Arbitrage Theory without a Reference Probability: * challenges of the model independent approach

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June 30, 2015

Abstract

In a model independent discrete time financial market, we discuss the richness of the family of martingale measures in relation to different notions of Arbitrage, generated by a class S of significant sets. The choice of S reflects into the intrinsic properties of the class of polar sets of martingale measures. In particular: if S reduces to a singleton, absence of Model Independent Arbitrage is equivalent to the existence of a martingale measure; for S being the open sets, absence of Open Arbitrage is equivalent to the existence of full support martingale measures. These results are obtained by adopting a technical filtration enlargement and by constructing a Universal Aggregator of all arbitrage opportunities.
Furthermore we prove the superhedging duality theorem, where trading is allowed with dynamic and semi-static strategies. We also show that the initial cost of the cheapest portfolio that dominates a contingent claim on every possible path might be strictly greater than the upper bound of the no-arbitrage prices. We therefore characterize the subset of trajectories on which this duality gap disappears and prove that it is an analytic set.
The talk is based on [BFM14, BFM15].

1 Extended abstract

The introduction of Knightian Uncertainty in mathematical models for Finance has recently renewed the attention on foundational issues such as option pricing rules, super-hedging, and arbitrage conditions.

We can distinguish two extreme cases:

1. We are completely sure about the reference probability measure P. In this case, the classical notion of No Arbitrage or NFLVR can be successfully applied.
2. We face complete uncertainty about any probabilistic model and therefore we must describe our model independently by any probability. In this case we might adopt a model independent (weak) notion of No Arbitrage

In the second case, a pioneering contribution was given in the paper by Hobson [Ho98] where the problem of pricing exotic options is tackled under model mis-specification. In his approach the key assumption is the existence of a martingale measure for the market, consistent with the prices of some observed vanilla options. In [DH07], Davis and Hobson relate the previous problem to the absence of Model Independent Arbitrages, by the mean of semi-static strategies. A step forward

*based on two joint papers

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towards a model-free version of the First Fundamental Theorem of Asset Pricing in discrete time was formerly achieved by Riedel [Ri11] in a one period market and by Acciao at al. [AB13] in a more general setup.

Between cases 1. and 2., there is the possibility to accept that the model could be described in a probabilistic setting, but we cannot assume the knowledge of a specific reference probability measure but at most of a set of priors, which leads to the new theory of Quasi-sure Stochastic Analysis. The idea is that the classical probability theory can be reformulated as far as the single reference probability P is replaced by a class of (possibly non-dominated) probability measures $\mathcal{P}'$. This is the case, for example, of uncertain volatility where, in a general continuous time market model, the volatility is only known to lie in a certain interval $[\sigma_m, \sigma_M]$.
In the theory of arbitrage for non-dominated sets of priors, important results were provided by Bouchard and Nutz [BN13] in discrete time. A suitable notion of arbitrage opportunity with respect to a class $\mathcal{P}'$, named $NA(\mathcal{P}')$, was introduced and it was shown that the no arbitrage condition is equivalent to the existence of a family $\mathcal{Q}'$ of martingale measure having the same polar sets of $\mathcal{P}'$.

Bouchard and Nutz [BN13] answer the following question: which is a good notion of arbitrage opportunity for all **admissible** probabilistic models $P \in \mathcal{P}'$ (i.e. one single H that works as an arbitrage for all admissible models) ? To pose this question one has to know **a priori** which are the admissible models, i.e. we have to exhibit a subset of probabilities $\mathcal{P}'$.

Our aim is to investigate arbitrage conditions and robustness properties of markets that are described independently of any reference probability or set of priors.

We consider a financial market described by a discrete time adapted stochastic process $S := (S_t)_{t\in I}$, $I = \{0, \dots, T\}$, defined on $(\Omega, \mathcal{F}, \mathbb{F})$, $\mathbb{F} := (\mathcal{F}_t)_{t\in I}$, with $T < \infty$ and taking values in $\mathbb{R}^d$. Note we are not imposing any restriction on S so that it may describe generic financial securities (for examples, stocks and/or options). Differently from previous approaches in literature, in our setting the measurable space $(\Omega, \mathcal{F})$ and the price process S defined on it are given, and we investigate the properties of martingale measures for S induced by no arbitrage conditions. The class $\mathcal{H}$ of admissible trading strategies is formed by all $\mathbb{F}$-predictable d-dimensional stochastic processes and we denote with $\mathcal{M}$ the set of all probability measures under which S is an $\mathbb{F}$-martingale and with $\mathcal{P}$ the set of all probability measures on $(\Omega, \mathcal{F})$. We introduce therefore a flexible definition of Arbitrage which allows us to characterize the richness of the set $\mathcal{M}$ in a unified framework.

