Aggregate Production Functions are NOT Neoclassical

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Abstract

The issue of whether production functions are consistent with the neoclassical postulates has been the topic of intensive debates. One of the issues is whether the production of an aggregate economic system can be represented as if it was a simple neoclassical well-behaved production function with inputs the aggregate capital K, homogeneous with output Y, and aggregate labor, L. Standard postulates are those of the marginal productivities - and the associated demand of labor and capital - are negatively related with the factor prices, namely the *wage rate* and the *profit rate*. The cases where the neoclassical properties do not hold are often regarded as anomalies. In this paper we use real input-output data and we search for these anomalies. We compute the aggregate values for capital, production and labour. We find that for the dataset considered the neoclassical postulates do not hold. We consider this to be a very robust result. The implication is that standard models relying on the neoclassical aggregate Cobb-Douglas-like production functions do not hold.

Keywords: Aggregate Neoclassical Production Function, Cobb-Douglas, CES, Technological Change, Macroeconomics, Growth.

JEL classifications: C61, C63, C67, O47

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1 Introduction

It is a widespread practice among most (macro) economists to use the *"neoclassical"* aggregate production function while constructing (macro)economic models. Routinely used models (Solow, Ramsey-Cass-Koopmans, endogenous growth, overlapping generations, real business cycles, aggregate demand and aggregate supply, dynamic stochastic general equilibrium, computable general equilibrium, and so on) are all based on aggregate CES production functions.

These models often represent a system which produces a large number of heterogeneous goods with a few index numbers (one number each for output Y, productive capital K, total quantity of employed labor L, and technological level or knowledge A). Samuelson has properly named this type of aggregate production function *Surrogate* or *As-if* production function Samuelson (1962, p.194 fn.1).

Such an aggregate representation may be useful, provided that the indexes have certain properties (see Irving Fisher (1922); Frisch (1936)).

At the end of the 1960s, it was concluded that aggregation could be problematic. The problems are of two types.

A type of problem is associated with the technical aggregation from micro to macro. That is: simple production functions when aggregated do not keep the same functional form (Fisher, 1969; Shaikh, 1974)[1].

Another type of problem is the value problem which was addressed during the two Cambridges debate[2]. The conclusion of this debate has been that there exist cases in which the aggregation from the many commodities space to a single *surrogate production function* (see Samuelson (1962)) may lead to a production function which is not well-behaved (this problem was admitted by Samuelson (1966) himself).

Solow, who acknowledged the problem (Solow (1976, p.138)), observed:

> ...I have to insist again that anyone who reads my 1955 article [Solow (1955)]will see that I invoke the formal conditions for rigorous aggregation not in the hope that they would be applicable ...but rather to suggest the *hopelessness of any formal justification of an aggregate production function in capital and labor*

Regardless of the widespread acknowledgment that aggregation could be problematic, the (generalized) Cobb-Douglas production function is widely used as an essential element of theories and as a fundamental tool for the empirical assessment of technological progress and productivity growth.

There are two reasons. The first is that although one is not assured that the aggregation will preserve always neoclassicla properties, there exist, at least in theory, sets of methods for which a *neoclassical surrogate production function* does in fact exist. The second is due to the fact that the economists questioning the neoclassical aggregate production function have been unable to convincingly demonstrate the empirical untenability of the production functions of the Cobb-Douglas type (or the generalized CES type) and have not been able, so it seems, to provide a valid and useful alternative.

The position taken by Sato (1974, p.383) is still representative of the state of affairs that is prominent today among the majority of economists. He argued:

> ...that there is a *not-too-small* world in which the neoclassical postulate [i.e. production function] is perfectly valid. So long as we live in that world, we need not to

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[1] For example let us assume two firms. The first produces output with the following production function $y_1 = f(k_1, \ell_1)$, the second with the following production function $y_2 = g(k_2, \ell_2)$. Both have the usual neoclassical properties, i.e., positive marginal productivities and positive marginal rates of substitution. It is possible that the function $f + g$ would not have the mentioned neoclassical properties.

[2] A relevant list of contributions to the two Cambridges debate may be found in Zambelli (2004)

give up the neoclassical postulate [i.e. production function] ... Nonetheless, it is important to realise that *there is another world in which the neoclassical postulate may not fare well or is contradicted. An empirical question is which of the two models is more probable.*

On the one hand, he admits the existence of the problem, and, on the other hand, he declares the belief that the world has *neoclassical* properties. In doing so, he makes the problem of aggregation as a type of *curiosum*, which is interesting from the theoretical point of view, but irrelevant for empirical applications.

This position has not been satisfactorily challenged by empirical demonstrations showing that the world is or is NOT neoclassical.

The empirically unchallenged belief that the world is *neoclassical* has led to a state of affairs in which productivity measurements (total and multiple factor productivities) and measurements of technological progress and of economic efficiency are all based on the aggregate neoclassical production function. In fact, the points of departure for these measures are still the neoclassical works of Solow (1957), Farrell (1957) (on this matter see Kao et al. (2014)).

Few authors have attempted to provide a measure of whether the production functions have neoclassical properties. Until recently (see Zambelli et al. (2014)) due to lack of data and of computational capacity a final assessment of whether production functions are *neoclassical* or not has not yet been made. Some authors have made an attempt to assess whether the world may exhibit neoclassical properties thorough numerical simulations (see, D'Ippolito (1987), Zambelli (2004)). Other have made some estimations of aggregate production functions Han and Shefold (2006) using real data, but have been unable to compute general cases so as to reach a definite conclusion on the matter Zambelli et al. (2014, fn. 17).

Thanks to the discovery of a powerful algorithm, the FVZ-algorithm Zambelli et al. (2014, Sec. 4), and the availability of a rather complete data set (Timmer (2012, WIOD))[3] here we compute surrogate production functions for a wide range of data belonging to 30 countries and for 15 years.

We have checked whether these production function have *neoclassical* properties. The results presented here are in our view rather conclusive: surrogate production functions do not have, contrary to what generally believed or postulated by faith, *neoclassical* properties.

2 The Aggregate *Neoclassical* Production Function - A Short Review

It is a widespread practice to assume that the production of a nation can be described with a *As-if* production function. The neoclassical aggregate production function is a mathematical relation that links the output with the inputs and which holds specific properties. We will follow the tradition set by xxxx J.B.Clark yyyy, Solow (1955, 1956, 1957), Arrow et al. (1961), Ferguson (1969), Shephard (1970). We consider the simple cases where there is one output Y and two physical inputs, K, L and factor productivity of knowledge variable, A.

$$Y = F(K, L) \tag{2.1}$$

There are three basic sets of assumptions that are considered to be necessary for the above functional form to be *neoclassic*.

First Set of Assumptions. *Law of positive, but decreasing marginal productivities.* These assumptions are those of *convexity*, *continuity* and *differentiability*. This translates to the fol-

[3] For the computations that will follow we have used the Data present in the WIOD Data Base. A description of the use of the available data can be found in Zambelli et al. (2014). See also below fn. **??**

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lowing properties for the function F: $\partial Y/\partial K = F_K(K,L) > 0$; $\partial^2 Y/\partial K^2 = F_{KK}(K,L) < 0$; $\partial Y/\partial L = F_L(K,L) > 0$, $\partial^2 Y/\partial L^2 = F_{LL}(K,L) < 0$;

which are characterized as the *Law of positive, but decreasing marginal productivities.*

Second Set of Assumptions. *Theory of Social Distribution based on Marginal Productivities.* It is assumed that markets operate in such a way that the wage rate w is equal to the marginal productivity of labor and the interest rate (or rental cost of capital) r is equal to the marginal productivity of capital. The *representative* producer is assumed to maximize profits given the constraints and that competition among producers would lead to choose the production level associated to the minimization of (factor) costs. Given the profit function

$$\Pi = pY - rK - wL \tag{2.2}$$

the first order conditions, assuming that the producer is a *price-taker*, i.e. prices are fixed[4] , we have that: the producer's decision problem is that of finding K^* and L^* maximize the profits, for given prices. The first order conditions for the maximization of profits are given by:

$$\frac{\partial \Pi}{\partial K} = p\frac{\partial Y}{\partial K} - r = 0 \Longrightarrow r = pF_K = p\frac{\partial Y}{\partial K} \tag{2.3}$$

$$\frac{\partial \Pi}{\partial L} = p\frac{\partial Y}{\partial L} - w = 0 \Longrightarrow w = pF_L = p\frac{\partial Y}{\partial L} \tag{2.4}$$

