A Dynamic Exchange Rate Model with Heterogenous Agents*

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May 15th, 2014
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Abstract

In this paper, we analyze an heterogenous agent model in which the fundamental exchange rate is endogenously determined by the real markets. The exchange rate market and the real markets are linked through the balance of payments. We have, analytically, determined that there exists at least a steady state in which the exchange rate is at its fundamental value and incomes of both countries are equal the autonomous components times the over-simplified multiplier (as in the Income-Expenditure model). That steady state can be unique and always unstable when all agents act as contrarians, while when agents act as fundamentalists is unique but its stability depends on reactivity. Finally, we demonstrate that the (in)stability of the economic system depends on both the reactivity of the markets and that of different agents involved.

Keywords: Complex Dynamics; Heteroneous Agents Models; Financial Markets.

JEL classification: C62, D84, E12, E32, G02

1 Introduction

In the last decades, models of financial markets have shown how complex dynamics of price fluctuations are related to the interactions between heterogenous agents. Heterogeneity is either related to the strategies applied (fundamentalists, chartists, contrarians, noise traders) or may emerge in the beliefs about the fundamental value (see for surveys Hommes, 2006, LeBaron 2006 and Westerhoff, 2009).

*Aknowledgements: We thank Marco Dardi who discuss with us an early version of the paper and the participants of two workshops held in Scuola Normale di Pisa in December 2013 and Univerisità Milano-Bicocca in February 2014.

In general, there is a lot of uncertainty on what the "true fundamental value" is. This is due to a strong subjective dimension of the estimation procedure: on the one hand agents do not necessarily follow the same 'structural model of the world', estimating the fundamental value in different ways; on the other hand, even when they use the same model, agents may hardly reach the same expectations because of different computational skills. The latter kind of heterogeneity is already incorporated in Brock and Hommes (1998) that contains a 3-type asset-pricing model with fundamentalists, upward and downward biased traders. We could interpret this example as a model with three groups of fundamentalists with trader types, one with the correct estimate of the fundamental and two other types with a wrong estimate of the fundamental. In this line of research, De Grauwe and Rovira (2012) and Rovira (2010) study the effect of *biased traders* (optimists or pessimists) in the exchange market, while Naimzada and Ricchiuti (2008, 2009) have developed a two-type version of biased traders model, with a market maker structure, in which the source of instability resides in the interaction of fundamentalists with heterogeneous beliefs about the fundamental value of the asset: complex dynamics arise for a greater distance in beliefs.

A key shortcoming of this literature is that the fundamental value is either fixed (as in the cases cited above) or time varying (as in Anufriev, 2013) but, as stated by De Grauwe and Rovira (2012), *'it still is exogenously determined. That is, it is not connected to the real part of the economy in any way'*. To the best of our knowledge only few papers have dealt with an endogenous fundamental value, in particular Lengnick and Wohltmann (2011), Westerhoff (2012) and Naimzada and Pireddu (2013a, 2013b).

Lengnick and Wohltmann (2011) combine an agent based model with a new-keynesian model. In their model, the fundamental value is fixed but it is assumed that the expectations vary over time following the unconditional standard deviation of the output gap. On the other hand, Westerhoff (2012) linked a stock market with heterogeneous speculators with a Keynesian goods market model for a closed economy. He assume that the performance of the stock market affects both consumption and investment and, at the same time, is affected by the national income. Naimzada and Pireddu (2013a) introduce two important differences to the Westerhoff's model: firstly, they use of a nonlinear adjustment for the real market and, secondly, they assume agents make their decision using a linear combination between an exogenous and an endogenous value of both national income and level of stock market. The parameter used to determinate the weighted average is a proxy of markets segmentation and, therefore, may be viewed as a policy that reduces the feed-back effects between markets.

In line with these models and recalling the De Grauwe and Rovira's (2012) suggestion, we propose and analyze an exchange rate model in which the long-run (fundamental) value of the exchange rate endogenously depends on the balance of payment. At the same time, the incomes of the two countries depend on the exchange rate. We combine a simple agent based model for the exchange rate (similar to those developed by De Grauwe and Rovira and Naimzada and Ricchiuti cited above) and the Keynesian goods market for two Countries (the Absorbtion Model)[1].

We have analytically found that there is at least a steady state in which the exchange rate is equal to its fundamental value, that is, the exchange rate for which the balance of payment is equal to zero, and incomes of both countries are equal the autonomous components times the over-simplified multiplier: they do not depend on the marginal propensity to import, so that a flexible exchange rate can isolate the two economic systems. Moreover, the steady state is unique and unstable when all agents act as contrarians, and it is unique and may be unstable when all agents act as fundamentalists depending on the reactivity of markets and agents.

[1]Our model is really close to that developed by Laursen and Metzler in 1950. The Laursen and Metzler model (1950) is static, while its continuous time dynamic version can be found in Gandolfo (1986).