Arbitrage *de la classe* $\mathcal{S}$. We look for a single strategy H in $\mathcal{H}$ which represents an Arbitrage opportunity in some appropriate sense. Let:

$$\mathcal{V}_H^+ = \{\omega \in \Omega \mid \ V_T(H)(\omega) > 0\},$$

where $V_T(H) = \sum_{t=1}^T H_t \cdot (S_t - S_{t-1})$ is the final value of the strategy H. It is natural to introduce several notion of Arbitrage accordingly to the properties of the set $\mathcal{V}_H^+$.

Definition 1 *Let $\mathcal{S}$ be a class of measurable subsets of Ω such that $\varnothing \notin \mathcal{S}$. A trading strategy $H \in \mathcal{H}$ is an Arbitrage de la classe $\mathcal{S}$ if*

- $V_0(H) = 0$, $V_T(H)(\omega) \geq 0\ \forall \omega \in \Omega$ *and* $\mathcal{V}_H^+$ *contains a set in* $\mathcal{S}$.

The class $\mathcal{S}$ has the role to translate mathematically the meaning of a "true gain". When a probability P is given (the "reference probability") then we agree on representing a true gain as $P(V_T(H) > 0) > 0$ and therefore the classical no arbitrage condition can be expressed as: no losses $P(V_T(H) < 0) = 0$ implies no true gain $P(V_T(H) > 0) = 0$. In a similar fashion, when a subset $\mathcal{P}'$ of probability measures is given, one may replace the P-a.s. conditions above with $\mathcal{P}$-q.s conditions, as in [BN13]. However, if we can not or do not want to rely on a priory assigned set of probability measures, we may well use another concept: there is a true gain if the set $\mathcal{V}_H^+$ contains a set considered *significant*. Families of sets, not determined by some probability measures, have been already used in the context of the first and second fundamental theorem of asset pricing respectively by Battig Jarrow [BJ99] and Cassese [C08] .

In order to investigate the properties of the martingale measures induced by No Arbitrage conditions of this kind we first study the structural properties of the market adopting a geometrical approach in the spirit of [Pl97] but with Ω being a general Polish space, instead of a finite sample space. In particular, we characterize the class $\mathcal{N}$ of the $\mathcal{M}$-polar sets i.e. those $B \subset \Omega$ such that there is no martingale measure that can assign a positive measure to B. In the model independent framework the set $\mathcal{N}$ is induced by the market since the set of martingale measure has not to withstand to any additional condition (such as being equivalent to a certain P). Once these polar sets are identified we explicitly build a process $H^\bullet$ which depends only on the price process S and satisfies:

- $V_T(H^\bullet)(\omega) \geq 0 \ \forall \omega \in \Omega$
- $N \subseteq \mathcal{V}_{H^\bullet}^+$ for every $N \in \mathcal{N}$.

This strategy is a measurable selection of a set valued process $\mathbb{H}$, that we baptize **Universal Arbitrage Aggregator** since for any P, which is not absolutely continuous with respect to $\mathcal{M}$, an arbitrage opportunity H^P (in the classical sense) can be found among the values of $\mathbb{H}$. More precisely,

Theorem 2 *If $P \in \mathcal{P}$ is not absolutely continuous with respect to $\sup_{Q\in\mathcal{M}} Q$ then there exists an $\mathbb{F}^P$-predictable trading strategy H^P which is a P-Classical Arbitrage and*

$$H^P(\omega) \in \mathbb{H}(\omega) \qquad P\text{-a.s.}$$

where $\mathcal{F}_t^P$ denote the P-completion of $\mathcal{F}_t$ and $\mathbb{F}^P := \{\mathcal{F}_t^P\}_{t\in I}$.

All the inefficiencies of the market are captured by the process $H^\bullet$ but, in general, it fails to be $\mathbb{F}$-predictable. To recover predictability we need to enlarge the natural filtration of the process S by considering a suitable technical filtration $\widetilde{\mathbb{F}} := \{\widetilde{\mathcal{F}}_t\}_{t\in I}$ which does not affect the set of martingale measures, i.e. any martingale measure $Q \in \mathcal{M}$ can be uniquely extended to a martingale measure $\widetilde{Q}$ on the enlarged filtration. One of the main contributions is given by the following:

Theorem 3 *Let $(\Omega, \widetilde{\mathcal{F}}_T, \widetilde{\mathbb{F}})$ be a suitably enlarged filtered space and let $\widetilde{\mathcal{H}}$ be the set of d-dimensional discrete time $\widetilde{\mathbb{F}}$-predictable stochastic process. Then*

$$\textit{No Arbitrage de la classe } \mathcal{S} \textit{ in } \widetilde{\mathcal{H}} \iff \mathcal{M} \neq \varnothing \textit{ and } \mathcal{N} \textit{ does not contain sets of } \mathcal{S}$$

In other words, properties of the family $\mathcal{S}$ have a dual counterpart in terms of polar sets of the pricing functional.