Equation eq.2.3 is the *demand for capital schedule* and eq. 2.4 is the *demand for labor schedule*. The physical world of production and that of the exchange maybe linked considering the Marginal Rate of Technical Substitution (MRTS) which is the change of one factor necessary so as for a change of another factor the production is along the same isoquant. We have the following relation:

$$0 = dY = \frac{\partial Y}{\partial K}dK + \frac{\partial Y}{\partial L}dL \Rightarrow MRTS = -\frac{dK}{dL} = \frac{\partial Y}{\partial L}\Big/\frac{\partial Y}{\partial K} \tag{2.5}$$

Substituting eq.2.3 and eq. 2.4 into 2.5 we can link a technical relation with factor prices:

$$MRTS = -\frac{dK}{dL} = \frac{w}{r} \tag{2.6}$$

Third Set of Assumptions. *Homogeneity of degree 1 and Constant Elasticity of Substitution - CES.* Arrow et al. (1961) introduce additional features. These are those of homogeneity of degree 1 - i.e. $\mathcal{A}F(\lambda K, \lambda L) = \lambda \mathcal{A}F(\lambda K, \lambda L) = \lambda Y$ - and that of the Constant Elasticity of Substitution. The elasticity of substitution is given by:

$$\sigma = -\frac{dK/K}{dL/L}\Big/\frac{dMRTS}{MRTS} = \frac{\partial ln(K/L)}{\partial ln(MRTS)} = \frac{\partial ln(K/L)}{\partial ln(w/r)} \tag{2.7}$$

Clearly with $\sigma = 1$ an increast of the capital-labor ratio will be matched by an exact increase in the *wage-profit ratio*. This is the case in which although it is possible to observe an increase in the capital-labour ratio this will be associated with constant shares ($\frac{wL}{pY}$ and $\frac{rK}{pY}$).

[4]A standard assumption is to assume that the prices p, w and r are independent of the quantities. This is an assumption which is not justified by actual operation of markets: clearly these prices are not independent of the quantities demanded or supplied. To assume the constancy of prices means that $\partial p/\partial L = \partial p/\partial K = \partial r/\partial L = \partial r/\partial K = \partial w/\partial L = \partial w/\partial K = 0$

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There are different functional forms that would be consistent with respect to the above assumptions. A widely adopted functional form is the Cobb-Douglas production function, which is a special case of the generalized CES-production function[5] :

$$Y = F(K, L) = \mathcal{A}K^{\alpha}L^{1-\alpha} \tag{2.8}$$

The isoquant associated with the above Cobb-Douglas production function is:

$$K = \Big(\frac{\overline{Y}}{AL^{1-\alpha}}\Big)^{1/\alpha} \tag{2.9}$$

where $\overline{Y}$ is a given constant level of output. The associated *wage-profit curve* is derived by substituting into eqs. 2.3 and 2.4 the marginal productivities associated with :

$$w = p(1-\alpha)\Big(\frac{p\alpha}{r}\Big)^{\frac{\alpha}{1-\alpha}} \tag{2.10}$$

Interesting is to note that while the position of the isoquant, eq. 2.9, is obviously dependent on the level of activity the *wage-profit curve* does not depend on it. The standard procedure is that of assuming the above and to estimate the parameters of the chosen production function. Here, for simplicity and for the sake of the exposition we assume as production function the one above, eq. 2. The parameters $\mathcal{A}$ and α are estimated from actual data. Figure 2.1 reports the aggregated data relative to the year 2009 of the values of per capital-labor ratio, K/L, and of the output per labor ratio, Y/L of 30 countries[6]. The requirement that the efficient production set should be *convex* leads to restrict the number of efficient points to be considered for the estimation of $\mathcal{A}$ and α. There are different methods for the estimation of the *best fit* production function. What it is important to realize is that the different methods (see Kao et al. (2014)) would be estimations that would be near the thick red line - i.e. near the convex set.

Today it is standard procedure to proceed as if the CES type aggregate production functions can represent production function of a national production system.

3 Samuelson's Surrogate Production Function

A very important and natural question is to ask whether it is sound to assume that a production system where there are many products and many different methods to produce them can be represented with a simple system like the Cobb-Douglas. We know that any system is producing a great variety of commodities.

Samuelson (1962) proposed a method meant to provide theoretical justification for the simplification known as aggregation.

> One need never speak of the *the* production function, but rather should speak of a great number of separate production functions, which correspond to each activity and which need have no smooth substitutability properties. All the technology of the economy could be summarized in a whole book of such production functions, each page giving the blueprint for a particular activity. Technological change can

[5] Arrow et al. (1961) have suggested the following generalized CES-Production function $Y = F(K, L) = \gamma_1[K^{\rho} + \gamma_2 L^{\rho}]^{1/\rho}$ where $\rho = (\sigma - 1)/\sigma$.

[6] The countries considered are: (AUS) Australia; (FIN) Finland; (KOR) Korea;(AUT) Austria; (FRA) France; (MEX) Mexico; (BEL) Belgium; (GBR) Great Britain; (NLD) Netherlands; (BRA) Brazil; (GRC) Greece; (POL) Poland; (CAN) Canada; (HUN) Hungary; (PRT) Portugal; (CHN) China; (IDN) Indonesia; (RUS) Russia; (CZE) Czech Republic; (IND) India; (SWE) Sweden; (DEU) Germany; (IRL) Ireland; (TUR) Turkey; (DNK) Denmark; (ITA) Italy; (TWN) Taiwan; (ESP) Spain; (JPN) Japan; (USA) United States.

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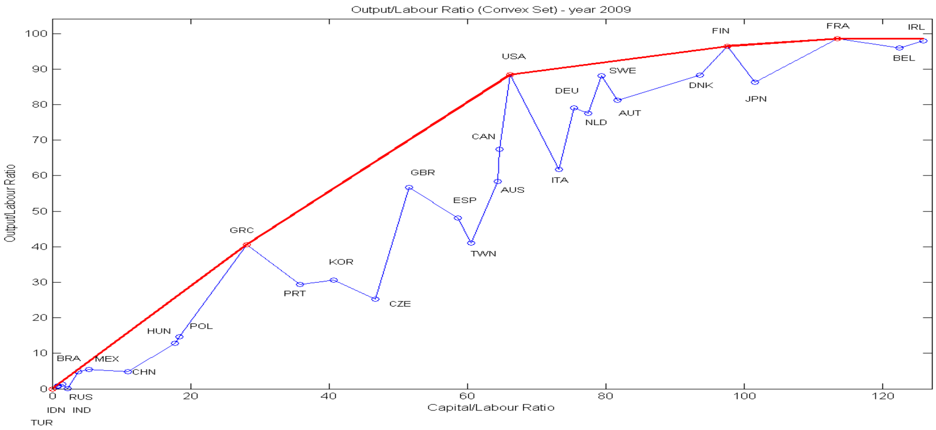

Figure 2.1: Output/Labor Ratio versus Capital/Labor Ratio - 2009, *Data from Timmer (2012). The thick line shows the boundary of the* convex set. *The points on the boundary are Greece, Unites States, Finland and France. The values for aggregated Capital and Labor are in euro 1995 as in WIOD Tables*

> be handled easily by adding new options and blue prints to the book ... Finally it is enough to assume that there is but one "primary" or non-producible factor of production, which we might as well call labor ... All other inputs and outputs are producible by the technologies specified in the blueprints Samuelson (1962, p.194)

The book of blueprints can be seen as different entries of the input-output tables. We observe from the actual tables that b_i units of commodity i can be produced with s_i different alternative methods.

$$\phi(z_i, :, i) : a_{i1}^{z_i}, a_{i2}^{z_i}, \ldots, a_{in}^{z_i}, \ell_i^{z_i} \mapsto b_i^{z_i} \tag{3.1}$$

where: $i = 1, \ldots, n; j = 1, \ldots, n; \quad z_i = 1, ..., s_i$; and $a_{ij}^{z_i} \in \mathbb{Q}$. $a_{i1}^{z_i}$ is the input of commodity j in producing a good i using a method z_i. s_i is the number of available methods for producing the good i and n is the number of goods.

