The origins of scale-free production networks

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1 Introduction

The scale-free nature of a wide range of socio-economic networks has been extensively documented in the recent literature [see e.g Barabási et al., 2009, Gabaix, 2009, Schweitzer et al., 2009]. An example of central concern for macroeconomics are production networks whose scale-free nature has recently been put forward by Acemoglu et al. [2012] as a potentially major driver of macroeconomic fluctuations [see also Battiston et al., 2007]. Relatedly, the scale-free distribution of firms' size [see Axtell, 2001] has also been identified as a key micro-economic source of aggregate volatility [see Gabaix, 2011].

It therefore seems problematic that the central tenet of economic theory with respect to the formation of structures, namely general equilibrium theory, has essentially nothing to say about the scale-free nature, or the nature in general, of the distribution of firms' size or this of production networks. Indeed, in a general equilibrium framework firms' size are either indeterminate (when there are constant returns to scale) or completely determinate by the primitives of the model (when there are decreasing returns to scale, the equilibrium size of the firm is completely determinate by its production technique.) In particular when firms have the same production technique, they have the same size at equilibrium.

The present paper addresses this wide gap in the theory through a dynamic extension of the general equilibrium model that accounts for three key stylized facts about the structure of the productive sector: firms' growth rates follow a Laplace distribution [see e.g Bottazzi and Secchi, 2006], firms' sizes are Zipf distributed and the degree distribution of production networks are scale-free.

The backbone of our approach is a model of monopolistic competition on the markets for intermediate goods, akin to the one introduced by Ethier [1982] (on the basis of Dixit and Stiglitz [1977]) and popularized by the endogenous growth literature [see e.g Romer, 1990]. In this framework, we represent supply relationships *** here supply relationship might be confusing (?) with respect

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to Acemoglu, JP etc. framework *** as the weighted edges of a network and consider out-of-equilibrium dynamics in which (i) demands are made in nominal terms and sellers adjust instantaneously their prices to balance real supply and nominal demand (ii) firms need time to adjust their production technologies (i.e the network weights) to prevailing market prices. When the set of relationships is fixed (i.e only the weights of the network can evolve), the identification with the underlying general equilibrium model is perfect in the sense that (i) the adjacency matrix of the network is in a one to one correspondence with the underlying ge economy (ii) the model does converge to the underlying general equilibrium. However, the context of interest for us is this where the technological structure is not fixed a priori and where, the different production goods being assumed substitutable, firms can, in the long-run, adjust their production technologies/ supply relationships (i.e the adjacency matrix) as a function of market prices. Then, we show that the model does not in general admit a steady-state but rather displays self-organized criticality [see Bak et al., 1987] (double-check) and settles in a regime where the distribution of firms 'size and the structure of the production network are scale-free.

Similarly to those of the existing literature [1] [see e.g Barabási and Albert, 1999, Gabaix, 1999], our results are mainly driven by a form of preferential attachement/proportional growth process: in a nutshell, the larger a firm is the easier it can accommodate, given its current capacity, a new consumer without substantially modifying its price and hence its competitive position.

However, our approach offers a much more systemic and comprehensive perspective that this existing in the literature. Indeed, from Kalecki [1945] and Simon et al. [1977] to more recent contributions such as Bottazzi and Secchi [2006], the problem of the distribution of firms' size has been approached almost solely through "island-models" in which the growth of each firm is studied in isolation and driven by exogenous shocks. On the contrary, in our model, growth opportunities are endogenous and we account for general equilibrium linkages.

More broadly, our paper contributes to the literature on the formation of socio-economic networks [see e.g Jackson et al., 2008] by providing micro-foundations for the emergence of scale-free networks which have been largely lacking in this literature, but for the notable exception of Jackson and Rogers [2007]. The paper also has close relationships with the infra marginal analysis pioneered by Xiaokai Yang [see Yang and Borland, 1991, Cheng and Yang, 2004] and Bak and co-authors's approach to the importance of self-organized criticality in economic networks [see Bak et al., 1993, Scheinkman and Woodford, 1994].

The remaining of the paper is organized as follows. In section 2, we propose a model of production networks as monopolistically competitive markets for intermediate goods. In section 3, we propose a numerical exploration of the dynamics of the model. Section 4 gives an analytical proof of the main results

[1][see also Bottazzi and Secchi, 2006] *we model this idea using a process whereby the probability for a given firm to obtain new opportunities depends on the number of opportunities already caught.*

and section 5 concludes.

2 The Model

2.1 A general equilibrium primer

We consider an economy consisting of a finite set of (monopolistically competitive) firms producing differentiated goods and of a representative household. We denote the set of firms by $M = \{1, \cdots, m\}$, the representative household by the index 0 and the set of agents by $N = \{0, \cdots, m\}$.

Our central concern is the endogenous formation of supply relationships between firms. Therefore, to assume away any exogenous determinism in this respect, we place ourselves in a setting where there is no a priori distinction between potential intermediary goods. More precisely, we consider that the production possibilities of firm i are given by a C.E.S production function of the form (todo mention the possibility of weights):

$$f_i(x_0, (x_j)_{j=1,\cdots,n_i}) = x_0^{\alpha}(\sum_{j=1}^{n_i} x_j^{\sigma})^{(1-\alpha)/\sigma} \tag{1}$$

where x_0 is the quantity of labor used in the production process, n_i the number of intermediary goods/components combined and x_j the quantity of input j used in the production process.

This representation assumes that each good can be used interchangeably in the production process . It is standard in models of monopolistic competition on the intermediate goods markets [see Ethier, 1982, Romer, 1990]. One of its key implications is that productivity grows with the number of components/suppliers[2].

As for the representative household, we consider that he supplies a constant quantity of labor (normalized to 1) and has preferences represented by a Cobb-Douglas utility function of the form $u(x_1, \cdots, x_m) = \prod_{i=1}^{m} x_i^{\alpha_{0,i}}$. He hence spends his income on each good $i \in M$ proportionally to $\alpha_{0,i}$ (we assume that $\forall i \in M, \alpha_{0,i} > 0$, so that the household consumes a positive quantity of each and every good).

