Homoclinic orbits and chaotic cycles in the Lucas model of endogenous growth

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Abstract

This paper shows that chaotic dynamics and sensitivity to initial conditions can characterize the standard Lucas's (1988) variant of the two-sector continuous time growth model. The most relevant implication, from a theoretical point of view, is probably the weakened role of the intertemporal equilibrium theory in providing indications about future economic conditions, given the initial state of the economy. Our "route to chaos" exploits the existence of a family of homoclinic orbits, connecting a saddle to itself in the effective dimension spanned by the dynamics of the model. In this situation, Shilnikov (1965; 1970) proved that if the saddle quantity is positive, irregular transient dynamics and high sensitivity to initial conditions result in specific systems. The characteristics of the chaotic dynamics in our specific context are highly interesting to point out: the growth rate of the economy exhibits smooth bursts of volatility, irregularly followed by periods of smaller-amplitude oscillations.

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1 Introduction

Over the last decades, the ability of the intertemporal equilibrium theory to provide indications about future economic conditions, given the initial state of the economy, has been questioned in many critical aspects. For instance, in the specific field of the two-sector, continuous-time, endogenous growth model, an influential literature has clearly established that in the presence of non-competitive elements (often an externality depending on the stock of either one of the capital goods), the "determinacy" of the perfect foresight equilibrium, namely the uniqueness of the collection of agents' actions, is not automatically warranted. On the contrary, multiple equilibrium paths can easily be shown to depart from given initial conditions of the state variables. The phenomenon can occur locally (cf, *inter al.*, Chamley, 1993; Benhabib and Farmer, 1994; 1996, Benhabib and Perli, 1994, Benhabib et al., 1994; Benhabib et al. 2000), or, as shown more recently, *in the large* (cf. *inter al.*, Mino, 2004; Mattana, Nishimura and Shigoka, 2006; Bella and Mattana, 2014) over a global range of initial growth rates; it implies that two economic systems, endowed with the same aggregate capital stock, may grow at different rates along different equilibrium paths, each of which depending upon the initial choice of a non-predetermined variable: the first period growth rate. However, despite the presence of indeterminacy, the growth rates of these economy are bound to converge asymptotically.

Consider now the case, in the same setting, the Ramsey-Euler system of equations arising from intertemporal maximization admits complex dynamics. In general terms, a dynamical system is said to be complex if it endogenously does not tend asymptotically to a fixed point, a limit cycle, or an explosion (Day, 1994), but rather to a complex geometric structure, part of the state-space spanned by the system, which "pulls" the dynamics into it (an attractor). When this happens, the dynamics suffers of "deep unpredicability" (Grandmont, 1985) and infinite precision in the measurement of initial conditions is needed if one is interested in predicting the motion. In this case, the ability of the intertemporal equilibrium theory to provide indications about future economic conditions is put at stake even more radically:

two economies, endowed with the same aggregate capital stock, but with different first period growth rates, will always experience different growth profiles, even asymptotically.

Quite surprisingly, the possible onset of complex dynamics in the continuous-time, two-sector endogenous growth model is, to the best of our knoweldge, completely overlooked (see, Boldrin et al., 2001). Yet, the mathematical theory behind continuous time complex behavior appears to readily apply to systems of dynamic laws emerging from the solution of the intertemporal maximization. Furthermore, the validity of the Transversality Condition can be easily checked along observable chaotic paths.

This paper is aimed to provide some first results in the field. Specifically, we show that a chaotic regime can emerge in the rather standard variant of the two-sector, continuous-time, endogenous growth setting: the Uzawa-Lucas endogenous-growth model (1988). The Ramsey-Euler equations arising from the competitive solution of the Lucas's model implies, in a conveniently reduced dimension, a strongly non-linear three-dimensional system which is already known to possess a rich spectrum of interesting dynamic phenomena that go, as the parameters of the model are tuned, from sub-critical and super-critical Hopf cycles (cf. Benhabib and Perli, 1994; Mattana and Venturi, 1999; Nishimura and Shigoka, 2006), to transcritical and fold bifurcations (cf. Barnett and Ghosh, 2014). The possibility of period-doubling bifurcations is also shown to exist in specific regions of the parameter space (cf. Barnett and Ghosh, 2014).

Our "route to complex dynamics" exploits the existence of a family of homoclinic orbits doubly asymptotic to a saddle-focus. The striking complexity of the dynamics near these homoclinic orbits has been discovered and investigated by Shilnikov (1965, 1970). The Shilnikov homoclinic bifurcation occurs already in vector fields of dimension three — the lowest possible phase-space dimension — and it is of codimension one, meaning that it is unfolded by a single parameter. Perhaps the most intriguing feature of a Shilnikov homoclinic bifurcation concerns the fact that it constitutes the simplest global phenomenon that can induce chaotic dynamics. What is interesting for us is that these chaotic dynamics is typically associated with a number of interesting, endogenously determined, phenomena, such as bursts, irregular cycles, spikes, and so on.

We can now explain the plan of this article. In the first section we present the economic model and derive the implied system of differential equations. In the second section, we mainly characterize the region of the parameters space that gives rise to a homoclinic orbit. The last section is devoted to show that chaotic behavior will be exhibited in a subset of the parameters space where the Shilnikov assumptions are satisfied and a homoclinic orbit occurs. In concluding the paper, we discuss the relevance and the implications of chaos to the endogenous growth theory.

2 The model

Consider the Problem

$$\underset{c(t),\ u(t)}{Max} \int_0^\infty \frac{c^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt \qquad (\mathcal{P})$$

$$\begin{aligned} \text{subject to} &: \\ \dot{k} &= Ak^{\beta}(uh)^{1-\beta} h_a^{\gamma} - c \\ \dot{h} &= \delta h(1-u) \\ k(0) &= k_0 > 0 \\ h(0) &= h_0 > 0 \end{aligned}$$

where c is per capita consumption, k is physical capital and h is human capital. Individuals have a fixed endowment of time, normalized to unity at each point in time, which is allocated to physical and human capital sectors according to the fractions u and $1-u$, respectively. Additionally, A and δ measure the technological levels respectively in the physical capital and human capital sectors; $\beta \in (0,1)$ is the share of physical capital in the physical capital sector; ρ is the time preference rate; σ is the inverse of the intertemporal elasticity of substitution. Finally $\gamma \in (0,1)$ is an externality parameter in the production of human capital. Therefore, the set of parameters $\theta \equiv (A, \beta, \delta, \gamma, \rho, \sigma)$ lives inside $\Theta = R_{++} \times (0,1) \times (0,1) \times R_{++}^3$.

Solution candidates for problem $\mathcal{P}$ can be obtained by means of the Prontryagin Maximum Principle. The discounted Hamiltonian is

$$\mathcal{H} = \frac{c^{1-\sigma}-1}{1-\sigma} + \mu_1 \left[Ak^{\beta}(uh)^{1-\beta} h_a^{\gamma} - c \right] + \mu_2 \delta h(1-u)$$

where μ_1 and μ_2 are the costate variables. The Ramsey-Euler first order conditions are

$$c^{-\sigma} = \mu_1 \tag{1.a}$$

$$\delta h \mu_2 = A(1-\beta)k^{\beta}u^{-\beta}h^{1-\beta}h_a^{\gamma}\mu_1 \tag{1.b}$$

$$\dot{\mu}_1 = \rho\mu_1 - A\beta k^{\beta-1}u^{1-\beta}h^{1-\beta}h_a^{\gamma}\mu_1 \tag{1.c}$$

$$\dot{\mu}_2 = \rho\mu_2 - A(1-\beta)ku^{1-\beta}h^{-\beta}h_a^{\gamma}\mu_1 - \mu_2\delta(1-u) \tag{1.d}$$

The Transversality Condition (TVC)

$$\lim_{t\to\infty}\left[e^{-\rho t}\left(\mu_1 k + \mu_2 h\right)\right] = 0 \tag{2}$$

must also be satisfied in order to rule out explosive paths. Since the (not maximized) Hamiltonian is jointly concave in both its state and control variables, by Mangasarian's condition the first order conditions are sufficient to solve problem $\mathcal{P}$.

Now, expressing the multipliers μ_1 and μ_2 in terms of their corresponding control variables c and u and considering that in equilibrium the solution path for h coincides with the given path h_a, we obtain the following (parametrized) four-dimensional system of first-order differential equations (cf. also Benhabib and Perli, 1994)

$$\begin{array}{rcl} \dot{c} & = & -\frac{\rho}{\sigma}c + \frac{A\beta}{\sigma}k^{\beta-1}u^{1-\beta}h^{1-\beta+\gamma}c \\ \dot{k} & = & Ak^{\beta}u^{1-\beta}h^{1-\beta+\gamma} - c \\ \dot{h} & = & \delta h\left(1-u\right) \\ \dot{u} & = & \frac{\delta(\beta-\gamma)}{\beta}u^2 + \frac{\delta(1-\beta+\gamma)}{\beta}u - \frac{c}{k} \end{array} \tag{$\mathcal{M}$}$$

Recall that c and u are jump variables, whereas k and h are the state variables of the model.

3 Long-run dynamics and stability properties

Let now g_z^* be the BGP growth rate of a generic variable z. Gathering information from system $\mathcal{M}$, we can show that

$$g_c^* = g_k^* = g_h^* = \delta(1-u^*) \equiv g^* \tag{3}$$

where g^* is the long-run growth rate of the economy. In Appendix, section 2, we show that the steady state satisfies the TVC in (2).