The set of methods for producing good i – i.e., the set of blueprints for the production of i – can be represented in matrix notation as $\Phi(1 : s_i, 1 : (n+2), i)$. The set of all the available methods is given by the following set of activities $\mathbf{\Phi} = \{\Phi(:, :, 1) \cup \Phi(:, :, 2) \ldots, \Phi(:, :, n)\}$ (see Zambelli et al. (2014)).

The cardinality of the above set of methods can be very large and subsets of the above methods can exhibit, in principle, a great variety of mathematical properties. Consequently, whether a production system has, for example, the convexity property and hence does approximate a neoclassical production function depends on the 'actual' structure of Φ (see Zambelli et al. (2014)). The set of all the available methods is given by the following set of activities $\mathbf{\Phi} = \{\Phi(:, :, 1) \cup \Phi(:, :, 2) \ldots, \Phi(:, :, n)\}$

Hence, a n-commodity output vector can be generated by using one combination of the methods, which belongs to set $\mathbf{\Phi}$. There are a total $\boldsymbol{s} = \prod_{i=1}^{n} s_i$ of these combinations. Given one of these combinations, $\mathbf{z} = [z_1, z_2, \ldots, z_n]'$, we have one production possibility. The heterogeneous production of a system would then depend on the employment used and the methods of production adopted. The triple ($\mathbf{A}^{\mathbf{z}}$, $\boldsymbol{L}^{\mathbf{z}}$, $\boldsymbol{B}^{\mathbf{z}}$, $\boldsymbol{x}$) is the standard representation of an input-output system where $\mathbf{A}^{\mathbf{z}}$ is the set of the inputs used, $\boldsymbol{L}^{\mathbf{z}}$ is the vector of the necessary labor and $\boldsymbol{B}^{\mathbf{z}}$

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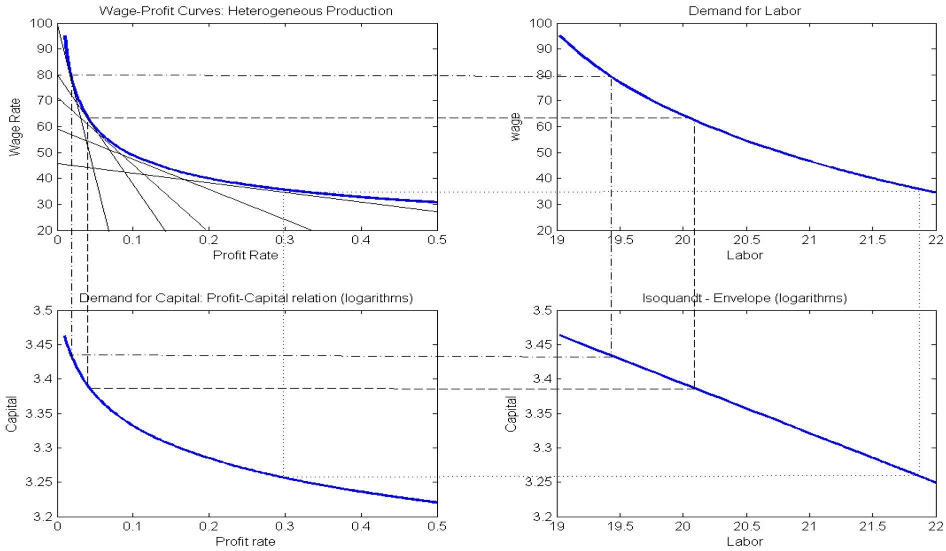

Figure 3.1: Cobb-Douglas: year 2009, *The graphs above are based on the estimate of the Cobb-Douglas where the data is the one of fig. 2.1. The estimated value for $\mathcal{A}$ is 34.6 and for α is 0.224. The North-West Graph* Cobb-Douglas wage-profit curve *is computed with eq. , the North-East Graph,* The Demand for Labor *curve is computed with eq. 2.4. The South-West Graph is* the Demand for Capital *curve derived from eq. 2.3. The South-East Graph is* Isoquandt *curve derived from eq. 2.9*

is the associated output. $\boldsymbol{x}$ is the vector defining the level of activity [7].

For a chosen system, $\mathbf{z}$, (i.e., a triple $\mathbf{B^z}, \mathbf{A^z}, \mathbf{L^z}$) the *production prices* that would assure accounting equilibrium are those that allow the following relation to hold:

$$\mathbf{A^z}(1+r)\mathbf{p} + \mathbf{L^z}w = \mathbf{B^z}\mathbf{p} \tag{3.2}$$

For a given uniform profit rate r and uniform wage rate w, there exists a price vector $\mathbf{p}$ that would allow the system to be remain productive for the subsequent periods as well:

$$\mathbf{p^z}(r,w) = [\mathbf{B^z} - \mathbf{A^z}(1+r)]^{-1}\mathbf{L^z}w \tag{3.3}$$

An important result in this context is that for a given combination of methods $\mathbf{z}$ (i.e., any triple $\mathbf{XB^z}, \mathbf{XA^z}, \mathbf{XL^z}$) the re-proportion activity level $\mathbf{x}$ does not influence the determination of the price vector $\mathbf{p}$. This is known as the *Non-Substitution Theorem*[8] in the literature.

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[7] $$\mathbf{A^z} = \mathbf{\Phi}(\mathbf{z}, 1:n, 1:n) = \begin{bmatrix} a_{11}^{z1} & a_{12}^{z1} & \dots & a_{1n}^{z1} \\ a_{21}^{z2} & a_{22}^{z1} & \dots & a_{2n}^{z2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}^{zn} & a_{n2}^{z1} & \dots & a_{nn}^{zn} \end{bmatrix}; \mathbf{L^z} = \mathbf{\Phi}(\mathbf{z}, n+1, 1:n) = \begin{bmatrix} \ell_1^{z1} \\ \ell_2^{z2} \\ \vdots \\ \ell_n^{zn} \end{bmatrix};$$

$$\mathbf{B^z} = diag(\mathbf{\Phi}(\mathbf{z}, n+2, 1:n)) = \begin{bmatrix} b_1^{z1} & 0 & \dots & 0 \\ 0 & b_2^{z2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_n^{zn} \end{bmatrix};$$

[8] On the origins of the *non-substitution-theorem*, see Arrow (1951), Koopmans (1951), Samuelson (1951). A more recent treatment is given in Mas-Colell et al. (1995), pp.159-60. See an exposition also in Zambelli (2004,

We then choose a *numéraire*, a vector composed of different proportion of the n produced goods forming the input-output tables,

$$\eta' \mathbf{p}^{\mathbf{z}}(r, w) = 1 \tag{3.4}$$

we are now in a position to define the *wage-profit curve*. By substituting 3.3 into 3.4 we obtain the *wage-profit curve* associated with the set of methods $\mathbf{z}$:

$$w^{\mathbf{z}}(r, \eta) = [\eta'[\mathbf{B}^{\mathbf{z}} - \mathbf{A}^{\mathbf{z}}(1+r)]^{-1}\mathbf{L}^{\mathbf{z}}]^{-1} \tag{3.5}$$

This is the *wage-profit curve* associated with system $\mathbf{z}$. Substituting 3.5 into 3.3 we obtain the price vector

$$\mathbf{p}^{\mathbf{z}}(r, \eta) = [\mathbf{B}^{\mathbf{z}} - \mathbf{A}^{\mathbf{z}}(1+r)]^{-1}\mathbf{L}^{\mathbf{z}}[\eta'[\mathbf{B}^{\mathbf{z}} - \mathbf{A}^{\mathbf{z}}(1+r)]^{-1}\mathbf{L}^{\mathbf{z}}]^{-1} \tag{3.6}$$

The price vector $\mathbf{p}^{\mathbf{z}}(r, \eta)$ is a function of the particular set of methods $\mathbf{z}$, of the profit rate r and of the *numéraire*.

Samuelson (1962) proposed to simplify the theory of production not by way of assumption, but by construction. The straight lines present in the North-West graph of 3.1 represent each different sets of methods $\mathbf{z}_j$ represent a *wage-profit relation*. The North-West graph is qualitatively equivalent to Figures 1 and 2 present in Samuelson (1962, p.195, p.197). As it can be seen there is an envelope that would be formed as the outer frontier of a large number of straight lines, i.e. a large number of different set of methods $\mathbf{z}_j$. He did originally assume of presuppose that the *wage-profit curves* were straight lines. This assumption was challenged during the *The Camabridgee Capital* Controversy (see Garegnani (1966); Pasinetti (1966); Bruno et al. (1966); Cohen and Harcourt (2003)). He had to give up this assumption during the debate that followed Samuelson (1966). The *Surrogate* or *As-if* production function is derived from the employment and the value of capital that is associated with the *wage-profit curves* that form the envelope.