As such, this model is incomplete. The micro-economic choices of the agents in terms of production or consumption can not be determined without further assumptions on the structure of interactions. In our firm-focused setting, these interactions are mainly characterized by the production network, which specifies the flows of goods between firms. The formation of these production networks is the key focus of the reminder of this paper.

A general equilibrium approach to the issue would consist, in our setting, in defining the production network through an adjacency matrix $A = (a_{i,j})_{i,j \in M}$

[2]This feature is also at the core of the infra-marginal approach to economic growth [see Yang and Borland, 1991] and of Adam Smith's original description of the effects of the division of labor.

such that $a_{i,j} = 1$ if j is a supplier of i and $a_{i,j} = 0$ otherwise. Consistency with equation (1) would then require that for all $i \in M$, $\sum_{j=1}^{m} a_{i,j} = n_i$ and, denoting by $S_i(A) := \{j \in M \mid a_{i,j} = 1\}$ the set of suppliers of firm i, the production function of firm i would be further specialized into:

$$f_i(x_0, (x_j)_{j\in S_i}) = x_0^{\alpha}(\sum_{j\in S_i(A)} x_j^{\sigma})^{(1-\alpha)/\sigma} \quad (2)$$

One could then define a general equilibrium of the economy $\mathcal{E}(A)$ associated to the production network A as follows.

Definition 1 *A general equilibrium of the economy $\mathcal{E}(A)$ is a collection of prices $(p_1^*, \cdots, p_m^*) \in \mathbb{R}_+^N$, production levels $(q_1^*, \cdots, q_m^*) \in \mathbb{R}_+^M$ and commodity flows $(x_{i,j}^*)_{i,j=0\cdots n} \in \mathbb{R}_+^{M\times M}$ such that:*

1. *Markets clear. That is one has for all $j \in N$, $q_j^* = \sum_{i=0}^{n} x_{i,j}^*$ (with $q_0^* = 1$ by normalization).*

2. *The representative consumer maximizes his utility. That is $(q_0^*, (x_{0,j}^*)_{j=1,\cdots,n})$ is a solution to*
$$\begin{cases} \max\ u_i((x_{0,j})_{j=1,\cdots,n}) \\ s.t\ \sum_{j=1}^{n} p_j^* x_{0,j}^* \leq 1 \end{cases}$$
(with the price of labor normalized to 1)

3. *Firms maximize profits. That is for all $i \in M$, $(q_i^*, (x_{i,j}^*)_{j\in S_i(A)})$ is a solution to*
$$\begin{cases} \max\ p_i^* q_i - \sum_{j\in S_i(A)} p_j^* x_{i,j} \\ s.t\ f_i((x_{i,j})_{j\in S_i(A)}) \geq q_i \end{cases}$$

Hence, in a general equilibrium setting, the adjacency structure of the production network is fixed and the magnitude of the physical flows between firms is determined at equilibrium. A particular case that has received widespread attention in the literature [see Acemoglu et al., 2012, Long and Plosser, 1983] is the Cobb-Douglas case (i.e when $\sigma \to$) in which the value of flows between firms at equilibrium is given by the corresponding exponents in the production function (uniformly equal to one in our framework).

Our aim in the following is to subsume this general equilibrium approach within an endogenous model of the formation of production networks.

2.2 An endogenous model of network formation

We consider a coupled model of network formation and out of equilibrium dynamics in which firms adaptively search for profit maximizing/cost minimizing input combinations. More precisely, we consider that time is discrete and indexed by $t \in \mathbb{N}$. Each agent $i \in N$ is initially endowed with a wealth $w_i^0 \in \mathbb{R}_+$

and a quantity of output $q_i^0 \in \mathbb{R}_+$ (normalized to 1 throughout in the case of the representative household). As for the production network, we assume its initial structure is given by the matrix of weights $\mathbb{A}^0 = (\alpha_{i,j}^0)_{i,j\in N}$, where $\alpha_{i,j}$ represents the share of agent i's expenses directed towards agent j.

We are concerned with the time evolution of the wealths $(w_i^t)_{i\in N}^{t\in\mathbb{N}}$, the quantities produced $(q_i^t)_{i\in N}^{t\in\mathbb{N}}$, the production network $\mathbb{A}^t = (\alpha_{i,j}^0)_{i,j\in N}^{t\in\mathbb{N}}$, as well as this of prices $(p_i^t)_{i\in N}^{t\in\mathbb{N}}$. This evolution is driven by the interplay between the workings of the market out of equilibrium and the evolution of the production network. More precisely, during each period $t \in \mathbb{N}$, the following sequence of events takes place:

1. Each agent i receives the nominal demand $\sum_{j\in N} \alpha_{i,j} w_j^t$.

2. Given the nominal demand $\sum_{j\in N} \alpha_{i,j} w_j^t$ and the output stock q_i^t, the market clearing price for firm i would be

$$\overline{p}_i^t = \frac{\sum_{j\in N} \alpha_{i,j} w_j^t}{q_i^t}. \tag{3}$$

   Now, we shall assume that prices adjust frictionally to their market-clearing values and hence consider that firm actually set their prices according to

$$p_i^t = \tau_p \overline{p}_i^t + (1-\tau_p) p_i^{t-1} \tag{4}$$

   where $\tau_p \in [0,1]$ is a parameter measuring the speed of price adjustment (the case $\tau_p = 1$ corresponding to instantaneous price adjustment).