As it is customary in the literature, we exploit these properties of the BGP to reduce the effective dimension of system in $\mathcal{M}$. Let us therefore consider the stationarizing transformations $X = kh^{-\frac{1-\beta+\gamma}{1-\beta}}$, and $Q = \frac{c}{k}$. Then, the motion generated by the competitive solution of Lucas's model implies the following three-dimensional system of first-order differential equations

$$\begin{aligned} \dot{X} &= AX^{\beta}u^{1-\beta} + \tfrac{\delta(1-\beta+\gamma)}{\beta}(1-u)X - QX \\ \dot{u} &= \tfrac{\delta(\beta-\gamma)}{\beta}u^2 + \tfrac{\delta(1-\beta+\gamma)}{\beta}u - Qu \\ \dot{Q} &= -\tfrac{\rho}{\sigma}Q + A\tfrac{\beta-\sigma}{\sigma}X^{\beta-1}u^{1-\beta}Q + Q^2 \end{aligned} \qquad (\mathcal{S})$$

Let now $P^* \equiv (X^*, u^*, Q^*)$ be values of (X, u, Q) such that $\dot{X} = \dot{u} = \dot{Q} = 0$, with $u^* \in (0,1)$. Benhabib and Perli (1994) show that, when $\theta \equiv (A, \beta, \delta, \gamma, \rho, \sigma)$ lies in one of the following two subsets of Θ

$$\hat{\Theta} = \left\{\theta \in \Theta : \rho \in (0,\delta); \sigma > 1 - \tfrac{\rho(1-\beta)}{\delta(1-\beta+\gamma)}; \gamma \in (0,1)\right\} \qquad (4)$$

$$\check{\Theta} = \left\{\theta \in \Theta : \rho \in (\delta, \delta + \tfrac{\gamma}{1-\beta}), \sigma \in (0, 1 - \tfrac{\rho(1-\beta)}{\delta(1-\beta+\gamma)}); \gamma \in (\tfrac{(1-\beta)(\rho-\delta)}{\delta}, 1)\right\} \qquad (5)$$

the steady state is unique, and $u^* = 1 - \frac{(1-\beta)(\rho-\delta)}{\delta[\gamma-\sigma(1-\beta+\gamma)]} \in (0,1)$.

Let now $\mathbf{J}$ denote the Jacobian of system S, evaluated at $(X, u, Q) = (X^*, u^*, Q^*) \equiv P^*$. Let furthermore

$$\det(\lambda\mathbf{I}-\mathbf{J}) = \lambda^3 - \mathrm{Tr}(\mathbf{J})\lambda^2 + \mathrm{B}(\mathbf{J})\lambda - \mathrm{Det}(\mathbf{J})$$

be the characteristic polynomial of $\mathbf{J}$, where $\mathbf{I}$ is the identity matrix and $\mathrm{Tr}(\mathbf{J})$, $\mathrm{Det}(\mathbf{J})$ and $\mathrm{B}(\mathbf{J})$, are Trace, Determinant and Sum of Principal Minors of second order of $\mathbf{J}$. It is straightforward to show that (cf Appendix 2, section 2)

$$\mathrm{Det}(\mathbf{J}) = j_{11}^* Q^* \tfrac{\gamma-\sigma(1-\beta+\gamma)}{\sigma(\beta-1)}\delta u^*$$

$$\mathrm{Tr}(\mathbf{J}) = \tfrac{2\beta-\gamma}{\beta}\delta u^*$$

$$\mathrm{B}(\mathbf{J}) = j_{11}^* Q^* + \tfrac{(\beta-\gamma)(\delta u^*)^2}{\beta}$$

We are now ready to recall from Benhabib and Perli (1994) the following propedeutical result.

Lemma 1 (*Properties of the real parts of the eigenvalues at the steady state*). *Let $\theta \in \hat{\Theta}$. Then,* $\mathbf{J}$ *has one negative eigenvalue and two eigenvalues with positive real parts. Let, complementarily, $\theta \in \check{\Theta}$. Then: i) if $\gamma > \beta$,* $\mathbf{J}$ *has one positive eigenvalue and two eigenvalues with negative real parts; ii) if $0 < \gamma \leq \beta$, there exist two subsets, $\check{\Theta}^A$ and $\check{\Theta}^B$, such that if $\theta \in \check{\Theta}^A$,* $\mathbf{J}$ *has one positive eigenvalue and two eigenvalues with negative real parts, whereas if $\theta \in \check{\Theta}^B$,* $\mathbf{J}$ *has three eigenvalues with positive real parts.*

Proof. The result follows from a simple application of the Routh-Hurvitz criterion in a R^3 ambient space. Calculations are in Appendix, section A.1. ■

As it will become clear later, we are specifically interested in the special situation in which the steady state is a saddle-focus, either stable or unstable. This, in turn, implies that the $\check{\Theta}^B \subset \Theta$ sub-region of the parameter space is of no interest in our case, whereas in the remaning parameter space $\Theta - \check{\Theta}^B$ there are the conditions for two eigenvalues with imaginary parts to emerge. However, before trying to determine the regions of the parameter space where a saddle-focus equilibrium emerges as a rest point of the dynamics in $\mathcal{S}$, we notice that when $\theta \in \hat{\Theta}$, subjective discounting (the ρ parameter) has not lower bound and can be chosen small; furthermore, the parameter measuring the inverse of the intertemporal elasticity of substituion (the σ parameter) can be chosen larger than 1.

Therefore, motivated by parameters plausibility, we point out the following.

Remark 1 *In the search for the existence of a saddle-focus rest point in system $\mathcal{S}$, we restrict attention to the $\hat{\Theta}$ sub-region of Θ.*

Consider, therefore, the following result.

Proposition 1 (*Existence of a saddle-focus rest point in system $\mathcal{S}$*). *Let $\theta \in \hat{\Theta}$ and denote $\bar{\Theta} \equiv \left\{\theta \in \hat{\Theta} : \mathbf{J} \text{ has one real and two complex eigenvalues}\right\}$. Then, $\bar{\Theta} \neq \varnothing$. By Lemma 1, since $\bar{\Theta} \subset \hat{\Theta}$, the unique rest point of system $\mathcal{S}$ is a saddle-focus.*

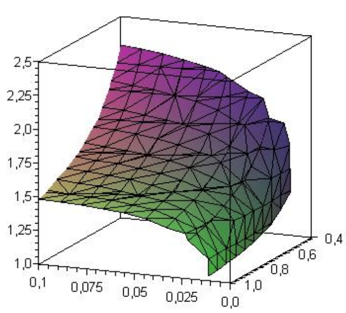
Proof. For the equilibrium to be a saddle-focus, we require a pair of eigenvalues with non-zero imaginary parts. Since parameters live inside $\bar{\Theta} \subset \hat{\Theta}$, by Lemma 1 the real parts of these eigenvalues must be negative. Solving the characteristic equation (4) provides the following three roots (Cardano formulas)

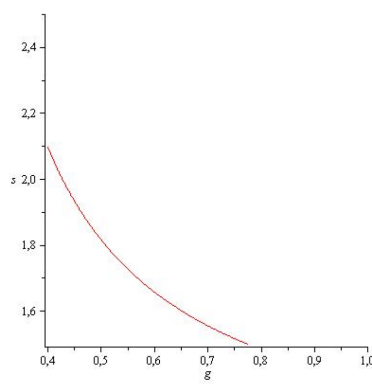
$$\begin{aligned}\lambda_1 &= \tfrac{Tr(\mathbf{J})}{3} + u + v \\ \lambda_{2,3} &= \tfrac{Tr(\mathbf{J})}{3} - \frac{u+v}{2} \pm \sqrt{3}\frac{u-v}{2} i\end{aligned}$$

where $u = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}}$, $v = \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}}$ and $\Delta = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$ is the discriminant. $p = \frac{3B(\mathbf{J}) - Tr(\mathbf{J})^2}{3}$ and $q = -\mathrm{Det}(\mathbf{J}) + 2\frac{\mathrm{Tr}(\mathbf{J})^3}{27} + \frac{\mathrm{Tr}(\mathbf{J})\mathrm{B}(\mathbf{J})}{3}$. $i = \sqrt{-1}$ is standard notation for imaginary unit. Therefore, for $\lambda_{2,3}$ to be complex conjugate, we need $\Delta > 0$. To complete the proof, consider the numerical example below. ■

Consider the following Example, implying that $\bar{\Theta} \neq \varnothing$.

Example 1 *Set* $(\beta, \rho) = (0.45, 0.01)$*. In figure* 1*, panel* (a)*, we depict the resulting surface of* (γ, δ, σ) *values such that* $\Delta = 0$*. Combinations of the* (γ, δ, σ) *parameters above the surface guarantee that* $\Delta > 0$*. Notice that, for a given technological level* δ *in the educational sector, there is a plausibility trade-off between the two remaining parameters* (γ, σ) *: that is to say, in order not to have a too high* σ*, a substantial degree of externality has to be accepted*[1]





Let us now set $\delta = 0.04$*; then the combinations of the remaining* (γ, σ) *parameters such that* $\Delta > 0$ *are above the two-dimensional curve depicted in*

[1]Extensive numerical simulations confirm this trade-off between γ, σ exists also for other combinations of the (β, ρ) parameters.

panel (b) of figure 1. To minimize the value of the externality factor, we choose $\sigma = 2.2$. Therefore, for $\gamma \in \bar{\Theta}$, we must choose $\gamma \gtrsim 0.3764796174$. Let us take $\gamma = 0.4$. Then, **J** *has one negative real eigenvalue and two complex conjugate eigenvalues with positive real parts. Specifically, we obtain*

$$\begin{aligned} \lambda_1 &\simeq -0.07343536069 \\ \lambda_{2,3} &\simeq 0.05351583945 \pm 0.006549619601i \end{aligned}$$

This economy (also taking $A = 0.2$) has $P^* \equiv (X^*, u^*, Q^*) \simeq (2.453, 0.756, 0.088)$ and $g^* = \delta(1 - u^*) \simeq 0.01$.