The value of aggregate net national product associated to a given set of methods, $\mathbf{z}$ is given by:

$$Y^{\mathbf{z}}_{val}(r, \mathbf{x}, \eta) = \mathbf{x}'[\mathbf{B}^{\mathbf{z}} - \mathbf{A}^{\mathbf{z}}]\mathbf{p}^{\mathbf{z}}(r, \eta) \tag{3.7}$$

The value of aggregate capital per worker is given by:

$$K^{\mathbf{z}}_{val}(r, \mathbf{x}, \eta) = \mathbf{x}'\mathbf{A}^{\mathbf{z}}\mathbf{p}^{\mathbf{z}}(r, \eta) \tag{3.8}$$

The per labor value of the net national product associated to a given set of methods, $\mathbf{z}$ is given by:

$$y^{\mathbf{z}}_{val}(r, \mathbf{x}, \eta) = \frac{Y^{\mathbf{z}}_{val}(r, \mathbf{x}, \eta)}{\mathbf{x}'\mathbf{L}^{\mathbf{z}}} = \frac{\mathbf{x}'[\mathbf{B}^{\mathbf{z}} - \mathbf{A}^{\mathbf{z}}]\mathbf{p}^{\mathbf{z}}(r, \eta)}{\mathbf{x}'\mathbf{L}^{\mathbf{z}}} \tag{3.9}$$

The value of capital per worker is given by:

$$k^{\mathbf{z}}_{val}(r, \mathbf{x}, \eta) = \frac{K^{\mathbf{z}}_{val}(r, \mathbf{x}, \eta)}{\mathbf{x}'\mathbf{L}^{\mathbf{z}}} = \frac{\mathbf{x}'\mathbf{A}^{\mathbf{z}}\mathbf{p}^{\mathbf{z}}(r, \eta)}{\mathbf{x}'\mathbf{L}^{\mathbf{z}}} \tag{3.10}$$

To different *activity levels* $\mathbf{x}$ there are associated different produced and distributed quantities and hence different consumption possibility. The same value of output, $y^{\mathbf{z}}_{val}(r, \mathbf{x}, \eta)$, or of capital, $k^{\mathbf{z}}_{val}(r, \mathbf{x}, \eta)$, may be associated to an enormous number of vectors $\mathbf{x}$[9]. We shall consider the activity level for which the value of aggregated net national product is highest.

footnote 2, p. 105)

[9]This is a very serious problem. When we search on whether the system follows neoclassical properties it would be also important to study the allocations implied. In the *neoclassical* literature this problem is not addressed by assuming the output and capital, and hence consumption, are all homogeneous. Obviously this cannot be done for the case of *heterogeneous* production

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For each tripple $(\mathbf{B^z}, \mathbf{A^z}, \mathbf{L^z})$, *numéraire*, η , and profit rate r there is an efficient activity vector, $\mathbf{x}^*$ which is determined with the following *linear programming* algorithm.

$$\max_{\mathbf{x}} \ \mathbf{x}'\big[\mathbf{B^z} - \mathbf{A^z}\big]\mathbf{p^z}(r,\eta) \tag{3.11}$$

$$\text{s.t.} \ \mathbf{x}'\big[(\mathbf{B^z} - \mathbf{A^z})\big] \geq \mathbf{0}' \tag{3.12}$$

$$\mathbf{x}'\mathbf{L^z} - \mathbf{e}'\mathbf{L^z} = \mathbf{0}' \tag{3.13}$$

$$\mathbf{x}' \geq \mathbf{0}' \tag{3.14}$$

The highest value of net national product per labor and the highest value of capital per labor product is given by:

$$y^{\mathbf{z}}_{val}(r, \mathbf{x}^*, \eta) \tag{3.15}$$

$$k^{\mathbf{z}}_{val}(r, \mathbf{x}^*, \eta) \tag{3.16}$$

The outer envelope of all m[10] possible *wage-profit curves* is the *wage-profit frontier*:

$$w^{\text{WPF}}_{\boldsymbol{E}}(r,\eta) = \max\left\{w^{\mathbf{z}_1}(r,\eta), w^{\mathbf{z}_2}(r,\eta), ..., w^{\mathbf{z}_m}(r,\eta)\right\} \tag{3.17}$$

where $\boldsymbol{E}$ is a subset of $\boldsymbol{\Phi}$, $(\boldsymbol{E} \subset \boldsymbol{\Phi})$.

The domain of $w^{\text{WPF}}_{\boldsymbol{E}}(r,\eta)$ is composed of v intervals. The junction between the different intervals are called *switch points* - points where the dominance of one *wage-profit curve* is replaced by another one.

$$r \in \Big[\big[0, \widehat{r}_1\big[\cup \big[\widehat{r}_1, \widehat{r}_2\big[, \ldots, \big[\widehat{r}_{v-2}, \widehat{r}_{v-1}\big] \cup \big[\widehat{r}_{v-1}, \mathcal{R}^{WPF}_{\boldsymbol{E}}\big]\Big] \tag{3.18}$$

where $\widehat{r}_k$ $(k = 1, 2, \ldots, v-1)$ are the switch points and $\mathcal{R}^{WPF}_{\boldsymbol{E}}$ is the maximum rate of profit of $w^{\text{WPF}}_{\boldsymbol{E}}(r,\eta)$. These intervals are relatively few with respect to the very large number of possible combination of methods.

Each interval, k, is the domain of a *wage-profit curve* that was generated by the set of methods $\mathbf{z}_{\{k\}}$. The whole set of methods that contribute to $w^{\text{WPF}}_{\boldsymbol{E}}(r,\eta)$ may be arranged in matrix notation as:

$$\mathbf{Z}^{\text{WPF}}_{\boldsymbol{E}} = \Big[\mathbf{z}^{\{1\}}, \mathbf{z}^{\{2\}}, \ldots, \mathbf{z}^{\{k\}}, \ldots, \mathbf{z}^{\{v\}}\Big] = \begin{bmatrix} z^{\{1\}}_{11} & z^{\{2\}}_{12} & \ldots & z^{\{v\}}_{1v} \\ z^{\{1\}}_{21} & z^{\{2\}}_{22} & \ldots & z^{\{v\}}_{2v} \\ \vdots & \vdots & \vdots & \vdots \\ z^{\{1\}}_{n1} & z^{\{2\}}_{n2} & \ldots & z^{\{v\}}_{nv} \end{bmatrix} \tag{3.19}$$

We have now all the elements defining *Samuelson's Surrogate Production Function*. The Suffix *WPF* would identify the values at the outer envelope defined by $\mathbf{Z}^{\text{WPF}}_{\boldsymbol{E}}$ 4.2. To sum up the relevant data is the following:

[10]The number of possible curves is enormous. In the database that we use, there are 31 sectors and 30 countries. This means that in order to determine the yearly *wage-profit frontier* we need to compute $31^{28} \approx 5.73 * 10^{41}$ *wage-profit curves*. Either one computes first all these curves or one should use an algorithm that reduces the computational time. We use this algorithm Zambelli et al. (2014)

Surrogate Production Function

$$\begin{aligned} &\textit{wage-profit frontier (eq. 3.17)}: && w^{\text{WPF}}(r,\eta) && (3.20)\\ &\textit{sectoral prices (eq. 3.6)}: && \mathbf{p}^{\text{WPF}}(r,\eta) && (3.21)\\ &\textit{aggregate output per worker 3.9}: && y_{val}^{\text{WPF}}(r,\mathbf{x}_{iso}^*(r),\eta) && (3.22)\\ &\textit{aggregate capital per worker 3.10}: && k_{val}^{\text{WPF}}(r,\mathbf{x}_{iso}^*(r),\eta) && (3.23) \end{aligned}$$