3. Whenever $\tau_p < 1$ markets do not clear (except if the system is at a stationary equilibrium). In case of excess demand, we assume that clients are rationed proportionally to their demand. In case of excess supply, we assume that the amount $\overline{q}_i^t := \sum_{j\in N} \alpha_{i,j} w_j^t / p_i^t$ is actually sold and that the rest of the output is stored as inventory. Together with production occurring on the basis of purchased inputs, this yields the following evolution of the product stock:

$$q_i^{t+1} = q_i^t - \overline{q}_i^t + f_i\Big(\frac{\alpha_{0,i} w_i^t}{p_0^t}, \big(\frac{\alpha_{j,i} w_i^t}{p_j^t}\big)_{j\in S_i(A)}\Big) \tag{5}$$

   N.B: in the case where $\tau_p = 1$, one necessarily has $\overline{q}_i^t = q_i^t$ and equation (5) reduces to

$$q_i^{t+1} = f_i\Big(\frac{\alpha_{0,i} w_i^t}{p_0^t}, \big(\frac{\alpha_{j,i} w_i^t}{p_j^t}\big)_{j\in S_i(A[t])}\Big) \tag{6}$$

4. As for the evolution of agents' wealth, it is determined on the one hand by their purchases of inputs and their sales of output. On the other hand, we assume that the firm sets its expenses for next period at $(1-\lambda)$ times its

current revenues and distributes the rest as dividends to the representative household. That is one has:

$$\forall i \in M,\ w_i^{t+1} = (1-\lambda)\overline{q}_i^t p_i^t \tag{7}$$

$$w_0^{t+1} = q_0^t p_0^t + \lambda \sum_{i \in M} \overline{q}_i^t p_i^t \tag{8}$$

Note that equation (8) can be interpreted as assuming that firms have myopic expectations about their nominal demand (i.e they assume they will face the same nominal demand next period) and target a fixed profit/dividend share $\lambda \in (0,1)$.

This first sequence of operations defines out of equilibrium dynamics for a given production network. As for the evolution of the network, it takes place at the end of the period according to two process: one governs the evolution of weights, the other this of the adjacency structure.

5. As for the evolution of weights, given prevailing prices the optimal input weights for a firm i are those that minimize production costs. Those are defined as the solution to the following optimization problem:

$$\begin{cases} \max & f_i(\frac{\alpha_0, i}{p_0^t}, (\frac{\alpha_j, i}{p_j^t})_{j \in S_i(A)}) \\ \text{s.t} & \sum_{j \in S_i(A)} \alpha_{j,i} = 1 \end{cases} \tag{9}$$

Now, as in the case of prices, we shall consider that the process of technological adjustment can be subject to frictions and that input weights are actually updated according to the following rule:

$$\alpha_i^{t+1} = \tau_w \overline{\alpha}_i^t + (1-\tau_w)\alpha_i^t \tag{10}$$

where $\overline{\alpha}_i^t \in \mathbb{R}^M$ denotes the solution of 9 and $\tau_w \in [0,1]$ measures the speed of technological adjustment of the production network.

6. As for the evolution of the adjacency structure, each firm independently receives the opportunity to change one of its suppliers with probability $\rho_{chg} \in [0,1]$. If the opportunity actually arises for firm i in period t, it selects randomly one of its most expensive supplier $\overline{j}_i$ and another random firm j among those to which it is not already connected. It then shifts its connection from firm $\overline{j}_i$ to firm j if and only if the price of j is cheaper than this of $\overline{j}_i$. In other words, the adjacency matrix A^t evolves according to:

$$\begin{gathered} a_{i,\bar{j}_i}^{t+1} = \begin{cases} 1 & \text{if } p_{i,\bar{j}_i} \leq p_{i,j} \\ 0 & \text{otherwise} \end{cases} \\ a_{i,j}^{t+1} = 1 - a_{i,\bar{j}_i}^{t+1} \end{gathered} \tag{11}$$

The actual weight of the new connection is then determined according to STAN ?

7. Finally, the possibility for a firm to lose connections implies that it can eventually be driven out of the market. Indeed, we consider that a firm that has lost all its connections toward other firms exits the market. To sustain competition in the economy, we assume that those exits are compensated by entries of new firms according to the following process. Every period, each (potential) firm that is out of the market independently enters with probability p_{new}. When entering, the firm is endowed with the following characteristics:

   - The number of suppliers is drawn from a binomial distribution $B(p, N_F)$. The success probability p is artificially adjusted in order to preserve on average the initial number of links L_0.
   - The price is initially set equal to the average price in the economy.
   - Each firm in the economy rewires to the newly created firm independently with probability $\bar{k}/n$, where $\bar{k}$ is the average number of clients at time 0.
   - The wealth of the firm is set equal to the average wealth of other firms[3] and its initial output stock is empty.

2.3 The linear case

In order to gain an understanding of the basic dynamics of the model, let us first consider the case where the network is fixed, both in terms of weights and adjacency structure. Note that this fixed weights assumption is equivalent to the assumption used in Acemoglu et al. [2012] that production functions are Cobb-Douglas (with the corresponding weights). In this respect, this simplified version of our model can be seen as an out of equilibrium extension of Acemoglu et al. [2012] (maybe better to make the connection with Cobb-Douglas)

The dynamic properties of this model are relatively straightforward. First, it is clear that the evolution of wealths follows a linear dynamic, which can be written matricially as:

$$w^{t+1} = R(\lambda)\mathbb{A}w^t \tag{12}$$

where

$$R(\lambda) := \begin{pmatrix} \lambda & \cdots & \cdots & \lambda \\ 0 & & & \\ \vdots & & (1-\lambda)I & \\ 0 & & & \end{pmatrix}$$

accounts for the redistribution of firms' revenues. The matrix of weights $\mathbb{A}$ being moreover row-stochastic, it is straightforward to check using Perron-Froebenius theorem that the linear system in 12 is globally asymptotically stable. Accordingly, as illustrated in Fig. 1, we observe convergence in our simulations towards

[3] To ensure conservation of money in the long term, this initial wealth of the firm is in practice considered as a loan that the firm has to reimburse before it can pay any dividend

a stationary equilibrium determined by $\overline{w} \in \mathbb{R}^N$ such that:

$$\overline{w} = R(\lambda) A \overline{w} \tag{13}$$

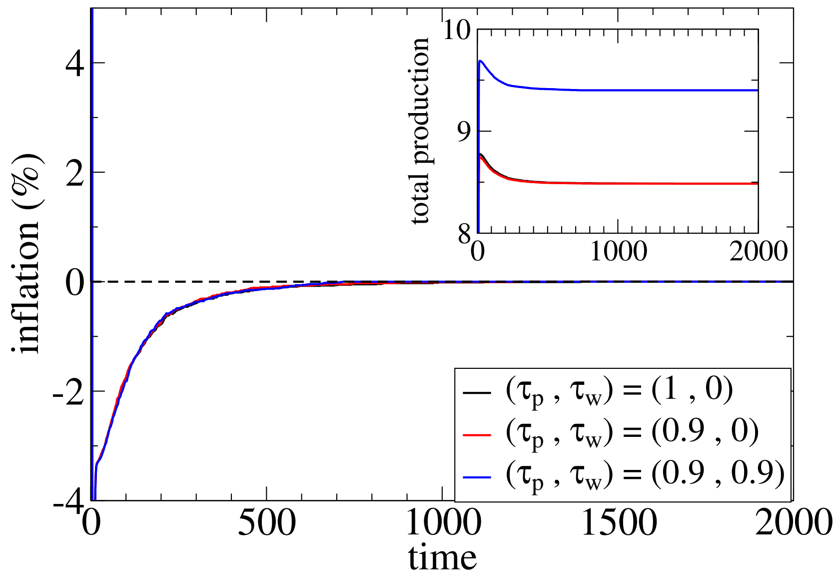

Figure 1: One time-step inflation rate and total production as a function of time for the basic model ($\rho_{chg} = 0$) and different values of τ_p, τ_w. Other parameters are: $\sigma = 0.5$, $\lambda = 0.05$, $M = 1000$.

2.4 Frictions and general equilibrium

Proceeding stepwise, we now consider the case where the adjacency structure of the network is fixed but the weights evolve according to equation (10). This setting is akin to the general equilibrium one introduced in section 2.1 but for the fact that firms aim at enforcing a mark-up proportional to λ on their production costs rather than at maximizing profits. More precisely, steady states of the dynamical system defined by equations (4) to (10) are mark-up equilibria in the following sense:

Definition 2 *A mark-up equilibrium of the economy $\mathcal{E}(A)$ is a collection of prices $(p_0^*, \cdots, p_n^*) \in \mathbb{R}_+^M$, production levels $(q_0^*, \cdots, q_n^*) \in \mathbb{R}_+^M$ and commodity flows $(x_{i,j}^*)_{i,j=0\cdots n} \in \mathbb{R}_+^{M\times M}$ such that:*

- *Markets clear. That is for all $i \in M$, one has*

$$q_i^* = \sum_{j=1}^{M} x_{i,j}^*.$$

- *The representative consumer maximizes his utility. That is* $(q_0^*, (x_{0,j}^*)_{j=1,\cdots,n})$ *is a solution to*

$$\begin{cases} max\ u_i((x_{0,j})_{j=1,\cdots,n}) \\ \\ s.t\ \sum_{j=1}^n p_j^* x_{0,j}^* \leq 1 \end{cases}$$

(with the price of labor normalized to 1*)*

- *Production costs are minimized. That is for all* $i \in M$, $(x_{i,j}^*)_{j=0\cdots n}$ *is the solution to*

$$\begin{cases} min & \sum_{j\in S_i(A)} p_j^* x_j \\ s.t & f_i(x_j) \geq q_i^* \end{cases}$$

- *Prices are set as a mark-up over production costs at rate* $\dfrac{\lambda}{1-\lambda}$. *That is one has for all* $i \in N$:

$$p_i^* = (1 + \frac{\lambda}{1-\lambda}) \frac{\sum_{j\in S_i(A)} p_j^* x_{i,j}^*}{q_i^*}$$

Note that for $\lambda = 0$, mark-up equilibria coincide with general equilibria in the sense of Definition 1. Indeed in a setting with constant returns to scale, profits are zero at a general equilibrium[4].

In this sense our model can be seen as a (dynamic) extension of the conventional general equilibrium approach. Yet, this identification between economic equilibria and steady states of our dynamical system only makes sense if these steady states are stable. We (first) investigate the issue numerically by performing, for different values of the elasticity of substitution σ, Monte-Carlo simulations in which we let vary the speeds of price and technology adjustment, i.e τ_p and τ_w.

The results of these simulations are reported in Figure 2 as phase diagrams in the (τ_p, τ_w) plane. As long as σ is in a neighborhood of 1, the system exhibits two distinct phases: a stable and an unstable one. In the stable phase, the system converges to equilibrium: excess demand vanishes and prices converge to their equilibrium values (see figure XX). In the unstable phase, there is a persistent mismatch between supply and demand as well as sustained volatility in the network (see Figure 3 ?): the system remains in disequilibrium.

The key determinant of stability is the relative speed of price (τ_p) and technological (τ_w) adjustment. The faster the relative speed of price adjustment, the more stable the system is. Yet, the stability range increases as the absolute speed of price adjustment decreases. Also, the size of the stable region increases as the elasticity of substitution decreases. There exists a critical value σ^* ($\sigma^* \sim 5/9$ for the parameter setting used in figure 2) such that for $\sigma \leq \sigma^*$, the unstable region disappears and the system converges to equilibrium independently of the speeds of price and technological adjustment.

[4] Yet in a dynamic setting like ours assuming $\lambda > 0$ seems necessary to prevent firms from remaining permanently at the brink of bankruptcy.

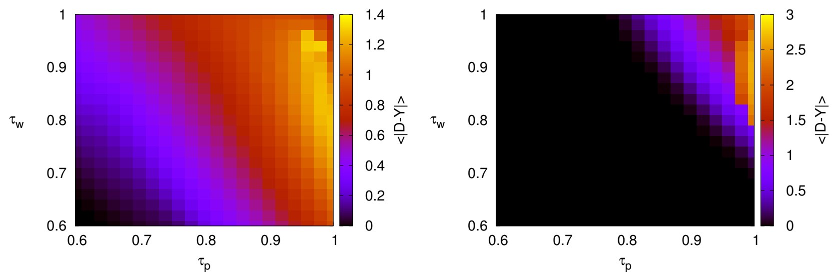

Figure 2: Color map of the stationary average mismatch between demand and supply in the (τ_p, τ_w) plane for $M = 2000$, $\lambda = 0.05$, $\sigma = 4/5$ (left) and $\sigma = 2/3$ (right). *** is probably better to have a diagram instead of a color map ***

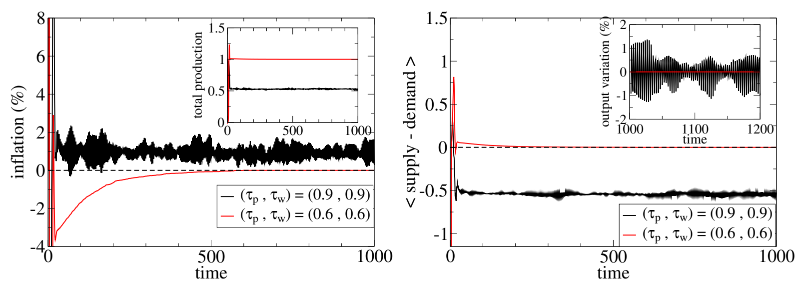

Figure 3: One time-step inflation rate, total production and average mismatch between supply and demand as a function of time for the basic model ($\rho_{chg} = 0$) and two different values of τ_p, τ_w corresponding to the stable / unstable region of Fig. 2. Other parameters are: $\sigma = 4/5$, $\lambda = 0.05$, $M = 1000$.