4 Shilnikov chaos

4.1 Preliminaries

Shilnikov (1965) establishes the following (see, Kutznetsov, 2000)

Theorem 2 (*Shilnikov,* 1965). *Consider the following dynamical system*

$$\frac{dY}{dt} = f(Y, \alpha), \quad Y \in R^3, \quad \alpha \in R^1$$

with f sufficiently smooth. Assume f has a hyperbolic saddle-focus equilibrium point $Y_0 = 0$ at $\alpha = 0$ implying that eigenvalues of the Jacobian $A = Df$ are of the form η and $\tau \pm \omega i$ where η, τ and ω are real constants with $\eta\tau < 0$. Assume that the following conditions also hold:
$(H.1)$ the saddle quantity $|\eta| - |\tau| > 0$;
$(H.2)$ there exists a homoclinic orbit Γ_0 based at Y_0.
Then:
(1) *the Shilnikov map, defined in the neighborhood of the homoclinic orbit of the system, possesses an infinite number of Smale horseshoes in its discrete dynamics;*
(2) *for any sufficiently small C^1-perturbation g of f, the perturbed system has at least a finite number of Smale horseshoes in the discrete dynamics of the Shilnikov map, defined in the neighborhood of the homoclinic orbit;*
(3) *both the original and the perturbed system exhibit horseshoe chaos.*

The application of Theorem 1 to specific contexts requires that conditions $(H.1)$ and $(H.2)$ are satisfied. For the case of the Lucas's model, we already know that, by Lemma 1, if $\theta \in \bar{\Theta}$, system S is a candidate for the presence of Shilnikov chaos since its (unique) rest point is a saddle-focus. Here below we shall ascertain whether the other conditions in Theorem 1 can be satisfied in specific regions of the relevant parameter space.

4.2 Sign of the saddle quantity

By Proposition 1, we know that if $\theta \in \bar{\Theta}$, $\mathbf{J}$ has one negative real eigenvalues and two complex eigenvalues with positive real part. For notational convenience, assume the eigenvalues are of the form $\lambda_1 = -\bar{\eta}$ and $\lambda_{2,3} = \bar{\tau} \pm \bar{\omega} i$ where $\bar{\eta}$, $\bar{\tau}$ and $\bar{\omega}$ are real constants. As clearly established by Theorem 1, the absolute value of the real eigenvalue has to be larger than the absolute value of the real part of the complex conjugate eigenvalues. In our case, we are in the position to show the following.

Lemma 2 *Consider the characteristic equation in* (4). *Let* $\theta \in \bar{\Theta}$. *Then, there are regions of the parameter space such that*

$$\Omega \equiv \left\{ \theta \in \bar{\Theta} : |\bar{\eta}| > |\bar{\tau}| \right\} \neq \varnothing$$

Proof. The result is already implied by the simulations in Example 1. ■

4.3 Existence of a family of homoclinic orbits

To prove the existence of a family of homoclinic orbits doubly asymptotic to the fixed point P^* we use the method of the undertermined coefficients (cf., *inter al.*, Shang et Han, 2005). We provide here a sketch of the steps required. Firsly, we need to put vector field $\mathcal{S}$ in normal form. As shown in Appendix, section A.2, we obtain the following (truncated) system of differential equations

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \begin{bmatrix} -\bar{\eta} & 0 & 0 \\ 0 & \bar{\tau} & -\bar{\omega} \\ 0 & \bar{\omega} & \bar{\tau} \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \\ + \begin{pmatrix} F_{1a} w_1 w_2 + F_{1b} w_1 w_3 + F_{1c} w_2 w_3 + F_{1d} w_1^2 + F_{1e} w_2^2 + F_{1f} \; w_3^2 \\ F_{2a} w_1 w_2 + F_{2b} w_1 w_3 + F_{2c} w_2 w_3 + F_{2d} w_1^2 + F_{2e} w_2^2 + F_{2f} \; w_3^2 \\ F_{3a} w_1 w_2 + F_{3b} w_1 w_3 + F_{3c} w_2 w_3 + F_{3d} w_1^2 + F_{3e} w_2^2 + F_{3f} \; w_3^2 \end{pmatrix} \quad (6)$$

where $w_1(\phi_1)$, $w_2(\phi_2)$ and $w_3(\phi_3)$ are transformations of the original X, u and Q variables, in turn depending on three ϕ_i, $i = 1, 2, 3$ free constants which can be conveniently chosen to mild/increase fluctuations in the eigenspace. The various $\bar{F}_i$ contain second-order nonlinear terms. For convenience of representation, let us denote the vector field in (6) in compact terms as

$$\dot{\mathbf{w}} = \mathbf{J}_N \mathbf{w} + \bar{F}(\mathbf{w})$$

Once the normal form is obtained, the method assumes that the analytical expressions of both the one-dimensional unstable manifold associated with $\lambda_1 = -\bar{\eta}$ and of the two-dimensional stable manifold associated with the complex eigenvalues $\lambda_{2,3} = \bar{\tau} \pm \bar{\omega} i$ can be polynomially approximated. As shown in Appendix, section 4, this implies that the transformed variables w_1, w_2 and w_3 evolve according to the following expressions

$$\begin{aligned} w_1 &= \left\{ \begin{array}{ll} \xi e^{-\bar{\eta} t} + \sum_{k=2}^{\infty} a_k(\xi) e^{-k\bar{\eta} t} & t \geq 0 \\ d_1 w_{11}(\psi,\zeta) + d_2 w_{12}(\psi,\zeta) & t \leq 0 \end{array} \right\} \\ w_2 &= \left\{ \begin{array}{ll} \sum_{k=2}^{\infty} b_k(\xi) e^{-k\bar{\eta} t} & t \geq 0 \\ d_1 w_{21}(\psi,\zeta) + d_2 w_{22}(\psi,\zeta) & t \leq 0 \end{array} \right\} \\ w_3 &= \left\{ \begin{array}{ll} \sum_{k=2}^{\infty} c_k(\xi) e^{-k\bar{\eta} t} & t \geq 0 \\ d_1 w_{31}(\psi,\zeta) + d_2 w_{32}(\psi,\zeta) & t \leq 0 \end{array} \right\} \end{aligned} \quad (7)$$

The $d_1 w_{i1}(\psi,\zeta) + d_2 w_{i2}(\psi,\zeta)$ elements, $i = 1,2,3$, are the linear superposition of the real and complex solutions of system (6) where d_1 and d_2 alternatively take the values 0 or 1. (ξ,ψ,ζ) is a triple of further free constants. In Appendix, section A.3, all relevant coefficients of the polynomials in (7) are computed as functions of the primitive parameters of the model.

Finally, the existence of the homoclinic orbit implies that, at $t = 0$, the polynomials analytically approximating the trajectory of a variable for $t \geq 0$ and $t \leq 0$ coincide for non-trivial value of the three arbitrary constants (ξ,ψ,ζ) (cf. Shang and Han, 2005). In our case, we obtain, at $t = 0$, the following relationship (cf. Appendix, section A.3)

$$\xi = \frac{F_{1c}\psi\zeta + F_{1e}\psi^2 + F_{1f}\zeta^2}{(4\bar{\tau} - \bar{\eta})^2 + \bar{\omega}^2}(2\bar{\tau} - \bar{\eta}) - \frac{F_{1d}}{\bar{\eta}} \frac{(4\bar{\eta}^2 + 4\bar{\eta}\bar{\tau} + \bar{\tau}^2 + \bar{\omega}^2)}{(F_{2d} + F_{3d})(\bar{\tau} - \bar{\omega} + 2\bar{\eta})}(\psi + \zeta) \quad (8)$$

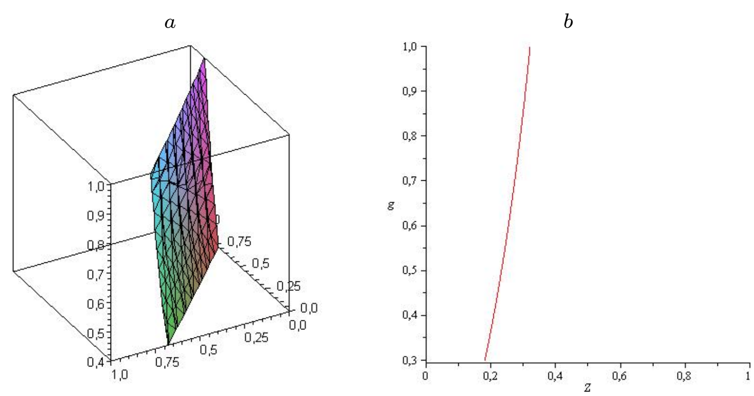
Let now γ be the bifurcation parameter. Then, we can prove the following crucial statement.

Lemma 3 (*Existence of a family of doubly asymptotic homoclinic orbits*). *Set the free constants* (ξ,ψ,ζ) *and the scaling factors* $(\varphi_1,\varphi_2,\varphi_3)$. *Denote the set* $\Upsilon \equiv \left\{\gamma \in \theta \in \Omega : \dot{\mathbf{w}} = \mathbf{J}_N \mathbf{w} + \bar{F}(\mathbf{w}) \text{ has a homoclinic orbit}\right\}$. *Then,* $\Upsilon \neq \varnothing$. *If* (6) *has a homoclinic orbit, also system* $\mathcal{S}$ *has a homoclinic orbit.*

Proof. As detailed in Appendix, section A.3, we apply the method of the undetermined coefficients to the system of dynamic laws implied by the Lucas's model. The expression in (8) is obtained. In Example 2 we show that $\Upsilon \neq \varnothing$. By topological equivalence, the results of homoclinic bifurcation in system (8) also apply to system $\mathcal{S}$. ■

Consider the following example.

