline{B}, \leq_{\overline{B}})$ be the quotient linear ordering $(B, \precsim_B)/\sim_B$. Finally, denote by $(C, \leq_C)$ the linear ordering obtained by taking the lexicographic product $(\overline{B}, \leq_{\overline{B}}) \times_{\text{lex}} (\mathbb{Z}, \geq) \times_{\text{lex}} (X, \leq_X)$, where $(\mathbb{Z}, \geq)$ denotes the reverse of the standard linear ordering $(\mathbb{Z}, \leq)$. Then $\big(\overline{A}, \leq_{\overline{A}}\big) \otimes_{\mathbb{Z}} (C, \leq_C)$ is a $\mathbb{Z}$-line. Now let $g\colon (X, \precsim) \to \big(\overline{A}, \leq_{\overline{A}}\big) \otimes_{\mathbb{Z}} (C, \leq_C)$ be a map such that the implication

$$f(x) = (a, n, b) \quad \Longrightarrow \quad g(x) = \big(\overline{a}, n, (\overline{b}, n, x)\big) \tag{4}$$

holds for each $(a, n, b) \in A \times \mathbb{Z} \times B$, where $\overline{a}$ and $\overline{b}$ denote, respectively, the equivalence classes of a in $\overline{A}$ and of b in $\overline{B}$. We claim that g is an embedding. To prove that g is order-preserving, let $x, x' \in X$. Denoted $f(x) := (a, n, b)$ and $f(x') := (a', n', b')$, we have:

$$\begin{aligned} x \prec x' &\iff f(x) \prec^{\oplus 1}_{\text{lex}} f(y) \\ &\iff a <_A a' \,\vee\, \big(a \sim_A a' \,\wedge\, n + 1 < n'\big) \,\vee\, \big(a \sim_A a' \,\wedge\, n + 1 = n' \,\wedge\, b <_B b'\big) \\ &\iff \overline{a} <_{\overline{A}} \overline{a'} \,\vee\, \big(\overline{a} = \overline{a'} \,\wedge\, n + 1 < n'\big) \,\vee\, \big(\overline{a} = \overline{a'} \,\wedge\, n + 1 = n' \,\wedge\, (\overline{b}, n, x) \prec_{\text{lex}} (\overline{b'}, n', x')\big) \end{aligned}$$

where the last equivalence holds because if $n + 1 = n'$, then

$$\begin{aligned} (\overline{b}, n, x) \prec_{\text{lex}} (\overline{b'}, n', x') &\iff \overline{b} <_{\overline{B}} \overline{b'} \,\vee\, \big(\overline{b} = \overline{b'} \,\wedge\, n > n'\big) \,\vee\, \big(\overline{b} = \overline{b'} \,\wedge\, n = n' \,\wedge\, x \prec_X x'\big) \\ &\iff \overline{b} <_{\overline{B}} \overline{b'}. \end{aligned}$$

Using (4), we obtain that $x \prec x'$ implies $g(x) \prec^{\oplus 1}_{\text{lex}} g(x')$, which shows that g is order-preserving. Since g is obviously injective (because $\leq_X$ is antisymmetric), it follows that g is an embedding, and the proof is complete. □

Note that in the special case that $\precsim$ is a total preorder, then both the transitive closure $\precsim_{\text{tc}}$ and the (classical) trace $\precsim_0$ are equal to $\precsim$. In this case, the correspondences $f(x) := (x, 0, x)$ and $g(x) := (x, 0, 0)$ define two embeddings of $(X, \precsim)$ into, respectively, $X \otimes_{\mathbb{Z}} X$ and $X \otimes_{\mathbb{Z}} \mathbf{1}$.

We conclude this section by deriving some interesting consequences of Theorem 5.6. Recall that a family of orderings $\mathcal{Z}$ is *universal* for an order-theoretic[27] property $\mathcal{P}$ if each element of $\mathcal{Z}$ has $\mathcal{P}$, and every ordering having $\mathcal{P}$ embeds into an element of $\mathcal{Z}$. Then we have:

Corollary 5.7 *$\mathbb{Z}$-lines are universal semiorders.*

PROOF. Apply Theorem 5.6: (ii) $\Rightarrow$ (i) yields that $\mathbb{Z}$-products are semiorders, and (i) $\Rightarrow$ (iii) implies that that every semiorder embeds into a $\mathbb{Z}$-line. □

The next result is an immediate consequence of Theorem 5.6 and the universality of $\mathbb{Q}$ for countable linear orders.

Corollary 5.8 *A countable simple preference is a semiorder if and only if it embeds into $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Finally, we obtain an upper bound on the dimension of semiorders [43].