The *Surrogate Isoquant* would be given by the values that would generatre the same vlaue of the output:

$$\begin{aligned} &\textit{Isoquant - aggregate output}: && \bar{\mathbf{Y}}_{iso}^{\text{WPF}}(r,\mathbf{x}_{iso}^*,\eta) = \mathbf{x}_{iso}^{*}{}'[\mathbf{B}^{\text{WPF}} - \mathbf{A}^{\text{WPF}}]\mathbf{p}^{\text{WPF}}(r,\eta) && (3.24)\\ &\textit{Isoquant - aggregate capital}: && K_{iso}^{\text{WPF}}(r,\mathbf{x}_{iso}^*,\eta) = \mathbf{x}_{iso}^{*}{}'\mathbf{A}^{\text{WPF}}\mathbf{p}^{\text{WPF}}(r,\eta) && (3.25)\\ &\textit{Isoquant - aggregate labor}: && L_{iso}^{\text{WPF}}(r,\mathbf{x}_{iso}^*,\eta) = \mathbf{x}_{iso}^{*}{}'\mathbf{L}^{\text{WPF}} && (3.26) \end{aligned}$$

4 Empirical Verification

4.1 Data and the Choice of the *Numéraire(s)*

We use data from the World Input-Output Database Timmer (2012) which is publicly available and it provides detailed input-output data at the industrial level for 35 industries from 1995-2011. The data set is composed of national input-output tables of 40 countries that includes 27 EU countries and 13 other major industrial countries. For more details regarding the construction of Input-Output tables in WIOD database, see Dietzenbacher(2013). The unique aspect of the SEA is that it offers data at the industry level.

In this article we have confined the analysis to a subset of 30 countries. Furthermore we have reduced the total sectors or industries to 31 . We are considering only those industries that belong to the core of the 'production' system [11]. The National Input-Output tables (NIOT) have been adjusted so as to include the imports of means of production. Hence, the methods associated with each sector would be the inputs of internally produced goods plus the inputs of the imported goods. All the current period values have been appropriately adjusted using price indexes. For this, we have used the data on price series that are available in the Social and Economic Accounts (SEA) section of the WIOD database (Timmer (2012)).

Once the above adjustments have been made, we organize the means of production, labour inputs and the gross output as in the multi-dimensional matrix $\mathbf{\Phi}$. This enables us to enumerate all the possible combinations of methods of production with the vectors $\mathbf{z}$ and associate them to production systems formed by the triple: $\mathbf{A}^{\mathbf{z}}$, $\mathbf{L}^{\mathbf{z}}$, $\mathbf{B}^{\mathbf{z}}$. We have used this information to compute the yearly *wage-profit frontier* and the global inter–temporal frontier (and hence also all the methods used at the frontier, $\mathbf{Z}_{\boldsymbol{E}}^{\text{WPF}}$).

4.2 An example: global frontier for 2009.

The knowledge of $\mathbf{Z}_{\boldsymbol{E}}^{\text{WPF}}$ allows the computation of the *wage-profit frontier*. For example the outer envelope of all the *wage-profit curves* for the year 2009 reproduced in the North-West Graph of Fig. 4.1 is computed on the basis of the 64 curves that in 2009 did *dominate* all the other $31^{28} \approx 5.73 * 10^{41}$ curves, $\mathbf{Z}_{\boldsymbol{E}_{\mathbf{2009}}}^{\text{WPF}}$, where $\boldsymbol{E}_{2009} \subset \mathbf{\Phi}$.

Samuelson assumed that $y_{val}^{WPF}(r,\mathbf{x}_{iso}^*(r),\eta)$, eq.3.22, $k_{val}^{\text{WPF}}(r,\mathbf{x}_{iso}^*(r),\eta)$ eq.3.23, would have *neoclassical* properties.

[11] The excluded sectors are: Public Administration and Defence, Compulsory Social Security; Education; Health and Social Work; Private Households with Employed Persons.

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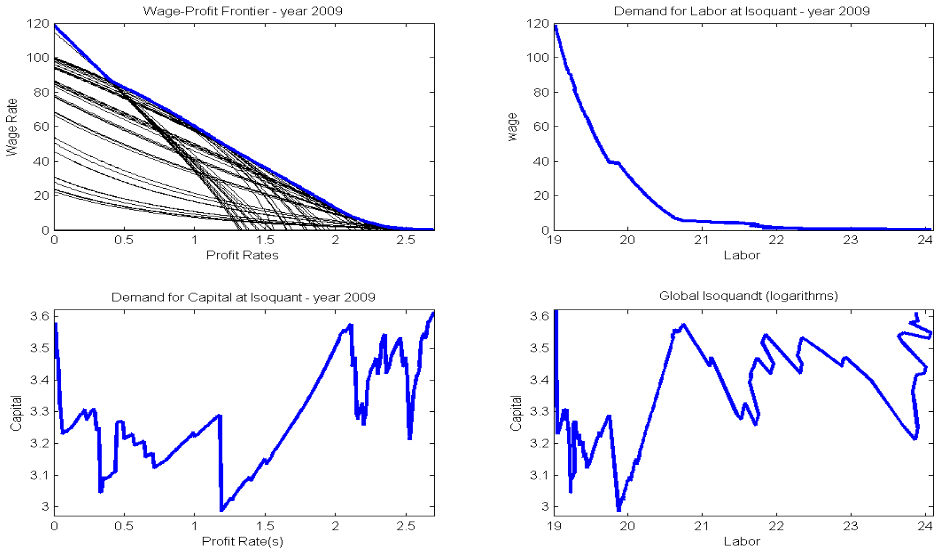

Figure 4.1: Year 2009: Aggregate Values and Heterogeneous Production. *The North-West Graph is the* wage-profit frontier *which is the envelope of the* wage-profit curves. *The North-East Graph,* The Demand for Labor at Isoquant *is the the quantity of labor necessary for the production of the same value quantity of the output, eq. 2.4. The South-West Graph is* the Demand for Capital *curve derived from eq. 2.3.The South-East Graph is* Isoquandt *curve derived from eq. 2.9*

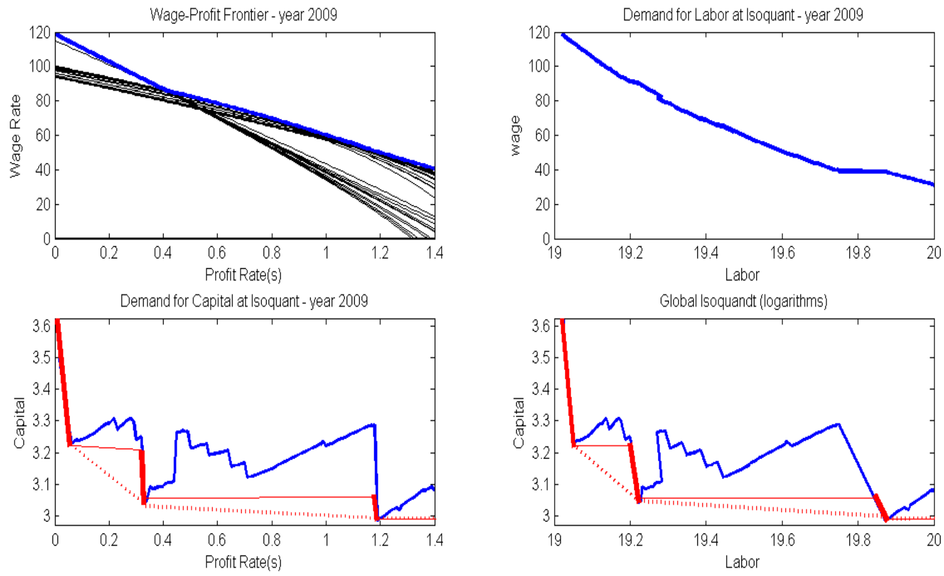

Figure 4.2: Year 2009. *Convexification.* Zoom-in of Fig. 4.1. The thick lines are the values Aggregate Values and Heterogeneous Production. *The North-West Graph is the coherent with the* neoclassical *postulates. The dashed line show* convexification.

We have computed the values of the system eqs. 3.20–3.23 from empirically observed data. We stress once more that we are able to perform these computations because we have for the first time the information of the methods present at the frontier, $\mathbf{Z}^{\mathrm{WPF}}_{\boldsymbol{E}_{2009}}$.

Fig. 4.1 reports some of the computations relative to 2009. The South-West Graph of Fig. 4.1 is the isoquant[12] which is computed following Samuelson 1962 *As-If* procedure. Clearly the isoquant fails to be coherent with respect to the neoclassical set of assumptions.