Example 2 *Set* $(A, \beta, \delta, \rho, \sigma) \simeq (0.2, 0.45, 0.04, 0.01, 2.2)$ *as in Example* 1*. Set, furthermore* $(\varphi_1, \varphi_2, \varphi_3) = \left(\frac{1}{500}, \frac{1}{50}, \frac{1}{20}\right)$*. Assume, finally* $\xi = \psi$*. Then equation* (9) *gives rise to the following rather regular surface in the* (γ, ψ, ζ) *coordinates* (*figure* 2*, panel a*)



Set now $\psi = \frac{1}{2}$*. In Figure* 2*, panel b, we portrait the combinations between the* ζ *constant and the externality factor, such that equation* (8) *is satisfied. Recall also, from Example* 1*, that with the parameters in this example, for the steady state to be a saddle-focus, we need* $\gamma \gtrsim 0.3764796174$*. This value corresponds to* $\zeta \cong 0.2$*, in figure* 5*, panel b. Therefore, setting* $\zeta = \bar{\zeta} \gtrsim 0.2$*, there exists a value of the bifurcation parameter* γ *such that system* (6) *has a homoclinic orbit doubly asymptotic to the origin.*

4.4 Chaotic motion

We are now ready to obtain our main results. Consider the following statement.

Proposition 3 (*Existence of spiral/Shilnikov chaos*)*. Recall Lemmas* 2 *and* 3 *and assume* $\theta \in \Upsilon$*. Then, system* $\mathcal{S}$ *admits chaotic solution trajectories.*

If we appropriately choose the scaling constants $(\varphi_1, \varphi_2, \varphi_3)$*, the TVC is satisfied along these paths.*

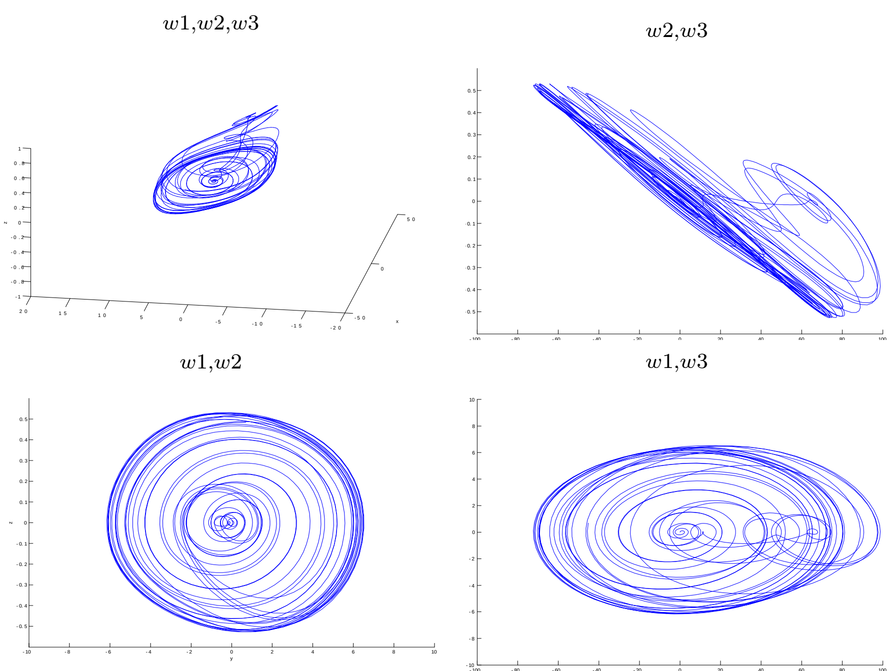
Proof. Lemmas 2 and 3 show that there are regions of the parameter space such that the dynamics implied by the standard Lucas's model of endogenous growth satisfies all necessary conditions of Theorem 1. In Appendix, section A.4, we discuss conditions such that the TVC is satisfied. ■

A major theoretical result also states that provided that the saddle quantity remains stricly positive, then the following result applies.

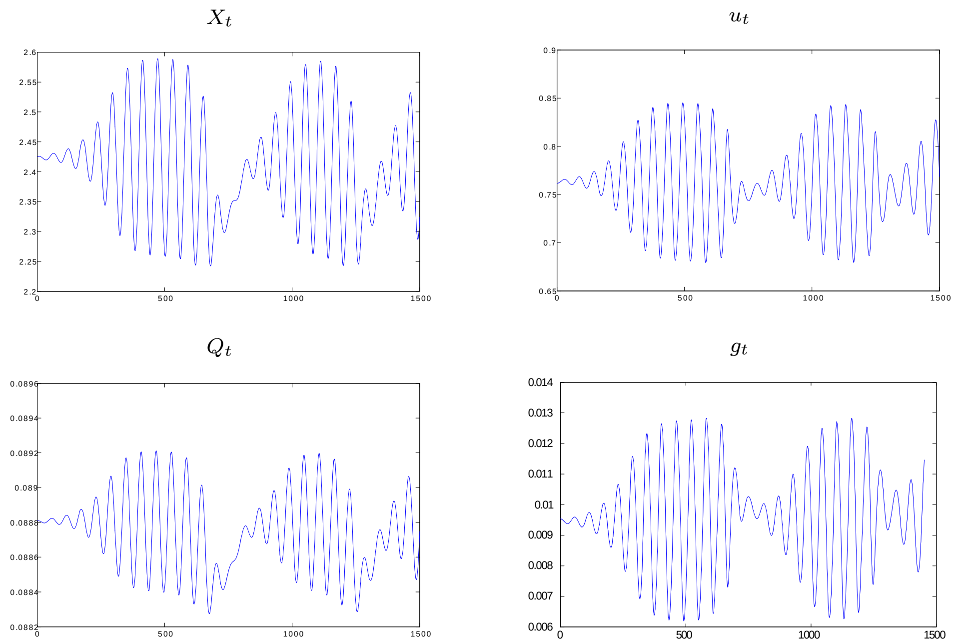
Remark 2 *The chaotic dynamics extends to a neighborhood around the (left and right) parameter value of the Shilnikov homoclinic bifurcation.*

We discuss here below an example of chaotic trajectories arising from system $\mathcal{S}$.

Example 3 *Set* $(A, \beta, \delta, \rho, \sigma) \simeq (0.2, 0.45, 0.04, 0.01, 2.2)$ *as in Example* 2*. Set also* $(\varphi_1, \varphi_2, \varphi_3) = \left(\frac{1}{500}, \frac{1}{50}, \frac{1}{20}\right)$ *and* $(\xi, \psi, \zeta) \simeq \left(\frac{1}{2}, \frac{1}{2}, \frac{21}{10}\right)$*. From Example* 2*, we know that if* $\gamma \simeq 0.45$*, this economy has* $P^* \equiv (X^*, u^*, Q^*) \simeq (2.425, 0.764, 0.089)$*,* $g^* \simeq 0.01$*, and presents spiral/Shilnikov chaos. In Figure* 3*, we plot the attractor in the transformed coordinates* (w_1, w_2, w_3) *and its projections. Initial conditions are set as* $w_1(0) = w_2(0) = w_3(0) = 0.01$*.*



We also present in figure 4, the chaotic time paths of the X_t, u_t and Q_t variables. The time path of growth rate of this economy is also depicted.



To estimate the volume of the parameter space implying chaotic dynamics, and also to have information on the general characteristics of these trajectories, we run several numerical simulations. The main results are the following:

i the onset of the homoclinic orbit is a very delicate situation, occurring in a small region of the parameter space. Typically what is required is a high value of the externality factor;

ii the existence of chaotic trajectories is relatively not sensitive to the choice of the initial conditions. In other words we can choose initial conditions relatively far away from the steady state;

iii any of the simulations has generated spikes; on the contrary, increases in volatility along the spiral attractor are relatively smooth and have never given rise to explosive bursts.

4.5 Conclusions

This paper applies the Shilnikov homoclinic theorem to the celebrated Lucas's (1988) endogenous growth model. To the best of our knowledge, it is the first time that complex dynamics is shown to arise in the specific setting of the continuous-time two sector endogenous growth model. In order to establish the presence of chaotic motion on the basis of Shilnikov work, we have first put the Ramsey-Euler equations implied by the model in normal form, and then have applied the undetermined coefficients method (cf. *inter al.* Shang and Han, 2005) to show that the dynamics support the existence of a transversal intersection between the stable manifold with the unstable manifold of a hyperbolic point (i.e. a homoclinic orbit, connecting a saddle to itself). Shilnikov (1965; 1970) has shown that, in this situation, if the saddle quantity is positive, infinitely many saddle limit cycles coexist at the bifurcation point; each of these saddle limit cycles had stable and unstable manifolds, which results in high sensitivity to initial conditions and irregular transient dynamics. The implications for the debate regarding the ability of intertemporal equilibrium theory to provide insights into future conditions, given the fundamentals of the economy are discussed.

Several numerical simulations have also allowed to infer some general characteristics of trajectories evolving in the chaotic attractor in out specific case. First of all, we find that the onset of the homoclinic orbit is a very delicate situation, occurring in very small region of the parameter space. Moreover, the attractor seems able to capture trajectories initialized relatively far away from the steady state. Last, but not least, we have not been able to generate spikes: on the contrary, volatility increases quite smoothly along the spiral attractor and never give rise to explosive bursts.