Corollary 5.9 (Rabinovitch, 1978) *The dimension of a strict semiorder is at most* 3.

PROOF. By Theorem 5.6, it suffices to show that the strict order $\prec^{\oplus 1}_{\text{lex}}$ on an arbitrary $\mathbb{Z}$-line $(A, \leq_A) \otimes_{\mathbb{Z}} (B, \leq_B)$ can be written as the intersection of three strict linear orders $<_1$, $<_2$ and $<_3$ on $A \times \mathbb{Z} \times B$. To define $<_1$, set:

- $\{a\} \times \mathbb{Z} \times B <_1 \{a'\} \times \mathbb{Z} \times B$ whenever $a <_A a'$;
- $\{a\} \times \{2n, 2n+1\} \times B <_1 \{a\} \times \{2n', 2n'+1\} \times B$ whenever $n < n'$;
- $<_1$ equal to $<_B$ on each $\{a\} \times \{2n, 2n+1\} \times B$.

To define $<_2$, set:

- $\{a\} \times \mathbb{Z} \times B <_2 \{a'\} \times \mathbb{Z} \times B$ whenever $a <_A a'$;
- $\{a\} \times \{2n-1, 2n\} \times B <_2 \{a\} \times \{2n'-1, 2n'\} \times B$ whenever $n < n'$;
- $<_2$ equal to $<_B$ on each $\{a\} \times \{2n-1, 2n\} \times B$.

To define $<_3$, set:

- $(a, n, b) <_3 (a', n', b')$ if $a <_A a'$ or ($a = a'$ and $n < n'$) or ($a = a'$, $n = n'$ and $b >_B b'$).

To complete the proof, we show that $\prec^{\oplus 1}_{\text{lex}}$ is equal to $<_1 \cap <_2 \cap <_3$. To prove one inclusion, let $(a, n, b) \prec^{\oplus 1}_{\text{lex}} (a', n', b')$. Then we have either $a <_A a'$, or ($a = a'$ and $n+1 < n'$), or ($a = a'$, $n+1 = n'$ and $b <_B b'$). In each case, we obtain $(a, n, b) <_i (a', n', b')$ for all $i = 1, 2, 3$, as claimed. For the reverse inclusion, assume that $(a, n, b) <_i (a', n', b')$ holds for each $i = 1, 2, 3$. By $<_3$, we have $a \leq_A a'$. If $a <_A a'$, then we are immediately done. So assume $a = a'$. By $<_3$, we have $n \leq n'$. If $n+1 < n'$, then we are done again. If $n+1 = n'$, then assume without loss of generality that n is even. By the definition of $<_1$, it follows that $b <_B b'$, and we are done. Finally, if $n = n'$, then by $<_3$ we get $b' <_B b$, whereas by $<_1$ we get $b <_B b'$. However, this is impossible. □

Figure 3 describes how the three strict linear orders $<_1$, $<_2$ and $<_3$, defined as in the proof of Corollary 5.9, distinguish from each other all the elements of $A \times \mathbb{Z} \times B$ having the same first coordinate. The gray areas, which collect one or two vertical slices together, are arranged in a linear order. Within each gray area, elements are ordered according to either $<_B$ (in $<_1$ and $<_2$) or its reverse ordering $>_B$ (in $<_3$).

[27] A property is *order-theoretic* if it is invariant under order-isomorphisms.

6 Conclusions and further directions of research

In this paper we have described an arbitrary semiorder as a subordering of a modified lexicographic product of three total preorders, where the chain $(\mathbb{Z}, \leq)$ of integers endowed with a shift operator is the middle factor. This modified lexicographic product, called *$\mathbb{Z}$-product*, is characterized by the fact that the lexicographic ordering is respected by the first and the third factor, but the middle factor introduces a threshold of discrimination. We prove that the family of $\mathbb{Z}$-products is universal for semiorders, in the sense that $\mathbb{Z}$-products are semiorders themselves, and each semiorder embeds into a $\mathbb{Z}$-product.

Let us quickly summarize the main steps of the representation of a semiorder by a $\mathbb{Z}$-product. Fix a semiordered structure $(X, \precsim)$, and denote by $\precsim_{\mathrm{tc}}$ and $\precsim_0$ the transitive closure and the trace of $\precsim$, respectively. Let $x, y \in X$ be such that $x \prec y$.

Step 1. Either $x \prec_{tc} y$ or $x \sim_{tc} y$ holds. (The case $y \prec_{tc} x$ is obviously impossible.) Further, if $x \prec_{tc} y$, then $x \prec y$.