The *wage-profit frontier* (North-West Graph) and the *Demand for Labor at Isoquant* (North-East Graph) are negatively sloped and this feature is independent from whether the set of methods Φ does or does not have neoclassical properties. These features would apply to any set. It is well known and proven that the *wage-profit frontier* would be always negatively sloped (see Sraffa (1960); Samuelson (1962)). Also the *Demand for Labor at Isoquant* is negatively sloped with respect to the wage rate. But this is too to be expected: as the wage rate decreases the most efficient methods of production might be those that utilize more labor. The neoclassical requirement is that as more labor is employed at isoquant less capital should be employed. By an inspection of the South-West graph of fig. 4.1 it is clear that this is obviously not the case. At least it is not the case for the whole domain.

At first a scholar trained thinking in terms of marginal productivities and of substitution among factors may find the positive relation at isoquants between labor and capital quite disturbing and counterintuitive.

Although this is a somewhat unpleasant result, it is an actual possibility and in fact, as we will see, it is the normal case. As the profit rate and the wage rate change there is a change in accounting equilibrium prices and eventually and/or consequently there would also be a change in the most efficient methods of production.

A temptation is to *'fix'* this by *convexification*. Figure 4.2, which is enlarged with respect to fig. 4.1, shows the *convexified* isoquant. The thick lines would be consistent with respect to the neoclassical assumptions, but the envelope would not be convex. It is the dotted lines, which exclude from the mapping most of the thick lines, which would be coherent with the *neoclassical* assumption. This might look as a reconciliation with standard *neoclassical* approach, but it is wrong. To *convexify* would mean to exclude from the feasible efficient production possibility frontier highly efficient solutions.

If the figures were relative to homogeneous production it would be the case that to produce with higher physical inputs. capital and labor, an output that could be produced with lower physical inputs would be highly inefficient. If we lived in the *homogeneous-one-good-world* this procedure would be correct. But we should keep in mind that we are plotting the values that correspond to a system of heterogeneous production: each point corresponds to a different activity level and, most importantly, to a different set of methods, hence a different net national product to be produced and distributed.

In Fig.4.3 we see the gains in the value of NNP per worker if we assume the most efficient methods (keeping fixed the employment level, sectoral distribution and Capital/Labor ratio as in 2009). We can see that all the countries, also those that after *convexification* as in Fig. 2.1 would have been considered most efficient according to standard methods, would be able to produce at a much higher level. To *convexify* would mean to force the systems to produce at highly sub-efficient levels.

It is now important to understand whether the results relative to 2009, can be extended to the whole period going from 1995 to 2009. Furthermore, it is also important to assess the dependence of the results with respect to some of the elements used for the investigation.

[12] We are now in the position of computing the isoquant associated with the *wage-profit frontier* which is derived from *heterogeneous* production.

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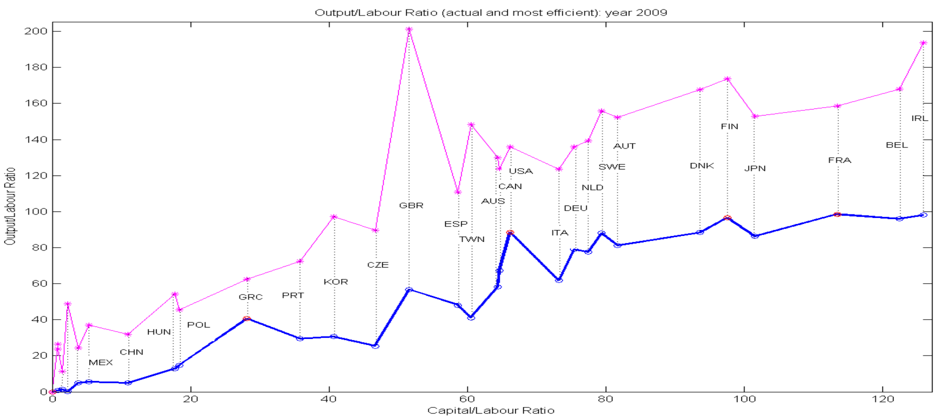

Figure 4.3: Output/Labor Ratio versus Capital/Labor Ratio - 2009, *Data from Timmer (2012). In the lowest line are the actual historical values as in Fig. 2.1. The highest lines are the values associated with the efficient production methods computed keeping the same as the historical observation labor distribution among the sectors and the same capital/labor ratio. The values for aggregated Capital and Output are in euro 1995 as in WIOD Tables*

4.3 Results and their Robustness

We are now in the position to verify whether the *Surrogate Production Function* has *neoclassical properties*. We have computed the values of the *Surrogate Production Function*, *As-If CES* or *As-If Cobb-Douglas* relative to the period going from 1995 to 2009 and with a set of different *numéraires*[13].

We have the following results.

1. **Neoclassic properties of the isoquants.** *Neoclassical* properties require that

$$\frac{\Delta K_{iso}^{\text{WPF}}(r, \mathbf{x}_{iso}^{*}, \eta)}{\Delta L_{iso}^{\text{WPF}}(r, \mathbf{x}_{iso}^{*}, \eta)} \leq 0 \tag{4.1}$$

   (a) **Global Result** $\forall r \in [0, \mathcal{R}_{\boldsymbol{E}}^{WPF}]$. A single *numéraire* where the above condition holds for the whole domain has not been found.

   (b) **Partial Result**. We compute the ratio of the sum of the intervals of the domain r where the above relation holds - numerator - with the total domain ($r \in [0, \mathcal{R}_{\boldsymbol{E}}^{WPF}]$) - denominator. The above relationships holds on average (of the *numéraires*) only for **6.95**% (standard deviation 0.79%) of the domain of the isoquant.

   Comment on the Isoquant. This is an important result that demonstrates that the *Surrogate Production Function* is not *neoclassic*.

2. **Capital/Labor Ratio**. Neoclassical Properties require that:

[13] The set of *numéraires* is composed of the standard commodities (see Sraffa (1960, pp.18-25)) relative to all the *wage-profit curves* that contribute to the 15 yearly *wage-profit frontiers*. For the year 1995 the frontier is formed of 79 curves for which we have 79 standard commodities, i.e. 79 *numéraires*, for 1996 of 74 curves, for 1997 of 59 curves and so on. For the 15 years we a total of 1004 standard commodities, hence a total of 1004 *numéraires*. For each standard commodity we have computed the values, eqs. 3.20–3.23 and eqs. 3.24–3.26, for each year. We have 15060 instances (1004 x 15 years)

$$\Delta k_{val}^{\text{WPF}}(r, \mathbf{x}^*(r), \eta) \leq 0 \tag{4.2}$$

(a) **Global Result** $\forall r \in [0, \mathcal{R}_{\boldsymbol{E}}^{WPF}]$. The Capital/Labor ratio is negatively sloped for the whole domain for **4.46**% of the instances (671 times over 15060).

(b) **Local Result**. We compute the ratio of the sum of the intervals of the domain r where the above relation holds - numerator - with the total domain ($r \in [0, \mathcal{R}_{\boldsymbol{E}}^{WPF}]$) - denominator. The above relationships holds on average **80**% (standard deviation 13.7%) of the domain.

Comment on the Capital/Labor Ratio. Clearly, as the profit rate increases the wage rate decreases. This means that the cost of labor would tend to decrease. Hence for most sectors it might become more convenient to shift towards methods of production where less capital in value is used. Nevertheless this is not to imply anything with respect to the cost of the capital used which would in turn depend on the new accounting equilibrium production prices, which do also change. It is still possible that the production prices are such that as total labor increases also capital increases. Only for **6.95%** of the cases it is true that $\Delta k_{val}^{\text{WPF}}(r, \mathbf{x}_{iso}^*(r), \eta) \leq 0$.

3. **Output/Capital Ratio**. Neoclassical Properties require that:

$$\Delta\Big(\frac{y_{val}^{\text{WPF}}(r, \mathbf{x}^*(r), \eta))}{k_{val}^{\text{WPF}}(r, \mathbf{x}^*(r), \eta))}\Big) \geq 0 \tag{4.3}$$

(a) **Global Result** $\forall r \in [0, \mathcal{R}_{\boldsymbol{E}}^{WPF}]$. The Output/Capital Ratio is never positively sloped for the whole domain.