5 APPENDIX

A.1 The Jacobian

Let **J** be the Jacobian of the right hand side of system (5) evaluated along the BGP. The single elements of **J** are as follows

$j_{11}^{*}=\frac{\partial \dot{X}}{\partial X}|_{BGP}=\left[\frac{\dot{X}}{X}-A\left(\beta-1\right)X^{\beta-1}u^{1-\beta}\right]|_{BGP}=-A\left(1-\beta\right)\left(X^{*}\right)^{\beta-1}\left(u^{*}\right)^{1-\beta}$

$j_{12}^{*}=\frac{\partial \dot{X}}{\partial u}|_{BGP}=\left[-A\left(1-\beta\right)X^{\beta}u^{-\beta}+\delta\frac{1-\beta+\gamma}{1-\beta}\right]|_{BGP}=-\frac{X^{*}}{u^{*}}\left(j_{11}^{*}-\delta\frac{1-\beta+\gamma}{1-\beta}u^{*}\right)$

$j_{13}^{*}=\frac{\partial \dot{X}}{\partial Q}|_{BGP}=\left[-X\right]|_{BGP}=-X^{*}$

$j_{21}^{*}=\frac{\partial\dot{u}}{\partial X}|_{BGP}=0$
$j_{22}^{*}=\frac{\partial\dot{u}}{\partial u}|_{BGP}=\left[\frac{\dot{u}}{u}+\delta\frac{\beta-\gamma}{\beta}u\right]|_{BGP}=\delta\frac{\beta-\gamma}{\beta}u^{*}$
$j_{23}^{*}=\frac{\partial\dot{u}}{\partial Q}|_{BGP}=\left[-u\right]|_{BGP}=-u^{*}$
$j_{31}^{*}=\frac{\partial\dot{Q}}{\partial X}|_{BGP}=\left[-A\left(1-\beta\right)\frac{\beta-\sigma}{\sigma}X^{\beta-2}u^{1-\beta}Q\right]|_{BGP}=j_{11}^{*}\frac{\beta-\sigma}{\sigma}\frac{Q^{*}}{X^{*}}$
$j_{32}^{*}=\frac{\partial\dot{Q}}{\partial u}|_{BGP}=\left[A\left(1-\beta\right)\frac{\beta-\sigma}{\sigma}X^{\beta-1}u^{-\beta}Q\right]|_{BGP}=-j_{11}^{*}\frac{\beta-\sigma}{\sigma}\frac{Q^{*}}{u^{*}}$
$j_{33}^{*}=\frac{\partial\dot{Q}}{\partial Q}|_{BGP}=\left[\frac{\dot{Q}}{Q}+Q\right]|_{BGP}=Q^{*}$
Therefore, we have

$$\mathbf{J}=\begin{bmatrix} j_{11}^{*} & -\frac{X^{*}}{u^{*}}\left(j_{11}^{*}-\delta\frac{1-\beta+\gamma}{1-\beta}u^{*}\right) & -X^{*} \\ 0 & \delta\frac{\beta-\gamma}{\beta}u^{*} & -u^{*} \\ \frac{\beta-\sigma}{\sigma}\frac{j_{11}^{*}Q^{*}}{X^{*}} & -\frac{\beta-\sigma}{\sigma}\frac{j_{11}^{*}Q^{*}}{u^{*}} & Q^{*} \end{bmatrix}$$

The eigenvalues of $\mathbf{J}$ are the solutions of its characteristic equation

$$\det\left(\lambda\mathbf{I}-\mathbf{J}\right)=\lambda^{3}-\mathrm{Tr}(\mathbf{J})\lambda^{2}+\mathrm{B}(\mathbf{J})\lambda-\mathrm{Det}(\mathbf{J})$$

where $\mathbf{I}$ is the identity matrix. $\mathrm{Tr}(\mathbf{J})$ and $\mathrm{Det}(\mathbf{J})$ are Trace and Determinant of $\mathbf{J}$, respectively. $\mathrm{B}(\mathbf{J})$ is the sum of principal minors of order 2. We obtain

$$\mathrm{Tr}(\mathbf{J})=\frac{2\beta-\gamma}{\beta}\delta u^{*}$$

$$\mathrm{Det}(\mathbf{J})=j_{11}^{*}\delta u^{*}Q^{*}\frac{\gamma-\sigma(1-\beta+\gamma)}{\sigma(\beta-1)}$$

$$\mathrm{B}(\mathbf{J})=j_{11}^{*}Q^{*}+\frac{\beta-\gamma}{\beta}(\delta u^{*})^{2}$$

A.2 System $\mathcal{S}$ in normal form

To put system $\mathcal{S}$ in normal form, a number of steps are required.
a) *Translation to the origin and second-order Taylor expansion.*
Consider first the following variables trasformation $\tilde{X}\equiv X-X^{*}$, $\tilde{u}\equiv u-u^{*}$, $\tilde{Q}=Q-Q^{*}$. Taylor expanding, system $\mathcal{S}$ becomes

$$\begin{pmatrix} \dot{\tilde{X}} \\ \dot{\tilde{u}} \\ \dot{\tilde{Q}} \end{pmatrix}=\mathbf{J}\begin{pmatrix} \tilde{X} \\ \tilde{u} \\ \tilde{Q} \end{pmatrix}+\begin{pmatrix} \tilde{f}_{1}(\tilde{X},\tilde{u},\tilde{Q}) \\ \tilde{f}_{2}(\tilde{X},\tilde{u},\tilde{Q}) \\ \tilde{f}_{3}(\tilde{X},\tilde{u},\tilde{Q}) \end{pmatrix}$$

where the various $\tilde{f}_{i}$ are non linear functions of the new variables $\tilde{X},\tilde{u},\tilde{Q}$.

b) *Transformation in eigen-coordinates.*

By Proposition 1, we know that if $\theta \in \bar{\Theta}$ $\mathbf{J}$ has one negative real eigenvalue $(\lambda_1 = -\bar{\eta})$ and two complex conjugate eigenvalues with positive real part $(\lambda_{2,3} = \bar{\tau} \pm \bar{\omega} i)$ where $\bar{\eta}$, $\bar{\tau}$ and $\bar{\omega}$ are positive real constants. To obtain the eigenbasis, therefore, we need to solve the following system

$$\begin{aligned} \mathbf{Ju} &= \bar{\tau}\mathbf{u} - \bar{\omega}\mathbf{v} \\ \mathbf{Jv} &= \bar{\omega}\mathbf{u} + \bar{\tau}\mathbf{v} \\ \mathbf{Jz} &= \bar{\eta}\mathbf{z} \end{aligned}$$

where $\mathbf{u} = [u_1, u_2, u_3]^{\mathrm{T}}$, $\mathbf{v} = [v_1, v_2, v_3]^{\mathrm{T}}$ and $\mathbf{z} = [z_1, z_2, z_3]^{\mathrm{T}}$ are (3×1) vectors. Possible candidates for the eigenvectors are

$$\mathbf{u} = \begin{pmatrix} \frac{\left(j_{11}^* - \bar{\tau} + v_3 j_{13}^*\right)}{\bar{\omega}} \varphi_1 \\ \frac{v_3 j_{23}^*}{\bar{\omega}} \varphi_1 \\ \frac{j_{31}^* + (j_{33}^* - \bar{\tau}) v_3}{\bar{\omega}} \varphi_1 \end{pmatrix}; \ \mathbf{v} = \begin{pmatrix} \varphi_2 \\ 0 \\ v_3 \varphi_2 \end{pmatrix}; \ \mathbf{z} = \begin{pmatrix} [j_{23}^* j_{12}^* \text{-} (j_{22}^* \text{-}) j_{13}^*] \varphi_3 \\ -(j_{11}^* - \bar{\eta}) j_{23}^* \varphi_3 \\ (j_{11}^* - \bar{\eta})(j_{22}^* - \bar{\eta}) \varphi_3 \end{pmatrix} \tag{A.1}$$

where $v_3 = [(\bar{\eta} - j_{11}^*)/j_{13}^*]$ and the three free constants φ_i $i = 1, 2, 3$ can be conveniently chosen[2]. Therefore, using $\mathbf{T} = [\mathbf{u}, \mathbf{v}, \mathbf{z}]$ to operate the coordinate change

$$\begin{pmatrix} \tilde{X} \\ \tilde{u} \\ \tilde{Q} \end{pmatrix} = \mathbf{T} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

we obtain

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \mathbf{J}_N \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} \bar{F}_1(w_1, w_2, w_3) \\ \bar{F}_2(w_1, w_2, w_3) \\ \bar{F}_3(w_1, w_2, w_3) \end{pmatrix} \tag{A.2}$$

where

$$\mathbf{J}_N \equiv \begin{bmatrix} -\bar{\eta} & 0 & 0 \\ 0 & \bar{\tau} & -\bar{\omega} \\ 0 & \bar{\omega} & \bar{\tau} \end{bmatrix}$$

is the linearization matrix in normal form when $\theta \in \bar{\Theta}$ where

$$\bar{F}_1 = F_{1a} w_1 w_2 + F_{1b} w_1 w_3 + F_{1c} w_2 w_3 + F_{1d} w_1^2 + F_{1e} w_2^2 + F_{1f}\ w_3^2 \tag{A.3a}$$
$$\bar{F}_2 = F_{2a} w_1 w_2 + F_{2b} w_1 w_3 + F_{2c} w_2 w_3 + F_{2d} w_1^2 + F_{2e} w_2^2 + F_{2f}\ w_3^2 \tag{A.3b}$$
$$\bar{F}_3 = F_{3a} w_1 w_2 + F_{3b} w_1 w_3 + F_{3c} w_2 w_3 + F_{3d} w_1^2 + F_{3e} w_2^2 + F_{3f}\ w_3^2 \tag{A.3c}$$

[2] Recall that these three constant have a crucial role in making coherent with economic dynamics the chaotic fluctuation of the X, u and Q variables.

is the (truncated) expression for the non-linear part. The $F_{i,j}(\theta, \varphi_i)$, $i = 1,2,3$ and $j = a, b...f$ coefficients are intricate combinations of the original parameters of the model, and also depend on the values of the three constants φ_i $i = 1,2,3$. Als coefficients have been computed, but not reported for the sake of a simple representation. They remain available upon request.