These results are reminiscent of those obtained in Bonart et al. [2014]: the larger the intrinsic volatility of the system (in our setting, the higher the elasticity of substitution), the slower the adjustment processes shall be for the system to be stable. As for the formation of networks, these results confirm that in absence of changes in the adjacency structure, the characteristics of the production networks are completely determined by exogenous technological constraints (represented by the production functions in our setting).

3 The endogenous formation of production networks

3.1 Steady-state analysis

In this section, we account for the increased flexibility in production technologies implied by models of monopolistic competition on the market for intermediary goods à la Ethier [1982] and Romer [1990]. In other words, we consider that the adjacency structure evolves according to Equation (11).

A steady state of the system would consist in vectors of wealths $\tilde{w}$, prices $\tilde{p}$, productions $\tilde{q}$ and in an adjacency matrix $\tilde{\mathbb{A}}$ satisfying the following properties.

- First, according to equation 11, each firm i buys only from the cheapest suppliers (otherwise it would rewire). That is, one has for all $i \in N$:

$$\max_{j \in S_i} \tilde{p}_j \leq \min_{k \notin S_i} \tilde{p}_k \tag{14}$$

- Second, firms only differ in terms of their number of suppliers n_i and, at a steady state, the larger the number of suppliers of a firm, the more productive and the cheaper it is (otherwise it could adopt the same production technique than any firm with a smaller number of suppliers and improve upon it by diversifying marginally). That is one has for all $i, j \in M$:

$$n_i > n_j \Rightarrow \tilde{p}_i > \tilde{p}_j \tag{15}$$

- Therefrom, one can deduce that at a steady-state only the firms with the maximal number of suppliers (the more productive according to equation 15) actually have consumers (according to equation 14). More precisely, let us denote by V the set of active firms in the steady state (i.e these actually having consumers), by $M_\mu := \{i \in M \mid n_i = \mu\}$ the set of firms with exactly μ suppliers, by m_μ the number of such firms, by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{\bar{k}}$ the decreasing sequence of μs for which $M_\nu \neq \emptyset$ and let $\nu_i = \sum_{j=1}^{i} m_{\mu_j}$. One then has:

$$M_{\mu_i} \subset V \Rightarrow \text{card}\{\ell \in M \mid n_\ell \geq \nu_{i-1}\} \geq m_{\mu_i} \tag{16}$$

  That is to say, for a firm with μ_i suppliers to have at least one consumer, there must be a firm that requires more than ν_{i-1} suppliers because its first ν_{i-1} suppliers are these that are more productive and hence cheaper than the μ_i suppliers firm.

A corollary of equation 16 is that there exists a steady state only if there are at least $m_{\mu_{\bar{k}}}$ firms that have more than $\nu_{\bar{k}-1}$ suppliers as otherwise there would always be a firm (in $M_{\bar{k}}$) without consumers. Such a firm would exit the market and be replaced by by an entering firm, hence contradicting the fact that the system is at a steady state. To clarify this condition, let us consider the case where all the firms have a distinct number of suppliers (that is $\bar{k} = m$

and $\forall i \; m_{\mu_i} = 1$). Then, there can be a steady state only if there exists a firm with exactly m suppliers, i.e a firm connected to every other firm. It is also worth noting that in this case the production network is a nested-split graph [see König et al., 2012, 2014] because every consumer of a firm in M_i also is a consumer of each of the cheaper firms (in M_j such that $j < i$).

This necessary condition for the existence of a steady-state is clearly extremely restrictive. It is not observed in simulations unless the system is initialized in a very peculiar state (e.g by letting all the firms exactly have the same number of suppliers). On the contrary, we generically observe sustained growth and decline of firms, entry and exit and changes in the micro-structure of the network. However, the system exhibits very robust distributional stylized facts that we investigate in the remaining of this paper.

3.1.1 Firms' demographics

A first major stylized fact of firms' demographics is that the growth rates of firms are distributed according to a "tent-shaped" double-exponential distribution [see Bottazzi and Secchi, 2006]. As illustrated in Fig. 4, our model generates exactly this type of Laplace distributions (Statistical test ???). For relatively short time intervals growth rates are indeed distributed $\sim \exp a|g - g_0|$) as found in empirical data. For longer time intervals there are fatter tails as one can expect given the final distribution of firms sizes. Following Arthur [1994], Bottazzi and Secchi put forward the fact that a Laplace type of distribution emerges because market success is cumulative or self-reinforcing. In their "island-based" model, this self-reinforcing process is hard-wired into the model: *"we model this idea using a process whereby the probability for a given firm to obtain new opportunities depends on the number of opportunities already caught."* In our setting, "self-reinforcing success" is also at play but it emerges endogenously. Indeed, the price-setting process (see equation 4) is such that whenever a firm gains a new consumer, its price increases (directly but also indirectly through the increase demand that it adresses to his own suppliers) and hence its competitiveness decreases. However, the larger the firm is the weaker the effect of an additional consumer is on its price and hence the more competitive it remains. Therefore, larger firms are more competitive and can seize more frequently new business opportunities. Hence, our model generates endogenously the self-reinforcing feedbacks introduced exogenously in Bottazzi and Secchi [2006] to generate a Laplacian distribution.