(b) **Local Result**. We compute the ratio of the sum of the intervals of the domain r where the above relation holds - numerator - with the total domain ($r \in [0, \mathcal{R}_{\boldsymbol{E}}^{WPF}]$) - denominator. The above relationships holds on average **1.4**% (standard deviation 0.79%).

Comment on the Output/Capital Ratio. This result is very surprising. As the profit rate increase the wage rate decreases and hence the cost of the labor used decreases. There is substitution in the methods of production. As part of the Cambridge Capital Controversy it is known as a possibility that as the profit rate increases it is possible that the output/capital ratio decreases, but the normal case should be the one in which it increases. But we see here that it increases for only for **1.4**% of the times. We remind the reader that the departure of this analysis is actual data and that we have computed the frontier most efficient methods.

Clearly, as the profit rate increases the wage rate decreases. This means that the cost of labor would tend to decrease. Hence for most sectors it would be more convenient to shift towards methods of production where less capital value is used. Nevertheless this is not to imply anything with respect to the cost of the capital used which would in turn depend on the new accounting equilibrium production prices, which do also change. It is still possible that the production prices are such that as total labor increases also capital increases. Only for 6% of the cases it is true that $\Delta k_{val}^{\text{WPF}}(r, \mathbf{x}_{iso}^*(r), \eta) \leq 0$. Therefore the normal case would be that as the profit rate increases the value od capital increases more than the produced output.

4. **Price Monotonicity**. The *neoclassical* principle requires that the sectoral prices change monotonically as the profit rate change. We have checked whether this happens.

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(a) **Global Result** $\forall r \in [0, \mathcal{R}_{\boldsymbol{E}}^{WPF}]$. There is no instance where the prices of all the sectors are monotonic functions of the profit rate.

(b) **Local Result**. The average of the percentage of the sectors for which the prices behave monotonically is **14%** (which is equivalent to 4-5 sectors on a total of 31 sectors where prices behave monotonically). The standard deviation is 6% (which is equivalent circa to one sector).

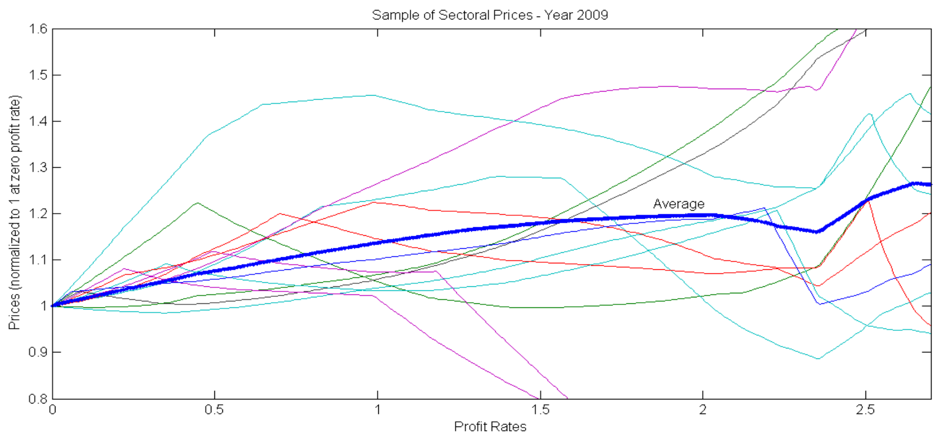

Figure 4.4: Normalized Sectorial Prices, Year 2009. The sectoral prices have been computed as in 3.22. Here they are normalized. The *numéraire* is the standard commodity relative to the first *wage-profit curve* of the 2009 *wage-profit frontier*

Comment on the Capital/Labor Ratio. The prices here are the prices associated to the *wage-profit frontier*. The fact that they do not behave monotonically is contrary to the notion of capital intensive or labor intensive methods which should be a necessary condition for a set of methods to be characterized as *neoclassical*. As an example we report a selection of sectoral prices (normalized to 1 for $r = 0$). relative to the year 2009 *wage-profit frontier*. For that year only two sectors have price functions that are monotonic and the remaining 29 have clear non monotonic properties. Fig. 4.4 shows only 13 these 29 functions, i.e. those that non only are non monotonic, but that do cross the average of all the prices: *neoclassical* characteristics would require that they should not.

5. **Elasticity of Substitution**. We have computed the actual elasticity of substitution for each year.

$$\sigma^{\mathrm{WPF}}(r,\eta) = \frac{\Delta ln(k_{val}^{\mathrm{WPF}}(r, \mathbf{x}^*(r), \eta))}{\Delta ln\left(\frac{w^{\mathrm{WPF}}(r,\eta)}{r}\right)} \tag{4.4}$$

For each set of methods associated to the yearly *wage-profit frontiers* and *numéraire.*

(a) **Result** . At switch points the elasticity would be very high (given the lack of smoothness or non differentiability). We have escluded the values at switch points. Here we present the results as the average across the profit rate and for all the instances. The average of the average elasticities has the value **0.06** (standard deviation 0.014).

Comment on Elasticity of Substitution. The averages of the standard deviations is 0.14, which indicates a high degree of variation as the profit rate changes. It is also clear that 0.06 is very far from 1.

5 Aggregate production functions are *NOT* neoclassical

Solow (1962), in an article with a clear title *Substitution and fixed proportions in the theory of capital* started the exposition by saying

> I have long since abandoned the illusion that participants in this debate actually communicate with each other.

But communication of one sort or the other has been always going on. Samuelson (1962) has tried to set the foundations so that a system of heterogeneous production could be represented *As-If* it was homogeneous production. Samuelson's Surrogate Production Function was challenged during the 60s' *Cambridges Capital Controversy*. The Special Issue of the Quarterly Journal of Economics, known also as the "QJE Symposium" ", was devoted entirely to the debate. In that occasion Samuelson admitted the existence of problems:

> Pathology illuminates healthy physiology. Pasinetti, Morishima, Bruno-Burnmeister-Sheshinski, Garegnani merit our gratitude for demonstrating that reswitching is a logical possibility in any technology, indecomposable or decomposable ... There often turns out to be no unambiguous way of characterizing different processes as more "capital-intensive," more "mechanized," more "roundabout" ... If all of this causes headaches for those nostalgic for the old time parables of neoclassical writing, we must remind ourselves that scholars are not born to leave an easy existence. We must respect, and appraise, the facts of life (Samuelson (1966, p.582-3)).

What was recognized as a pathology was the possibility that for some regions in the domain of the profit rate r it could be the case that the change in the value of the capital/labor ratio $\Delta k_{val}^{\mathrm{WPF}}(r, \mathbf{x}^*(r), \eta)$ would be positive, and not negative as required by *neoclassical theory*[14].

Like Samuelson also other did admit the existence of this problem, but it was considered to be a *pathology*, or a *perversity* or a *paradox*. Lacking of empirical evidence there has been a general tendency to declare a sort of faith on the tenability of the *neoclassical* cases (see on this Carter (2011)). At the end of the 60s Ferguson wrote:

> [The] validity [of the Cambridge Criticism of neoclassical theory] is unquestionable, but its importance is an empirical or an econometric matter that depend upon the amount of substitutability there is in the system. Until *econometricians have the answer for us* placing reliance upon neoclassical economic theory is *a matter of faith. I personally have the faith* (Ferguson (1969, p. xv; emphasis added)).

It so happened that also others declared their *faith*. Surprisingly *econometricians* never really delivered a satisfactory answer. Macroeconomic theory has adopted the *Robinson Crusoe* type of models where capital is homogeneous with output and the Cobb-Douglas or CES production

[14] This characteristic is also known as *Capital Reversing* or *Wicksell Effect*. One special case of *Capital Reversing* is that of *reswitching*, i.e. the case where methods of production that were efficient for high profit rates would be efficient again for low profit rates. In the case of *reswitching* this feature is independent of the chosen *numéraire* η. It is a case which is indisputably non *neoclassical*. Unfortunately the attention of many authors has been on *reswitching* and not on *capital reversing*. Just for the record for the 15 yearly *wage-profit frontier* that we have computed which are the envelope of more than 50 *wage-profit curves* we did not find a single instance of *reswitching*, but as we have seen *capital reversing* is extremely frequent.

function is assumed to hold, but has no empirical justification: aggregation problems are ruled out by assumption.