A.3 Proof of Lemma 3 (existence of a homoclinic orbit)

To verify whether system (A.2) admits a homoclinic bifurcation, we apply the method of the undetermined coefficients (cf., for a clear exposition, Shang and Han, 2005). The method assumes that the analytical expressions of both the one-dimensional unstable manifold associated with $\lambda_1 = -\bar{\eta}$ and of the two-dimensional stable manifold associated with the complex conjugate eigenvalues $\lambda_{2,3} = \bar{\tau} \pm \bar{\omega} i$ can be polynomially approximated. The method proceeds as follows. Let the equations

$$w_1 = a_0 + \sum_{k=1}^{\infty} a_k e^{k\lambda t}; \quad w_2 = b_0 + \sum_{k=1}^{\infty} b_k e^{k\lambda t}; \quad w_3 = c_0 + \sum_{k=1}^{\infty} c_k e^{k\lambda t} \tag{A.4}$$

represent the analytic expression of the manifold associated with the generic eigenvalue λ, where a_m, b_m, c_m are unknown coefficients. Time derivatives of (A.4) yeld

$$\dot{w}_1 = \sum_{k=1}^{\infty} a_k k \lambda e^{k\lambda t}; \quad \dot{w}_2 = \sum_{k=1}^{\infty} b_k k \lambda e^{k\lambda t}; \quad \dot{w}_3 = \sum_{k=1}^{\infty} c_k k \lambda e^{k\lambda t} \tag{A.5}$$

Equating (A.2) with (A.5), and taking into account (A.4), we derive the following expression

$$\begin{pmatrix} \sum_{k=1}^{\infty} a_k k\lambda e^{k\lambda t} \\ \sum_{k=1}^{\infty} b_k k\lambda e^{k\lambda t} \\ \sum_{k=1}^{\infty} c_k k\lambda e^{k\lambda t} \end{pmatrix} = \mathbf{J}_N \begin{pmatrix} a_0 + \sum_{k=1}^{\infty} a_k e^{k\lambda t} \\ b_0 + \sum_{k=1}^{\infty} b_k e^{k\lambda t} \\ c_0 + \sum_{k=1}^{\infty} c_k e^{k\lambda t} \end{pmatrix} + \tag{A.6}$$

$$+ \begin{matrix} F_{1a}\left(a_0 + \sum_{k=1}^{\infty} a_k e^{k\lambda t}\right)\left(b_0 + \sum_{k=1}^{\infty} b_k e^{k\lambda t}\right) \\ F_{2a}\left(a_0 + \sum_{k=1}^{\infty} a_k e^{k\lambda t}\right)\left(b_0 + \sum_{k=1}^{\infty} b_k e^{k\lambda t}\right) \\ F_{3a}\left(a_0 + \sum_{k=1}^{\infty} a_k e^{k\lambda t}\right)\left(b_0 + \sum_{k=1}^{\infty} b_k e^{k\lambda t}\right) \end{matrix} +$$

$$+ \begin{matrix} F_{1b}\left(a_0 + \sum_{k=1}^{\infty} a_k e^{k\lambda t}\right)\left(c_0 + \sum_{k=1}^{\infty} c_k e^{k\lambda t}\right) \\ F_{2b}\left(a_0 + \sum_{k=1}^{\infty} a_k e^{k\lambda t}\right)\left(c_0 + \sum_{k=1}^{\infty} c_k e^{k\lambda t}\right) \\ F_{3b}\left(a_0 + \sum_{k=1}^{\infty} a_k e^{k\lambda t}\right)\left(c_0 + \sum_{k=1}^{\infty} c_k e^{k\lambda t}\right) \end{matrix} +$$

$$+ \begin{matrix} F_{1c}\left(b_0 + \sum_{k=1}^{\infty} b_k e^{\bar{\eta} k t}\right)\left(c_0 + \sum_{k=1}^{\infty} c_k e^{k\lambda t}\right) \\ F_{2c}\left(b_0 + \sum_{k=1}^{\infty} b_k e^{\bar{\eta} k t}\right)\left(c_0 + \sum_{k=1}^{\infty} c_k e^{k\lambda t}\right) \\ F_{3c}\left(b_0 + \sum_{k=1}^{\infty} b_k e^{\bar{\eta} k t}\right)\left(c_0 + \sum_{k=1}^{\infty} c_k e^{k\lambda t}\right) \end{matrix} +$$

$$+ \begin{matrix} F_{1d}\left(a_0 + \sum_{k=1}^{\infty} a_k e^{k\lambda t}\right)^2 + F_{1e}\left(b_0 + \sum_{k=1}^{\infty} b_k e^{k\lambda t}\right)^2 + F_{1f}\left(c_0 + \sum_{k=1}^{\infty} c_k e^{k\lambda t}\right)^2 \\ F_{2d}\left(a_0 + \sum_{k=1}^{\infty} a_k e^{k\lambda t}\right)^2 + F_{2e}\left(b_0 + \sum_{k=1}^{\infty} b_k e^{k\lambda t}\right)^2 + F_{2f}\left(c_0 + \sum_{k=1}^{\infty} c_k e^{k\lambda t}\right)^2 \\ F_{3d}\left(a_0 + \sum_{k=1}^{\infty} a_k e^{k\lambda t}\right)^2 + F_{3e}\left(b_0 + \sum_{k=1}^{\infty} b_k e^{k\lambda t}\right)^2 + F_{3f}\left(c_0 + \sum_{k=1}^{\infty} c_k e^{k\lambda t}\right)^2 \end{matrix}$$

The system is initialized at the origin. Namely

$$\begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{A.7}$$

The method allows to derive the unknown coefficients a_m, b_m, c_m of the analytical expression for the homoclinic orbit by comparing elements with same exponential at the right and left hand sides of (A.6), for k sequentially set to $1, 2 ... p$ where p is the required degree of approximation. Once the analytical expression is obtained, doubly convergence to $\mathbf{0}$ in R^3, that is to say for $t \to \pm\infty$ has to be searched by tuning the bifurcation parameter.

Let us first find the analytical expression for $t > 0$. In this case, the solution trajectory converges to the origin along the stable one-dimensional manifold spanned by the real eigenvalue. The method requires the following steps.

a) Set $k = 1$. From the expression (A.6), the vector $(a_1, b_1, c_1)'$ can be retrieved comparing elements with same exponential $e^{\lambda t}$. We have

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \lambda e^{\lambda t} = \mathbf{J}_N \begin{pmatrix} a_0 + a_1 e^{\lambda t} \\ b_0 + b_1 e^{\lambda t} \\ c_0 + c_1 e^{\lambda t} \end{pmatrix}$$

Taking into account (A.7), we also have

$$[\lambda \mathbf{I} - \mathbf{J}_N] \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{A.8}$$

where $\mathbf{I}$ is the identity matrix. Evaluating (A.8) at $\lambda = -\bar{\eta}$ and solving for the unknown constants a_1,b_1,c_1, we obtain the following result

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} \xi \\ 0 \\ 0 \end{pmatrix}$$

where ξ is an arbitrary constant.

b) Set $k = 2$. From (A.6), keeping only elements with $e^{2\lambda t}$, we obtain

$$[2\lambda \mathbf{I} - \mathbf{J}_N] \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} F_{1d}\xi^2 \\ F_{2d}\xi^2 \\ F_{3d}\xi^2 \end{pmatrix} \tag{A.9}$$

Again evaluating (A.9) at $\lambda = -\bar{\eta}$ and solving for the unknown coefficients a_2, b_2 and c_2, we have

$$\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \xi^2 \begin{bmatrix} -\frac{1}{2\lambda+\eta} & 0 & 0 \\ 0 & \frac{\tau-2\lambda}{4\lambda^2-4\lambda\tau+\tau^2+\omega^2} & \frac{\omega}{4\lambda^2-4\lambda\tau+\tau^2+\omega^2} \\ 0 & -\frac{\omega}{4\lambda^2-4\lambda\tau+\tau^2+\omega^2} & \frac{\tau-2\lambda}{4\lambda^2-4\lambda\tau+\tau^2+\omega^2} \end{bmatrix} \begin{pmatrix} F_{1d} \\ F_{2d} \\ F_{3d} \end{pmatrix}$$

In general terms, the procedure can be further iterated for $k = 3, 4...p$, till the desired degree of approximation is reached. For the purposes of this paper, however, merely concerned with the *existence* of the orbit, $k = 2$ is sufficient[3]. So, for $t > 0$, the coordinates of the variables (not far away from the steady state) can be approximated as follows

$$w_1 = \xi e^{-\bar{\eta}t} + a_2(\xi)\, e^{-2\bar{\eta}t};\; w_2 = b_2(\xi)\, e^{-2\bar{\eta}t};\; w_2 = c_2(\xi)\, e^{-k\bar{\eta}t} \tag{A.10}$$

where all parameters are known and ξ can be arbitrarily fixed.