On a more structural level, key stylized facts about firms are the Zipf distribution of their size [see Axtell, 2001] and the presence of fat-tails in the degree distribution of production networks [see Atalay et al., 2011, Acemoglu et al., 2012]. As illustrated in Fig. 5 both features are clearly matched in the long-run by our model. Both the distribution of firms' sizes and the distribution of incoming links (i.e. the number of clients) are characterized by a power-law tail with exponent close to 2.1 (emphasize this is independent of the initial shape of the network)

Yet, as noted by Bottazzi and Secchi [2006], these long-run properties can not

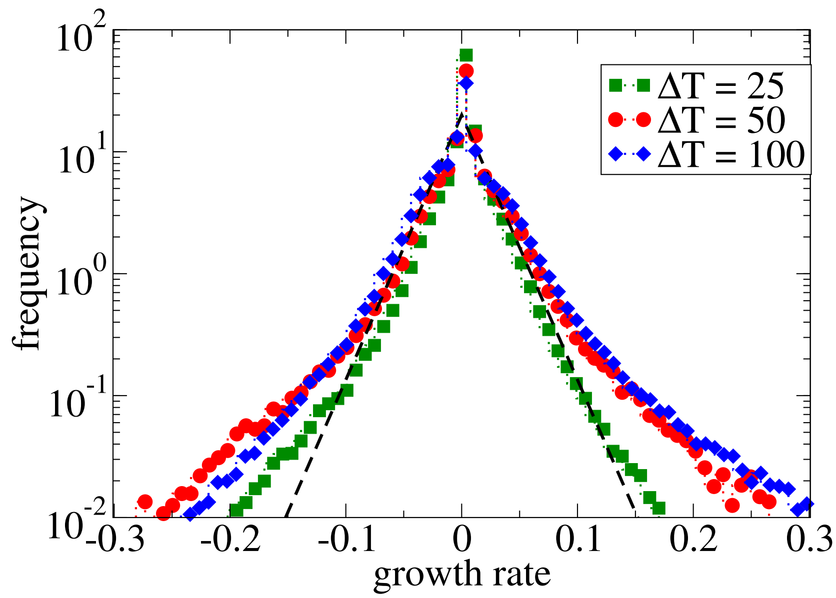

Figure 4: Distribution of firms' growth rates after 2 10^6 time steps for the model with $\rho_{chg} = \rho_{new} = 0.05$ and $\sigma = 1/2$. Different symbols / colors correspond to different time intervals to compute growth rates (for each firm we histogram only the last 30 rates). Other parameters are: $\tau_p = \tau_w = 0.8$ and $M = 10000$. *** change again this figure... ***

be explained by an exponential distribution of firms growth-rate. Accordingly, in our setting, "self-reinforcing success" in price competition is at-play in the short-run but in the long-run competition materializes mainly throughout the entry-exit process and the evolution of the network structure.

3.1.2 A master-equation approach to the formation of production networks

N.B The key driver of the fat-tail degree distribution is the fact that (i) the number of opportunities are fixed (ii) the probability to seize a new opportunity is independent of the size, (iii) the probability to lose an opportunity increases proportionally to the size. Hence to maintain a stable distribution, one must have very large firms that can lose large number of links.

In order to understand how these dynamics processes influence the formation on the network structure (and incidentally the distribution of firms' size), we adopt a meso-scale approach and study the evolution of the degree distribution of the network through a master equation. That is we investigate, the dynamics of the relative frequency of firms of degree k in the network, in other words we study the variation of $P(k, t)$ the probability to have a firm of degree k in the network at time t.

In our setting, the probability for a firm of losing or gaining a link depends only on the price. More precisely, let us denote by $i_j(t)$ the id of the jth most expensive firm at time t. One then denotes by $\pi_j^+(t)$ and $\pi_j^-(t)$ respectively the probability that firm $i_j(t)$ receives a new incoming link and loses an incoming link. One has:

$$\pi_j^+(t) = \frac{1}{n}\frac{n-j}{n-1} \tag{17}$$

$$\pi_j^-(t) = \frac{d_{j(t)}}{n\overline{d}}\frac{j-1}{n-1} \tag{18}$$

where $d_{j(t)}$ denotes the (in)degree of firm $j(t)$, and $\overline{d}$ the mean degree (so that $n\overline{d}$ is the total number of links). Then, if one denotes by $\rho_k(t)$ and $\mu_k(t)$ respectively the probability that a firm of degree k respectively gains and loses a link at time t, one has:

$$\rho_k(t) = \frac{1}{n(n-1)} \sum_{\{j|d_j=k\}} n-j \tag{19}$$

$$\mu_k(t) = \frac{k}{(n-1)n\overline{d}} \sum_{\{j|d_j=k\}} j-1 \tag{20}$$

The degree distribution of the network then obeys the following master-equation

$$\frac{\partial P(k,t)}{\partial t} = P(k-1,t)\rho_{k-1}(t) + P(k+1,t)\mu_{k+1}(t) - P(k,t)(\rho_k(t)+\mu_k(t)) \tag{21}$$

Therefrom, one can deduce that the stationary distribution of degrees satisfies the following equation:

$$P_{k-1}\rho_{k-1} + P_{k+1}\mu_{k+1} - P_k\rho_k - P_k\mu_k = 0 \tag{22}$$

Moreover, it is clear from equations 17 and 18 that at a stationary distribution, the position in the price ordering must decrease with the degree. Hence, if one denotes by η_k the number of firms of degree k at the stationary state and by $\nu_k = \sum_{i=1}^{k} \nu_i$, it must be that the ν_k firms of degree k have positions $n-\nu_{k-1}$ to $n-\nu_{k-1}-\nu_k+1 = n-\nu_k+1$ in the price ordering. It is then standard calculus to check that:

$$\sum_{\{j|d_j=k-1\}} n-j = \frac{1}{2}(2\nu_{k-2}+\eta_{k-1}-1)\eta_{k-1} \tag{23}$$

$$\sum_{\{j|d_j=k\}} j-1 = \frac{1}{2}(2n-2\nu_{k-1}-\eta_k-1)\eta_k \tag{24}$$

We then focus on the detailed balanced condition, which is the sufficient condition for a distribution to be stationary given by