The highly questionable procedure of assuming a priori a Cobb-Douglas like production functions has not been abandoned also when authors have seriously disputed the statistical validity of the empirical estimation Simon and Levy (1963); Simon (1979); Shaikh (1974) of the aggregated function itself. What these authors have shown is that the seemingly robust estimation results are due to the *"Laws of Algebra"*, i.e., practically any data can be fitted with a Cobb-Doublas like production function. Hence the Cobb-Douglas and the CES production functions cannot be taken as representations of production. These findings have been ignored by most of the profession. One reason is that if it is the case that a Cobb-Douglas can fit almost anything, it is also the case that it could fit also production data that could be Cobb-Douglas. A tautology does not need to be verified. We have shown, see Fig. 4.3, that to convexify is not only methodologically improper, but it leads also to wrong conclusions with respect to the efficiency and the estimates of potential output. Through convexification information on efficient heterogeneous methods of production is lost.

In this article we have computed the methods belonging to the *wage-profit frontier* and we have computed all the possible *surrogate as-if* aggregated values following the rigorous and robust methods as suggested in Samuelson (1962) (see eqs.3.20–3.23 and eqs.3.24–3.26).

Subsequently, in the previous section, 4.3, we have verified whether the surrogate values have *neoclassical* properties. It has been shown that the *As-if aggregate production functions* are never *neoclassical* for the whole domain: the computed isoquants are never neoclassic and the capital/labor ratio is negatively sloped for the whole domain only for 5% of the *numéraires*[15]. Furthermore, also the Elasticity of Substitution, eq.4.4, is far from being constant and is very far from being equal to 1.

6 Conclusion

In this paper we have been able to verify, for the first time, whether the empirically generated data justify the artificial construction and use of the *"Af-If Neoclassical Production Function"*. The set of methods Φ, from which the most efficient methods Z have been extracted is not neoclassical in the sense that the aggregated values are far from being coherent with respect to set of *assumptions* described in section 2.

The negative results are robust and demonstrate that aggregate production functions are not *neoclassical*. We stress once more that this verification is done here for the first time. The reason is that it is only now that the *wage-profit frontier* for a large number of methods (and countries) could be computed Zambelli et al. (2014) thanks to an algorithm that reduced the computational complexity. The conclusion is that negatively sloped capital/labor ratios are not at all pathologies and isoquants are not at all neoclassical. The implications are important.

The *demand for capital* at isoquant would be always similar to the one depicted in the South-West graph of fig. 4.1.

The results presented here might turn out to be surprising for all those trained inside the *neoclassical* framework. The fact that aggregate CES or Cobb-Douglas production functions do not hold neoclassical properties has very strong implications. Just to mention one: the idea that the interest rate is associated with the rate of profit which is determined by a notion of marginal productivity of capital, as in eq. xxxx. is not sustainable. Consequently also the notion of the wage rate being determined by the marginal productivity of labor is not meaningful. These are

[15] In Zambelli (2004) an investigation was conducted with virtual production systems. In that context it was computed that the capital/labor ratio would be negatively sloped 40% of the times. Here, with real data, it is shown that it is negatively sloped only 5% of the times and it is never independent of the *numéraire* chosen.

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two of the *"fundamentals"* of an economic system.

If the results presented here hold this means that also the standard notion of *"fundamentals"* has to be subjected to revision. Surely what has to be reconsidered is th notion of a *neoclassical* aggregate production function.

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Appendix A discussion on possible deficiencies of the method used

1. **High degree of methods of production substitutability**. It can be said that the set of methods $\mathbf{\Phi}$ is made of all possible methods or blueprints coming from nations that have very different characteristics. To assume that the different methods could be combined so as to form the methods at the frontier $\mathbf{Z}^{\text{WPF}}_{\boldsymbol{E_{year}}}$, where $\boldsymbol{E}_{year} \subset \mathbf{\Phi}$ is a very strong assumption. An assumption that requires high degree of substitutability across national systems of production.

   Response. The set of methods $\mathbf{Z}^{\text{WPF}}_{\boldsymbol{E_{year}}}$ does represent a benchmark. To consider this benchmark non feasible because some methods of production have only local applicability is reasonable. Alternatively one could pick a subset $\boldsymbol{E'_{year}} \subset \boldsymbol{E_{year}} \subset \mathbf{\Phi}$ and derive the set of methods at the frontier associate to it, $\mathbf{Z}^{\text{WPF}}_{\boldsymbol{E'}_{year}}$. We have tried to construct subsets of Φ with respect to regional vicinity (for example North America, South America, East Asia, South Asia, North Europa, South Europa and so on) and the results are qualitatively similar to the results presented here. On one extreme we have the maximum degree of substitutability, as we have in this paper, and on the other we would have no substitutability. This would be the case in which the methods are Nation dependent. In that case we would operate only with *wage-profit curves*. Also in that very simple extreme the question can be posed on whether the system properties that are necessary for them to be considered *neoclassical*. For reasons of space we do not deal with that issue here, but we have studied the problem in the companion piece Zambelli (2014). There we consider the data for 15 years and 30 countries. We have studied the 450 different systems and have studied their characteristics with variants of the aggregate values as with the eqs.3.20–3.23 and eqs.3.24–3.26, where we replace the *wage-profit frontiers* with the *wage-profit curves*. Also for the assumption of non-substitutability the conclusion is that there is empirical evidence in support of non *neoclassical* properties.

2. **Fixed versus flexible proportions of the means of production.** Here we have considered the entries of the input-output tables, eq. 3.1, as methods of production (blue prints). There are two alternative ways to deal with the observations: one is to consider each observation as just one method of production and the other to consider all the observations as points of the same production function.

   $$f(a^{z_i}_{i1}, a^{z_i}_{i2}, \ldots, a^{z_i}_{in}, \ell^{z_i}_{i}) = b^{z_i}_{i} \tag{6.1}$$

   where $z_i = 1, ..., s_i$. In this case there would be s_i observations from which to estimate the above function.

   Response. It has to be pointed out that the important issue is not whether there is substitutability among means of production, but whether the production function f has the required *neoclassical* properties. The estimation of f has to be done properly. To assume *a priori* that f is like a Cobb-Douglas or that it is a CES production function is statistically improper (see Simon and Levy (1963); Simon (1979); Shaikh (1974)).

   In the approach that we have presented here we are not making *restrictions* on the structure of Φ. Therefore the set of the most efficient methods of production $\mathbf{Z}^{\text{WPF}}_{\boldsymbol{E'}_{year}}$ could embed substitutions in the means of production coherent with *neoclassical* assumptions, but this is not the case. The isoquants are all of the types shown in fig. 4.1, and not at all like the ones of fig. 3.1. This should provide evidence that if the function f exists it may not necessarily be neoclassical.

3. **Fixed versus Circulating Capital**. Neoclassical production functions have as inputs

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fixed capital and labor while here we are working exclusively with circulating capital, i.e. the means of production used during the year. **Response.** Our measurement of capital is based on eq. 3.10 $K^{\mathbf{z}}_{val}(r, \mathbf{x}, \eta) = \mathbf{x}'\mathbf{A}^{\mathbf{z}}\mathbf{p}^{\mathbf{z}}(r, \eta)$. This might not be satisfactory, but it is difficult to figure out alternative ways to measure capital. In the case of the aggregate neoclassical production function capital is homogeneous with output. Estimates of the so-called fixed capital are based on the *perpetual inventory methods* which uses the yearly flow values of the input-output tables. Although attempts to find robust indexes (or proxies) to measure capital as if it was a physical magnitude are made the point of departure are transformations that directly use the input-output tables observations. A fraction of the produced value of output is assumed to be used for the fixed capital formation (investment) while a fraction δ is the assumed depreciation of capital. The issue is whether as the factor prices change the use of the factors of production should be negatively related. This should occur also for the circulating capita, which is an important correlated component with respect to the total value of capital.

4. **Production prices are not market prices**. It might be argued that the prices that we use to compute the values of aggregate output and capital are not market prices but virtually generated prices and hence are not relevant.

   Response. Production prices are the prices that would be necessary for the system to be able to reproduce itself. They are accounting equilibrium prices. Hence market prices should have values around these prices. Furthermore, these production prices should as well follow the scarcity principle. This means that as the profit rate increases the relative prices of the more capital intensive sectors should decrease faster than the prices of the more labor intensive capital. They should anyway change monotonically as the profit rate and wage rate change.

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