We can now proceed to obtain the analytical expression of the orbits for $t < 0$. In this case, the solution trajectory converges to the origin along

[3] We are not interested here to derive and plot with precision the homoclinic orbit.

the unstable manifolds spanned by the complex conjugate eigenvalues $\lambda_{2,3} = \bar{\tau} \pm \bar{\omega} i$. Choosing $\lambda_2 = \bar{\tau} + \bar{\omega} i$, we obtain the following.

c) Set $k = 1$. From the expression (A.6), the vector $\left(\bar{a}_1, \bar{b}_1, \bar{c}_1\right)'$ can be retrieved comparing elements with same exponential $e^{\lambda t}$. We have

$$\begin{pmatrix} \bar{a}_1 \\ \bar{b}_1 \\ \bar{c}_1 \end{pmatrix} \lambda e^{\lambda t} = \mathbf{J}_N \begin{pmatrix} \bar{a}_0 + \bar{a}_1 e^{\lambda t} \\ \bar{b}_0 + \bar{b}_1 e^{\lambda t} \\ \bar{c}_0 + \bar{c}_1 e^{\lambda t} \end{pmatrix}$$

Taking into account (A.7), we also have

$$(\lambda \mathbf{I} - \mathbf{J}_N) \begin{pmatrix} \bar{a}_1 \\ \bar{b}_1 \\ \bar{c}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{A.11}$$

valuating (A.8) at $\lambda_2 = \bar{\tau} + \bar{\omega} i$, the values of the unknown coefficients can be obtained as

$$\begin{pmatrix} \bar{a}_1 \\ \bar{b}_1 \\ \bar{c}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \varsigma + \psi i \\ \psi + \varsigma i \end{pmatrix}$$

where ς and ψ are arbitrary constants and $i = \sqrt{-1}$.

d) Set $k = 2$. From (A.6), matching elements with $e^{2\lambda t}$, and simplifying, we can easily derive

$$\begin{pmatrix} \bar{a}_2 2\lambda \\ \bar{b}_2 2\lambda \\ \bar{c}_2 2\lambda \end{pmatrix} = \mathbf{J}_N \begin{pmatrix} \bar{a}_2 \\ \bar{b}_2 \\ \bar{c}_2 \end{pmatrix} + \begin{pmatrix} F_{1c}\bar{b}_1\bar{c}_1 + F_{1e}\bar{b}_1^2 + F_{1f}\bar{c}_1^2 \\ F_{2c}\bar{b}_1\bar{c}_1 + F_{2e}\bar{b}_1^2 + F_{2f}\bar{c}_1^2 \\ F_{3c}\bar{b}_1\bar{c}_1 + F_{3e}\bar{b}_1^2 + F_{3f}\bar{c}_1^2 \end{pmatrix}$$

or

$$[2\lambda \mathbf{I} - \mathbf{J}_N] \begin{pmatrix} \bar{a}_2 \\ \bar{b}_2 \\ \bar{c}_2 \end{pmatrix} = \begin{pmatrix} F_{1c}\bar{b}_1\bar{c}_1 + F_{1e}\bar{b}_1^2 + F_{1f}\bar{c}_1^2 \\ F_{2c}\bar{b}_1\bar{c}_1 + F_{2e}\bar{b}_1^2 + F_{2f}\bar{c}_1^2 \\ F_{3c}\bar{b}_1\bar{c}_1 + F_{3e}\bar{b}_1^2 + F_{3f}\bar{c}_1^2 \end{pmatrix}$$

Evaluating λ at $\lambda_2 = \bar{\tau} + \bar{\omega} i$, the unknown coefficients $\bar{a}_2$ $\bar{b}_2$ and $\bar{c}_2$ can be obtained as

$$\begin{aligned} \begin{pmatrix} \bar{a}_2 \\ \bar{b}_2 \\ \bar{c}_2 \end{pmatrix} &= [2\lambda \mathbf{I} - \mathbf{J}_N]^{-1} \begin{pmatrix} F_{1c}\bar{b}_1\bar{c}_1 + F_{1e}\bar{b}_1^2 + F_{1f}\bar{c}_1^2 \\ F_{2c}\bar{b}_1\bar{c}_1 + F_{2e}\bar{b}_1^2 + F_{2f}\bar{c}_1^2 \\ F_{3c}\bar{b}_1\bar{c}_1 + F_{3e}\bar{b}_1^2 + F_{3f}\bar{c}_1^2 \end{pmatrix} = \\ &= \begin{pmatrix} \frac{F_{1c}\bar{b}_1\bar{c}_1 + F_{1e}\bar{b}_1^2 + F_{1f}\bar{c}_1^2}{2(\bar{\tau}+\bar{\omega}i)+\bar{\eta}} \\ \frac{(2(\bar{\tau}+\bar{\omega}i)-\bar{\tau})(F_{2c}\bar{b}_1\bar{c}_1 + F_{2e}b_1^2 + F_{2f}\bar{c}_1^2) + (F_{3c}\bar{b}_1\bar{c}_1 + F_{3e}\bar{b}_1^2 + F_{3f}\bar{c}_1^2)\bar{\omega}}{[(2(\bar{\tau}+\bar{\omega}i)-\bar{\tau})^2+\bar{\omega}^2]} \\ -\frac{\bar{\omega}(F_{3c}\bar{b}_1\bar{c}_1 + F_{3e}\bar{b}_1^2 + F_{3f}c_1^2) + (2(\tau+\bar{\omega}i)-\bar{\tau})(F_{2c}\bar{b}_1\bar{c}_1 + F_{2e}\bar{b}_1^2 + F_{2f}\bar{c}_1^2)}{[(2(\bar{\tau}+\bar{\omega}i)-\bar{\tau})^2+\bar{\omega}^2]} \end{pmatrix} \end{aligned}$$

where $\bar{b}_1$ and $\bar{c}_1$ are as in (A.9).

As explained above, the procedure can be iterated for higher values of k. However, for the purposes of this paper, we can stop the computation at $k = 2$.

Therefore, for $t < 0$, coordinates of the variables (not far away from the steady state) can be approximated as follows

$$\begin{aligned}
w_1 &= (\varsigma - \psi i)\, e^{(\bar{\tau}+\bar{\omega}i)t} + (\bar{a}_2 + \bar{a}_2 i)\, e^{(\bar{\tau}+\bar{\omega}i)t} \\
w_2 &= (\varsigma + \psi i)\, e^{(\bar{\tau}+\bar{\omega}i)t} + \left(\bar{b}_2 + \bar{b}_2 i\right) e^{(\bar{\tau}+\bar{\omega}i)t} \\
w_3 &= (\bar{c}_2 + \bar{c}_2 i)\, e^{(\bar{\tau}+\bar{\omega}i)t}
\end{aligned}$$

According to the Euler formulae, we have

$$\begin{aligned}
w_1 &= e^{\bar{\tau}t}\left[\varsigma\cos(\bar{\omega}t) + \psi\sin(\bar{\omega}t) + (\varsigma\sin(\bar{\omega}t) - \psi\cos(\bar{\omega}t))\, i\right] + \\
&\quad + e^{2\bar{\omega}t}\left[\bar{a}_2^1\cos(2\bar{\omega}t) - \bar{a}_2^2\sin(2\bar{\omega}t)\right] + i\left[\bar{a}_2^1\sin(2\bar{\omega}t) + \bar{a}_2^2\cos(2\bar{\omega}t)\right] \\
w_2 &= e^{\bar{\tau}t}\left[\varsigma\cos(\bar{\omega}t) - \psi\sin(\bar{\omega}t) + (\varsigma\sin(\bar{\omega}t) + \psi\cos(\bar{\omega}t))\, i\right] + \\
&\quad + e^{2\bar{\omega}t}\left[\bar{b}_2^1\cos(2\bar{\omega}t) - \bar{b}_2^2\sin(2\bar{\omega}t)\right] + i\left[\bar{b}_2^1\sin(2\bar{\omega}t) + \bar{b}_2^2\cos(2\bar{\omega}t)\right] \\
w_3 &= e^{2\bar{\omega}t}\left[\bar{c}_2^1\cos(2\bar{\omega}t) - \bar{c}_2^2\sin(2\bar{\omega}t)\right] + i\left[\bar{c}_2^1\sin(2\bar{\omega}t) + \bar{c}_2^2\cos(2\bar{\omega}t)\right]
\end{aligned}$$

Now, let us use the principle of superposition saying that, for a generic autonomous system $\dot{W} = g(W)$, if $W_1 + W_2 i$ is a complex solution, then W_1 and W_2 are real solutions of the same system. In our case, we have

$$\begin{aligned}
w_{11} &= e^{\bar{\tau}t}\left[\varsigma\cos(\bar{\omega}t) + \psi\sin(\bar{\omega}t)\right] + e^{2\bar{\omega}t}\left[\bar{a}_2^1\cos(2\bar{\omega}t) - \bar{a}_2^2\sin(2\bar{\omega}t)\right] \\
w_{21} &= e^{\bar{\tau}t}\left[\varsigma\cos(\bar{\omega}t) - \psi\sin(\bar{\omega}t)\right] + e^{2\bar{\omega}t}\left[\bar{b}_2^1\cos(2\bar{\omega}t) - \bar{b}_2^2\sin(2\bar{\omega}t)\right] \\
w_{31} &= e^{2\bar{\omega}t}\left[\bar{c}_2^1\cos(2\bar{\omega}t) - \bar{c}_2^2\sin(2\bar{\omega}t)\right]
\end{aligned}$$

and

$$\begin{aligned}
w_{12} &= e^{\bar{\tau}t}\left[(\varsigma\sin(\bar{\omega}t) - \psi\cos(\bar{\omega}t))\right] + e^{2\bar{\omega}t}\left[\bar{a}_2^1\sin(2\bar{\omega}t) + \bar{a}_2^2\cos(2\bar{\omega}t)\right] \\
w_{22} &= e^{\bar{\tau}t}\left[(\varsigma\sin(\bar{\omega}t) + \psi\cos(\bar{\omega}t))\right] + e^{2\bar{\omega}t}\left[\bar{b}_2^1\sin(2\bar{\omega}t) + \bar{b}_2^2\cos(2\bar{\omega}t)\right] \\
w_{32} &= e^{2\bar{\omega}t}\left[\bar{c}_2^1(\varsigma,\psi)\cos(2\bar{\omega}t) - \bar{c}_2^2(\varsigma,\psi)\sin(2\bar{\omega}t)\right] + i\left[\bar{c}_2^1\sin(2\bar{\omega}t) + \bar{c}_2^2\cos(2\bar{\omega}t)\right]
\end{aligned}$$