$$\forall k \in \mathbb{N}\ P_k\mu_k = P_{k-1}\rho_{k-1} \tag{25}$$

or equivalently

$$\frac{P_{k-1}}{P_k} = \frac{\mu_k}{\rho_{k-1}} \tag{26}$$

Using equations 23 and 24 and the fact that ${}^{P_k}/_{P_{k-1}} = {}^{\eta_k}/_{\eta_{k-1}}$, one gets:

$$\frac{k}{\bar{\bar{d}}} \frac{(2n - 2\nu_{k-1} - \eta_k - 1)\eta_k}{(2\nu_{k-2} + \eta_{k-1} - 1)\eta_{k-1}} = \frac{\eta_{k-1}}{\eta_k} \tag{27}$$

or equivalently

$$\frac{1}{\bar{\bar{d}}} \frac{(2n - 2\nu_{k-1} - \eta_k - 1)}{(2\nu_{k-2} + \eta_{k-1} - 1)} = \frac{1}{k}(\frac{\eta_{k-1}}{\eta_k})^2 \tag{28}$$

which eventually yields after division by n :

$$\frac{1}{\bar{\bar{d}}} \frac{(2 - 2\Pi_{k-1} - P_k - {}^1/_n)}{(2\Pi_{k-2} + P_{k-1} - {}^1/_n)} = \frac{1}{k}(\frac{p_{k-1}}{p_k})^2 \tag{29}$$

where $\Pi_k = \sum_{\ell=1}^{k} P_\ell$. It is then clear that $\lim_{k\to+\infty} \Pi_k = 1$ and that P_k is negligible with respect to $1 - \Pi- = \sum_{\ell=k+1}^{+\infty} P_\ell$ as $k \to +\infty$. Therefrom, one ca deduce that as $k \to \infty$:

$$\Pi_k \sim 1 - \frac{\bar{\bar{d}}}{2k} \tag{30}$$

Taking a continuous approximation and differentiating, it is clear that π asymptotically follows a power-law with exponent -2.

*** START ALTERNATIVE ***

Therefrom, one can deduce that the stationary distribution of degrees satisfies the following equation:

$$P_{k-1}\rho_{k-1} + P_{k+1}\mu_{k+1} - P_k\rho_k - P_k\mu_k = 0 \tag{31}$$

Moreover, it is clear from equations 17 and 18 that at a stationary distribution, the position in the price ordering must decrease with the degree. Hence, if one denotes by $F_k = \sum_{i<k} P_i$ the probability of having a firm with degree strictly smaller than k, one has for the transition probabilties

$$\begin{aligned} \rho_k &= \frac{F_k}{N} \\ \mu_k &= (1 - F_{k+1})\frac{k}{N\bar{\bar{d}}}. \end{aligned} \tag{32}$$

We then focus on the detailed balanced condition, which is the sufficient condition for a distribution to be stationary and is given by

$$\forall k \in \mathbb{N}\ P_k\mu_k = P_{k-1}\rho_{k-1} \tag{33}$$

and therefore substituting Eq. 32 one easily finds

$$\frac{k}{\bar{d}} \frac{(1 - F_k - P_k)}{F_k - P_{k-1}} = \frac{P_{k-1}}{P_k} \tag{34}$$

which in the limit of $k \gg 1$ yields $F_k \sim k^{-1}$.

***END ALTERNATIVE ***

This is in very strong agreement with the simulation results depicted in figure 5 and sheds light on a possible endogenous mechanism for the origin of scale-free production networks. We also confirm analytical results by simulating numerically the evolution of Eq. 21 through Eq. 32. In order to do that we start with $P(k,t)$ binomially distributed as $B(n, d/n)$ with $d = 20$ and let evolve the distribution until it reaches a stationary state (approximately 10^6 steps). Formally the power-law tail arises because of the factor k appearing in the ratio between the probability of losing and gaining a link (see equations 20 and 19 respectively). Two processes are at play. On the one hand the most competitive firms tend to attract links and hence there is a tendency towards concentration on the most competitive firms. On the other hand, large firms are the most affected by competition from a new entrant because their chance to lose a customer is proportional to their size. This second process can be seen as a from of inverted preferential attachment process [see Barabási and Albert, 1999] where asymptotically large firms lose connections proportionally to their degree whereas in Barabási and Albert [1999] finite-size firms gain links proportionally to their degree.

4 Conclusion

We have a very simple dynamic model which suggests that a simple dynamic extension of the monopolistic competitive framework (considered in endogenous growth theory) is consistent with main stylized facts about firms' demographics and can give a systemic perspective on their origin. The key assumption about the dynamics is that prices adjust, in general, faster than technology (recall results about convergence to equilibrium)....

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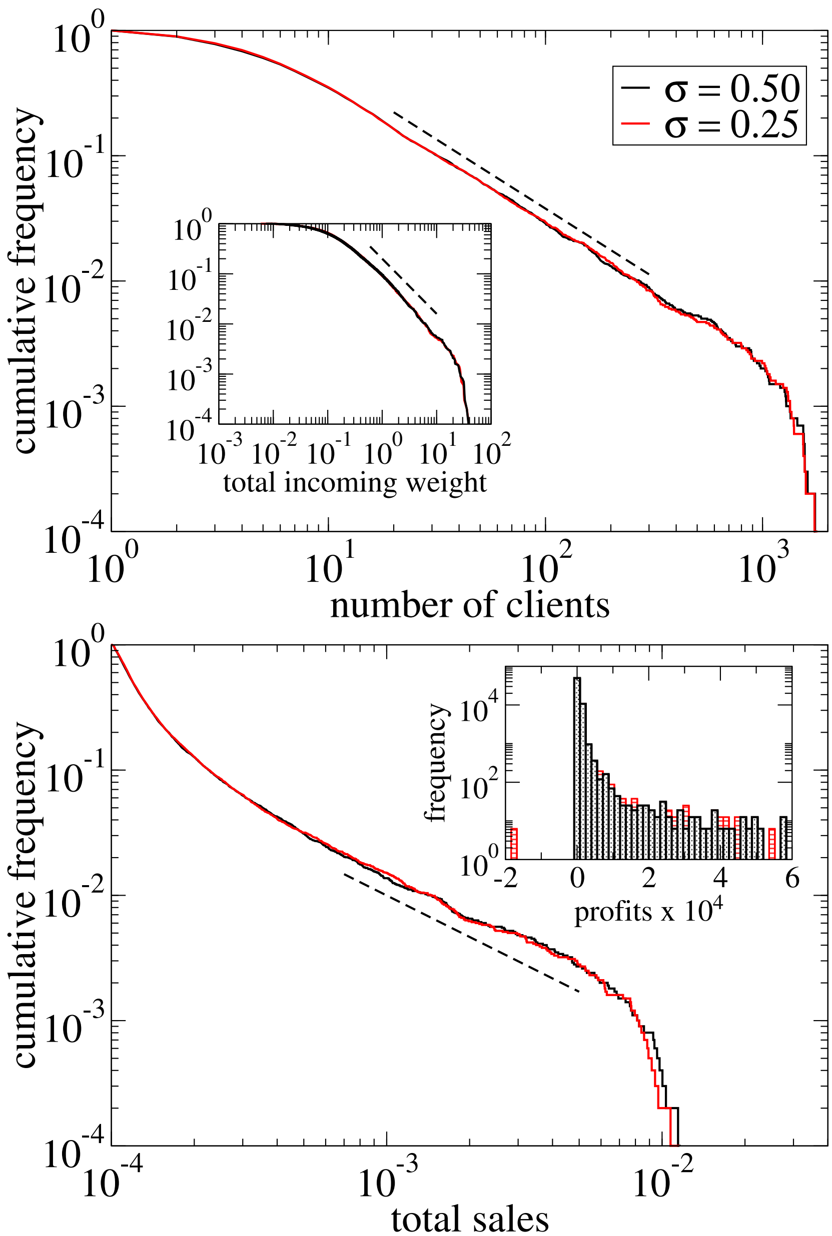