Step 2. In order to analyze what occurs in the case $x \sim_{tc} y$, a special function $\zeta \colon X \to \mathbb{Z}$ is obtained. This function, called a *linear monotonic integer slicer* (LMIS), has the following properties: for each $u, v \in X$ such that $u \sim_{tc} v$, we have (S1) $u \prec v \implies \zeta(u) < \zeta(v)$, (S2) $\zeta(u) + 1 < \zeta(v) \implies u \prec v$, and (S3) $\zeta(u) < \zeta(v) \implies u \prec_0 v$. The definition of ζ is reminiscent of constructions related to the Scott-Suppes representation of a semiorder. As a consequence, whenever $x \prec y$ and $x \sim_{tc} y$ holds, then we have either $\zeta(x) + 1 < \zeta(y)$ or $\zeta(x) + 1 = \zeta(y)$. Moreover, $\zeta(x) + 1 < \zeta(y)$, combined with $x \sim_{tc} y$, implies $x \prec y$.

Step 3. Thus, it remains to establish a procedure that allows us to "distinguish" x from y whenever they are in consecutive slices, that is, $\zeta(x) + 1 = \zeta(y)$. To that end, we construct a total preorder on X, called a *sliced trace*, which depends on the function ζ. This total preorder, denoted by $\precsim_\zeta$, has the property that the equivalence $x \prec_\zeta y \iff x \prec y$ holds for each $x, y \in X$ such that $x \sim_{\mathrm{tc}} y$ and $\zeta(x) + 1 = \zeta(y)$.

The main result of this paper characterizes semiorders as those simple preferences that are embeddable in a $\mathbb{Z}$-product. In fact, we show that a semiorder embeds in the $\mathbb{Z}$-product having the transitive closure of the semiorder as its first factor, and a sliced trace as its third factor. Further, special $\mathbb{Z}$-products – called *$\mathbb{Z}$-lines*, and characterized by the fact that the extreme factors are linear orders – turn out to be universal semiorders, too. Finally, as a corollary of the universality of $\mathbb{Z}$-lines, we derive that the dimension of a strict semiorder is at most three.

Future research on the topic goes in two main directions. First, we believe that our descriptive approach naturally prompts suitable extensions of several results on semiorders that are scattered throughout the literature. For instance, we conjecture that positive-threshold GNR and GUR representations of semiorders à la Beja-Gilboa (see Theorems 3.7, 3.8, 4.4, and 4.5 in [5]) can be obtained as particular cases of more general representations. We also conjecture that Candeal and Induráin's internal characterization of the Scott-Suppes representability of a semiorder (see Lemma 3.4 and Theorem 3.6 in [11], as well as Theorem 4.11 and Corollary 4.12 in [9]) is a special case of forms of utilities with values in a suitable $\mathbb{Z}$-line.[28]

In another direction of research, we are currently working on an extension of the descriptive characterization of a semiorder to other quasi-transitive preferences, which satisfy a *weak* (m, n)-

[28] We have just proved that this conjecture holds true: see Example 5.4.

Ferrers property[29] in the sense of Giarlotta and Watson [31]. For instance, it would be of some interest to identify suitable $\mathbb{Z}$-line representations of enhanced forms of semiorders, such as *strong interval orders* (weakly $(3,2)$-Ferrers) and *strong semiorders* (weakly $(3,2)$- and $(4,1)$-Ferrers), which have a special geometric/combinatorial interpretation.[30] In the same direction, one could identify universal types of pairs $(\precsim_1, \precsim_2)$ of binary relations such that $\precsim_1$ is a preorder and $\precsim_2$ is a well-structured extension of $\precsim_1$, for instance special types of *NaP-preferences*.[31]

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[29] Recall that a reflexive relation $\precsim$ is *weakly (m,n)-Ferrers* if $(x_1 \precsim \ldots \precsim x_m \ \wedge \ y_1 \precsim \ldots \precsim y_n)$ implies $(x_1 \precsim y_n \ \vee \ y_1 \precsim x_m)$. See Figure 3 in [31] for a summary of the implications among combinations of weak (m,n)-Ferrers properties.

[30] See Figure 5 in [25] for a typical geometric form of strong interval orders and strong semiorders.