Therefore, near the unstable manifold, the solution of the generic system $\dot{W} = g(W)$ can be characterized as follows

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = d_1 \begin{pmatrix} w_{11} \\ w_{21} \\ w_{31} \end{pmatrix} + d_2 \begin{pmatrix} w_{12} \\ w_{22} \\ w_{32} \end{pmatrix}$$

where $(d_1, d_2) = (0,1)$ or $(1,0)$.

Therefore, going back to the solution of our system of differential equations, we can write

$$w_1 = \begin{cases} \xi e^{-\bar{\eta}t} + a_2 e^{-2\bar{\eta}t} & t \geq 0 \\ d_1 \left\{ e^{\bar{\tau}t} \left[\varsigma \cos(\bar{\omega}t) + \psi \sin(\bar{\omega}t) \right] + e^{2\bar{\omega}t} \left[\bar{a}_2 \cos(2\bar{\omega}t) - \bar{a}_2 \sin(2\bar{\omega}t) \right] \right\} + \\ d_2 \left\{ e^{\bar{\tau}t} \left[(\varsigma \sin(\bar{\omega}t) - \psi \cos(\bar{\omega}t)) \right] + e^{2\bar{\omega}t} \left[\bar{a}_2 \sin(2\bar{\omega}t) + \bar{a}_2 \cos(2\bar{\omega}t) \right] \right\} & t \leq 0 \end{cases}$$

$$w_2 = \begin{cases} b_2 e^{-2\bar{\eta}t} & t \geq 0 \\ d_1 \left\{ e^{\bar{\tau}t} \left[\varsigma \cos(\bar{\omega}t) - \psi \sin(\bar{\omega}t) \right] + e^{2\bar{\omega}t} \left[\bar{b}_2 \cos(2\bar{\omega}t) - \bar{b}_2 \sin(2\bar{\omega}t) \right] \right\} + \\ d_2 \left\{ e^{\bar{\tau}t} \left[(\varsigma \sin(\bar{\omega}t) + \psi \cos(\bar{\omega}t)) \right] + e^{2\bar{\omega}t} \left[\bar{b}_2 \sin(2\bar{\omega}t) + \bar{b}_2 \cos(2\bar{\omega}t) \right] \right\} & t \leq 0 \end{cases}$$

$$w_2 = \begin{cases} c_2 e^{-2\bar{\eta}t} & t \geq 0 \\ d_1 \left\{ e^{2\bar{\omega}t} \left[\bar{c}_2^1 \cos(2\bar{\omega}t) - \bar{c}_2 \sin(2\bar{\omega}t) \right] \right\} \\ d_2 \left\{ e^{2\bar{\omega}t} \left[\bar{c}_2^1 \cos(2\bar{\omega}t) - \bar{c}_2 \sin(2\bar{\omega}t) \right] + i \left[\bar{c}_2 \sin(2\bar{\omega}t) + \bar{c}_2 \cos(2\bar{\omega}t) \right] \right\} & t \leq 0 \end{cases}$$

At $t = 0$, the coordinates of the variables along the unstable manifold equate the coordinates of the variables along the stable manifold. This leads to the following system

$$\begin{aligned} \xi + a_2 &= d_1 (\zeta + \bar{a}_2) \bar{a}_2 + d_2 \\ b_2 &= d_1 \left[\psi + \bar{b}_2(\zeta, \psi) \right] + d_2 \left[\zeta + \bar{b}_2(\zeta, \psi) \right] \\ c_2 &= d_1 \left[\zeta + \bar{c}_2(\zeta, \psi) \right] + d_2 \left[-\psi + \bar{c}_2(\zeta, \psi) \right] \end{aligned}$$

Finally, recalling the values of the various coefficients obtained above, we obtain the following surface

$$\xi(\psi, \zeta) = \frac{F_{1c}\psi\zeta + F_{1e}\psi^2 + F_{1f}\zeta^2}{(4\tau - \eta)^2 + \omega^2}(2\tau - \eta) - \frac{F_{1d}}{\eta} \frac{(4\eta^2 + 4\eta\tau + \tau^2 + \omega^2)}{(F_{2d} + F_{3d})(\tau - \omega + 2\eta)}(\psi + \zeta) \tag{A.12}$$

representing the combinations of the $0 < (\xi, \psi, \zeta) < 1$ parameters giving rise to a family of homoclinic orbits.

Proposition 2 can now be clarified. It implies that, given the structural parameters of the model (belonging to the non-empty subset satisfying the requirement H.1 of the Shilnikov theorem), given the constants φ_1, φ_2 and φ_3 in the choice of the eigenvectors, then, we can pick up the value of the externality factor γ such that (A.9) in Appendix is satisfied and one homoclinic orbit emerges. To complete the information, the exact form of the F_{ij}

coefficients necessary for the computation) is the following

$$\begin{aligned}
F_{1c} &= V^* z_1 + [V^* \tfrac{u^*}{X^*} + \tfrac{\delta(1-\beta+\gamma)}{1-\beta}] z_2 - z_3 - v_3 z_1) \\
F_{1d} &= \tfrac{1}{2} V^* u_1^2 + [V^* \tfrac{u^*}{X^*} + \tfrac{\delta(1-\beta+\gamma)}{1-\beta}] u_1 u_2 - u_1 u_3 \\
F_{1e} &= \tfrac{1}{2} V^* - v_3 \\
F_{1f} &= \tfrac{1}{2} V^* z_1^2 + [V^* \tfrac{X^*}{u^*} + \tfrac{\delta(1-\beta+\gamma)}{1-\beta}] z_1 z_2 - z_1 z_3 \\
F_{2d} &= \eta u_2^2 - u_1 u_3 \\
F_{3d} &= \tfrac{\beta-\sigma}{\sigma} V^* Q \left[\tfrac{1}{2} \tfrac{(\beta-1)(\beta-2)}{\beta X^*} + \tfrac{1}{2X^* u^{*-1+\beta}} + \tfrac{1-\beta}{\beta X^* u^*} \right] u_1^2 + u_3^2 + \tfrac{\beta-\sigma}{\beta\sigma} V^* \left[-\tfrac{1}{X^* u^*} + 1 \right] u_1 u_3
\end{aligned}$$

where $V^* = \beta(\beta-1) X^{*\beta-2} u^{*1-\beta}$.

A.4 The TVC holds along some chaotic paths

The TVC requires the present value of the state variables to converge to zero as the planning horizon proceeds towards infinity. Given an optimal path, the necessity of the TVC reflects the impossibility of finding an alternative feasible path for which each state variable deviates from the optimum and increases discounted utility.

In the present context, satisfying the TVC implies the following

$$\underset{t\to\infty}{lim} \left[e^{-\rho t} \left(\lambda_1 k + \lambda_2 h \right) \right] = 0 \tag{A.13}$$

Benhabib and Perli (1994) show that, at the steady state $P^* \equiv (X^*, u^*, Q^*)$

$$\lim_{t\to\infty} \left(-\rho + \tfrac{\dot{\lambda}_1}{\lambda_1} + \tfrac{\dot{k}}{k} \right) = \lim_{t\to\infty} \left(-\rho + \tfrac{\dot{\lambda}_2}{\lambda 2} + \tfrac{\dot{h}}{h} \right) = -\delta u^*$$

As a consequence, at the steady state, the TVC holds (provided that $\theta \in \hat{\Theta} \cup \check{\Theta}$).

Moreover, Mattana and Venturi (1999) and Nishimura and Shigoka (2006) show that the TVC is also satisfied in case the economy converges to a Hopf cycle, well located in the R^3 ambient space, provided that the solution trajectories are in a convenient small neighborhood of the steady state.

For the case of solution trajectories resulting in spiral/Shilnikov chaos, we proceed as follows. First, we observe that an extension in R^3 of the argument in Mattana and Venturi (1999) and Nishimura and Shigoka (2006) can be used to assure that (A.13) is always satisfied along the homoclinic orbit. More formally, assume we can appropriately choose the scaling factors in

(A.1) so that the homoclinic orbit is contained in an open sphere with radius r around the steady state $P^* \equiv (X^*, u^*, Q^*)$ such that $0 < u_{|_r} < 1$. Then along every homoclinic orbit contanined in the sphere, the present value of the state variables will always tend to zero as the planning horizon proceeds towards infinity.
Secondly, we need to show that $0 < u_t < 1$ along a specific chaotic paths emerging after the rupture of the homoclinic orbit. In this regard, it suffices to recall from Theorem 1, that the chaotic trajectory is confined in a small neighborhood of the homoclinic orbit.

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