Figure 5: Basic firms statistics after 3 10^6 time steps for the model with $\rho_{chg} = \rho_{new} = 0.05$. *Top:* Cumulative frequency for the number of incoming links (clients) and total incoming weight (sum over all clients weights). *Bottom:* Cumulative frequency for total sales and profits histogram. In both graphs red lines are for $\sigma = 0.25$ while black solid lines are for $\sigma = 0.5$. Dashed black lines are a guide for the eye and correspond to $f(x) \sim x^{-1.1}$. Other parameters are: $\tau_p = \tau_w = 0.8$ and $M = 10000$.

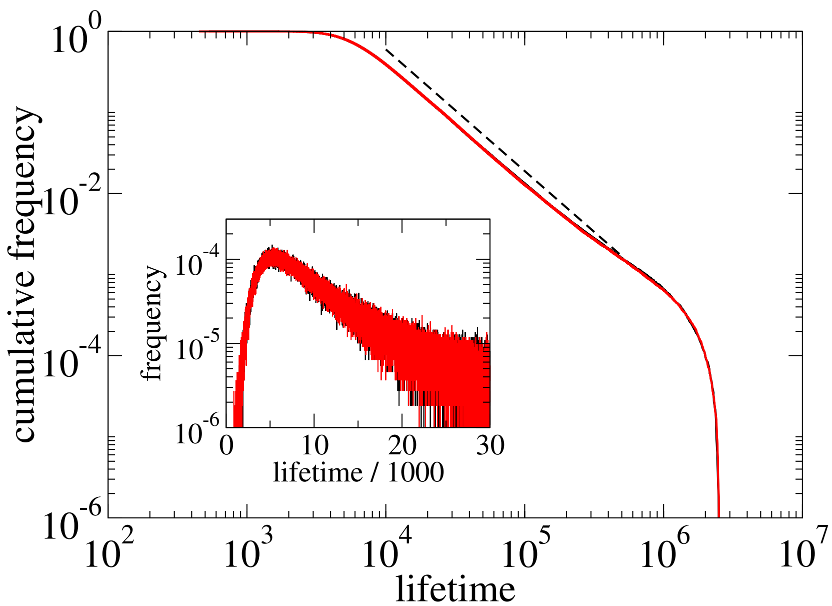

Figure 6: Other firms statistics after 3 10^6 time steps for the model with $\rho_{chg} = \rho_{new} = 0.05$ (as in Fig. 5). Frequency and cumulative frequency for firms lifetime. For longer lifetimes the distribution has an exponential decay (see inset) followed by a power law tail with exponent ~ 2.5. Other parameters are: $\tau_p = \tau_w = 0.8$ and $N = 10000$.

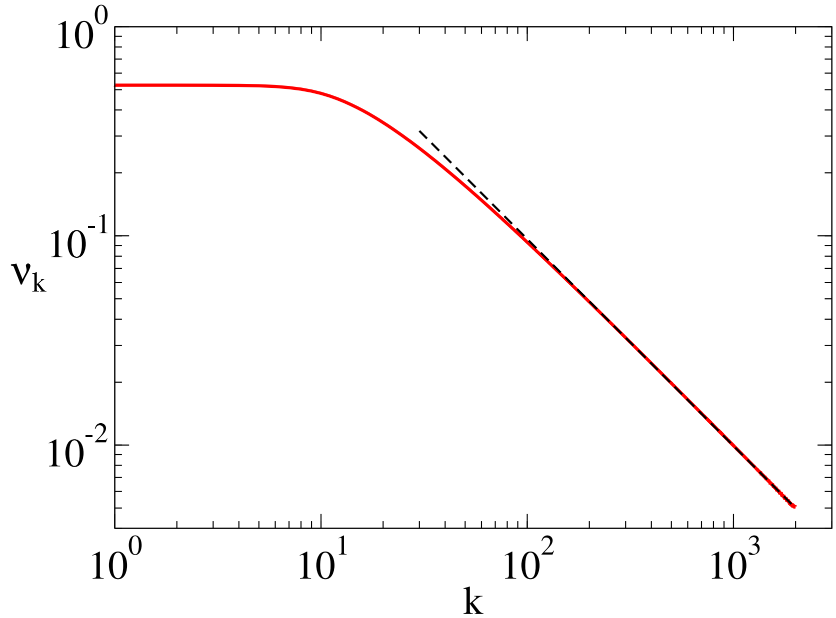

Figure 7: Numerical results for the stationary cumulative distribution of Eq. 21 with $n = 2000$ (red line). The exponent of the dashed black line is 0.989.