We further provide our version of the Fundamental Theorem of Asset Pricing: the equivalence between absence of Arbitrage de la classe $\mathcal{S}$ in $\widetilde{\mathcal{H}}$ and the existence of martingale measures $Q \in \mathcal{M}$ with the property that $Q(C) > 0$ for all $C \in \mathcal{S}$.

Model Independent Arbitrage. When $\mathcal{S} := \{\Omega\}$ then the Arbitrage de la classe $\mathcal{S}$ corresponds to the notion of a Model Independent Arbitrage. As Ω never belongs to the class of polar sets $\mathcal{N}$, from Theorem 3 we directly obtain the following result.

Theorem 4

$$\textit{No Model Independent Arbitrage in } \widetilde{\mathcal{H}} \Longleftrightarrow \mathcal{M} \neq \varnothing.$$

An analogous result has been obtained in [AB13] when considering a single risky asset S as the canonical process on the path space $\Omega = \mathbb{R}_+^T$, a possibly uncountable collection of options $(\varphi_\alpha)_{\alpha \in A}$ whose prices are known at time 0, and when trading is possible through semi-static strategies (see also [Ho11] for a detailed discussion). Assuming the existence of an option φ_0 with a specific payoff, equivalence in Theorem 4 is achieved in the original measurable space $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{H})$. In our setup, although we are free to choose a $(d+k)$-dimensional process S for modeling a finite number of options (k) on possibly different underlying (d), the class $\widetilde{\mathcal{H}}$ of admissible strategies are dynamic in every S^i for $i = 1, \dots d+k$. In order to incorporate the case of semi-static strategies we would need to consider restrictions on $\widetilde{\mathcal{H}}$ and for this reason the two results are not directly comparable.

Arbitrage with respect to open sets and market feasibility. In the topological context, in order to obtain full support martingale measures, the suitable choice for $\mathcal{S}$ is the class of open sets. This selection determines the notion of Arbitrage with respect to open sets, which we shorten as "Open Arbitrage":

- **Open Arbitrage** is a trading strategy $H \in \mathcal{H}$ such that $V_0(H) = 0$, $V_T(H)(\omega) \geq 0 \ \forall \omega \in \Omega$ and $\mathcal{V}_H^+$ contains an open set.

This concept admits the following dual reformulation:

$$\begin{aligned}&\text{An Open Arbitrage consists in a trading strategy } H \in \mathcal{H} \text{ and a non empty}\\ &\text{weakly open set } \mathcal{U} \subseteq \mathcal{P} \text{ such that for all } P \in \mathcal{U},\ V_T(H) \geq 0\ P\text{-a.s. and } P(\mathcal{V}_H^+) > 0.\end{aligned} \quad (1)$$

We relate this notion of arbitrage opportunity to the question: which are the markets that are feasible in the sense that the properties of the market are nice for "**most**" probabilistic models? Clearly this problem depends on the choice of the feasibility criterion, but to this aim we do not need to exhibit a priori a subset of probabilities. On the opposite, given a market (described without reference probability), the induced set of No Arbitrage models (probabilities) for that market will determine if the market itself is feasible or not. What is needed here is a good notion of "**most**" probabilistic models.
More precisely given the price process S defined on $(\Omega, \mathcal{F})$, we introduce the set $\mathcal{P}_0$ of probability measures that exhibit No Arbitrage in the classical sense:

$$\mathcal{P}_0 = \{P \in \mathcal{P} \mid \text{No Arbitrage with respect to } P\}. \quad (2)$$

When

$$\overline{\mathcal{P}_0}^{\tau} = \mathcal{P}$$

with respect to some topology τ the market is feasible in the sense that any "bad" reference probability can be approximated by No Arbitrage probability models. It is possible to show that this property is equivalent to the existence of a full support martingale measure if we choose τ as the weak* topology.

Feasible markets and absence of Open Arbitrage can be characterized in terms of existence of full support martingale measures. We denote with $\mathcal{P}^+ \subset \mathcal{P}$ the set of full support probability measures.