[31] A *NaP-preference* (*necessary and possible preference*) is a pair $(\precsim_1, \precsim_2)$ of binary relations on the same ground set such that its *necessary* component $\precsim_1$ is a preorder, its *possible* component $\precsim_2$ is a quasi-transitive completion of the first, and the two relations jointly satisfy suitable forms of transitive coherence and mixed completeness [25, 26, 27]. Under the *Axiom of Choice*, a NaP-preference is characterized by the existence of a set of total preorders whose intersection and union give, respectively, the necessary component and the possible component (Theorem 3.4 in [27]). Special types of NaP-preferences (having, e.g., a semiorder as possible component) are studied in [25].

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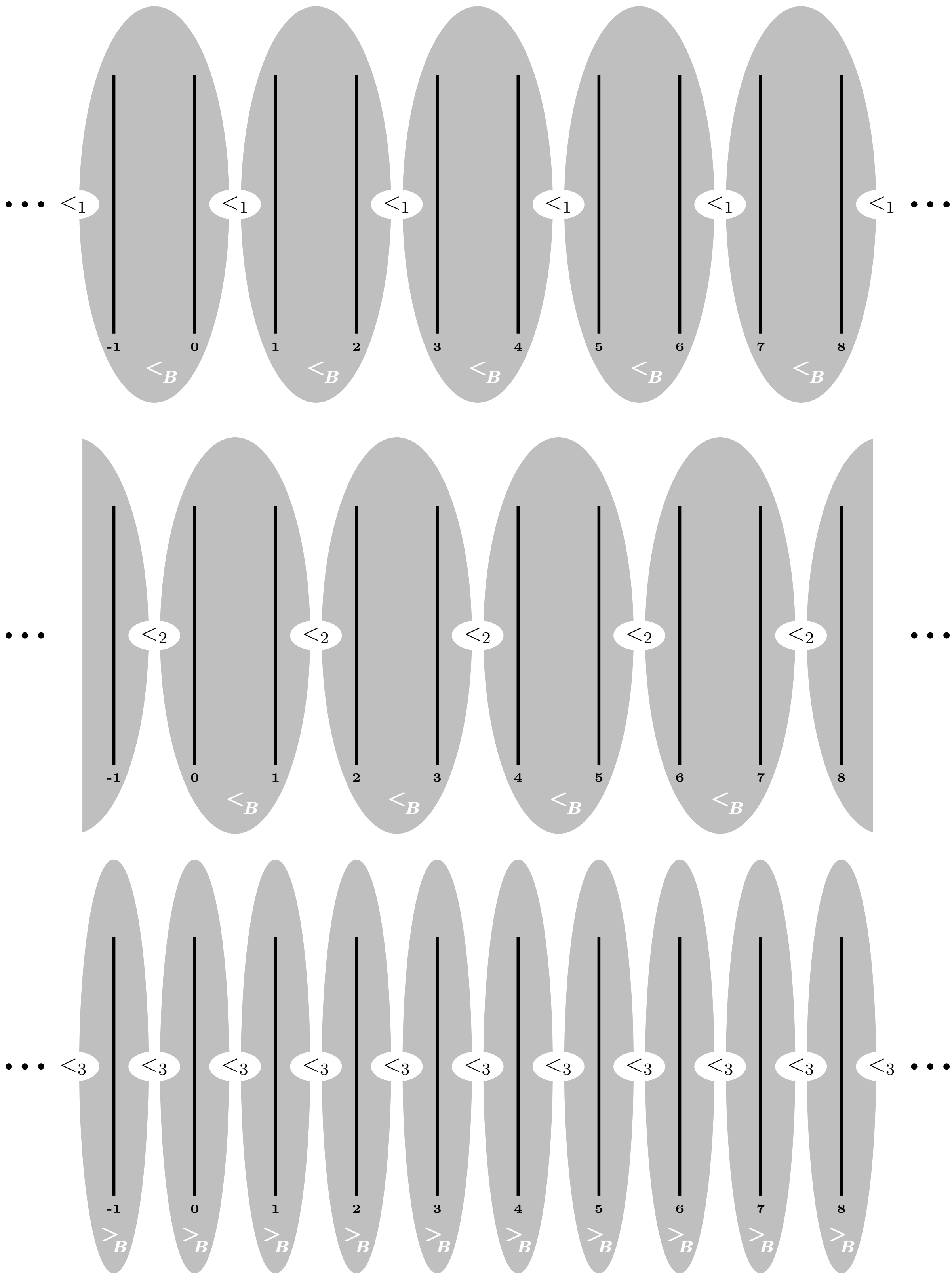

Figure 3: The strict linear orders $<_1$, $<_2$ and $<_3$ on $A \times \mathbb{Z} \times B$ constructed in the proof of Corollary 5.9: how they act on each slice $\{a_0\} \times \mathbb{Z} \times B$ (for a fixed $a_0 \in A$).