Theorem 5 *The following are equivalent:*

1. *The market is feasible, i.e* $\overline{\mathcal{P}_0}^{\sigma(\mathcal{P},C_b)} = \mathcal{P}$*;*
2. *There exists* $P \in \mathcal{P}_+$ *s.t. No Arbitrage w.r.to* P *(in the classical sense) holds true;*
3. $\mathcal{M} \cap \mathcal{P}_+ \neq \varnothing$*;*
4. *No Open Arbitrage holds with respect to admissible strategies* $\widetilde{\mathcal{H}}$*.*

Riedel [Ri11] already pointed out the relevance of the concept of full support martingale measures in a probability-free set up. Indeed in a one period market model and under the assumption that the price process is continuous with respect to the state variable, he showed that the absence of a one point arbitrage (non-negative payoff, with strict positivity in at least one point) is equivalent to the existence of a full support martingale measure. This equivalence is no longer true in a multiperiod model (or in a single period model with non trivial initial sigma algebra), even for price processes continuous in ω. We consider a multi-assets multi-period model without ω-continuity assumptions on the price processes and we develop the concept of open arbitrage, as well as its dual reformulation, that allows for the equivalence stated in the above theorem.

Dual representation of the Superhedging price. We consider the set of *efficient beliefs* $\Omega_* \subset \Omega$ defined as the set of $\omega \in \Omega$ such that there exists $Q \in \mathcal{M}$ such that $Q(\{\omega\}) > 0$. The set Ω_* is therefore the complement of the maximal polar set induced by $\mathcal{M}$. Theorem 1.4 in [AB13] proves (under a restrictive technical assumption) that in a model independent framework the superhedging price can be defined through ω-by-ω superhedging strategies, without introducing a duality gap. Nevertheless we provide a counterexample which shows that dropping this technical assumption Ω_* becomes the maximal set on which it is possible to obtain a superhedging duality.

Theorem 6 (Superhedging) *Let* $g : \Omega \mapsto \mathbb{R}$ *be an* $\mathcal{F}$*-measurable random variable. Then*

$$\begin{aligned} & \inf \{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T \geq g \ \mathcal{M}\text{-q.s.}\} \\ = \ & \inf \{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T(\omega) \geq g(\omega) \ \forall \omega \in \Omega_*\} \\ = \ & \sup_{Q \in \mathcal{M}_f} E_Q[g] = \sup_{Q \in \mathcal{M}} E_Q[g], \end{aligned}$$

where

$$\Omega_* := \{\omega \in \Omega \mid \exists Q \in \mathcal{M} \text{ s.t. } Q(\omega) > 0\}. \tag{3}$$

This result coincide with the superhedging duality proved in [BN13] for the quasi-sure framework only in the case that the class $\mathcal{P}'$ contains all the probability measures on Ω. In such a case Ω_* coincides with Ω and the No-Arbitrage condition is the strongest possible. Moreover in the quasi-sure approach both the product structure of Ω and the properties on the set of reference probabilities $\mathcal{P}'$ play a crucial role.

We now allow for the possibility of static trading in a finite number of options. Let us add to the previous market k options $\Phi = (\phi^1, ..., \phi^k)$ which expires at time T and assume without loss of generality that they have zero initial cost. We assume that each ϕ^j is an $\mathcal{F}$-measurable random variable. Define $h\Phi := \sum_{j=1}^k h^j \phi^j$, $h \in \mathbb{R}^k$, and

$$\mathcal{M}_\Phi := \{Q \in \mathcal{M}_f \mid E_Q[\phi^j] = 0 \ \forall j = 1, ..., k\} = \{Q \in \mathcal{M}_f \mid E_Q[h\Phi] = 0 \ \forall h \in \mathbb{R}^k\}, \tag{4}$$

which are the options-adjusted martingale measures, and

$$\Omega_\Phi := \{\omega \in \Omega \mid \exists Q \in \mathcal{M}_\Phi \text{ s.t. } Q(\omega) > 0\} \subseteq \Omega_*. \tag{5}$$

We have by definition that for every $Q \in \mathcal{M}_\Phi$ the support satisfies $supp(Q) \subseteq \Omega_\Phi$. We define the superhedging price when semi-static strategies are allowed by

$$\pi_\Phi(g) := \inf\left\{x \in \mathbb{R} \mid \exists (H,h) \in \mathcal{H} \times \mathbb{R}^k \text{ such that } x + (H \cdot S)_T(\omega) + h\Phi(\omega) \geq g(\omega) \ \forall \omega \in \Omega_\Phi\right\}. \tag{6}$$

With the same methodology used in the proof of Theorem 6 we can obtain the superhedging duality with semi-static strategies:

Theorem 7 (Super-hedging with options) *Let $g : \Omega \mapsto \mathbb{R}$ and $\phi^j : \Omega \mapsto \mathbb{R}$, $j = 1, ..., k$, be $\mathcal{F}$-measurable random variables. Then*

$$\pi_\Phi(g) = \sup_{Q \in \mathcal{M}_\Phi} E_Q[g].$$

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