2 Set up of the model

There are two countries (Europe and US), who freely trade without restrictions and who have both a flexible exchange rate regime. As in Laursen and Metzler (1950) we assume that *'foreign exchange may be purchased and sold freely for all purposes except capital movement'*: that is the financial account is closed and the balance of payment depends only from the current account[2]. Therefore, the two Keynesian goods markets are linked only through imports and exports, whose value depend on the exchange rate. We firstly present the exchange rate determination and then the two Keynesian goods markets.

2.1 Exchange Rate Determination

Let us define $E(t)$ the exchange rate at time t between the two countries, expressed as dollars per euros. The actual exchange rate is determined by a market maker who looks at the excess of demand of euros in the system, so that exchanges are possible out of equilibrium. The law of motion of the exchange rate is as follows:

$$E(t+1) = E(t) + g_3(ED(t))$$

where g_3 is the speed of adjustment of the market maker with respect to the excess of demand $ED(t)$. In line with the Heterogenous Agents Models, we assume that there are agents with heterogenous strategies (fundamentalists vs. contrarians, the difference will be explained below) and/or beliefs about the fundamental value. Moreover, similarly to Kirman (1998), our model involves agents who act on the currency market on the basis of the forecast developed by *gurus/experts* who have developed their own mechanism to predict the future exchange rate, based on the current exchange rate and its fundamental value, that is the one that makes the balance of payment be equal to zero.

Particularly, given that the balance of payment is the difference between exports and imports, that exports of country 1 are the imports of country 2 (and vice versa) and that, for every $i \in \{1, 2\}$, the imports of a country i at time t is equal to $mpi_i Y_i(t)$, where mpi_i is the marginal propensity to import of country i and $Y_i(t)$ is its income at time t, we have that the balance of payment is equal to zero when

$$\frac{mpi_2 Y_2(t)}{E(t)} - mpi_1 Y_1(t) = 0;$$

and the fundamental value $\tilde{F}(t)$ of the exchange rate at time t is straightforward:

$$\tilde{F}(t) = \frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}.$$

Assuming there are n gurus, where $n \in \mathbb{N}$, we suppose that, for every $j \in \{1, \ldots, n\}$, the guru j determines its forecasting on the exchange rate at time t, say $\tilde{E}_j(t)$, looking at the ratio $\tilde{F}(t)$ and the exchange rate $E(t)$ and computing:

$$\tilde{E}_j(t) = E(t) + k_j\left(\tilde{F}(t), E(t)\right) = E(t) + k_j\left(\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}, E(t)\right)$$

where k_j is a bias based on their own set of information. We assume that, for every $j \in \{1, \ldots, n\}$, $k_j : \mathbb{R}^2 \to \mathbb{R}$ is a regular function such that, for every $s \in \mathbb{R}$, $k_j(s, s) = 0$. Therefore the only constraint on k_j is that when the guru j observes that $\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)} = E(t)$,

[2] We know that nowadays it is a strong assumption, but we prefer to relax it in a further work.

she believes that in the future the exchange rate will not change. According to the related literature on the topic, we also say that guru j is fundamentalist if

$$\text{for every } (z_1, z_2) \in \mathbb{R}^2, \ \text{sign}(k_j(z_1, z_2)) = \text{sign}(z_1 - z_2), \tag{1}$$

while we say she is contrarian if

$$\text{for every } (z_1, z_2) \in \mathbb{R}^2, \ \text{sign}(k_j(z_1, z_2)) = -\text{sign}(z_1 - z_2), \tag{2}$$

We also denote by $k : \mathbb{R}^2 \to \mathbb{R}^n$ the function such that, for every $(z_1, z_2) \in \mathbb{R}^2$, $k(z_1, z_2) = (k_j(z_1, z_2))_{j=1}^n$.

Moreover, the financial operators carry out their demand of currency based on the suggestions of a specific guru j, specifically, the agents that follow the guru j at time t are $\pi_j(t)$. In this literature quotas may be either fixed (Day and Huang, 1990) or variable; in the latter case agents can switch from one expert to the next, following for example an adaptive belief system (as in Brock and Hommes, 1998). The switching mechanisms are usually based either on profitability (the agents choose the trading rule or the guru according to its capacity to make profits) or on fitness (agents choose the rule or the guru that indicate the closer proximity to the actual price). We do not assume a specific functional form for the switching mechanism, requiring just that:

$$\pi_j(t) = \lambda_j \left(k \left(\tilde{F}(t), E(t) \right) \right),$$

the only assumptions on λ_j are that for every $j \in \{1, \dots, n\}$, $\lambda_j : \mathbb{R}^n \to (0,1)$ is $C^1(\mathbb{R}^n)$ and, for every $x \in \mathbb{R}^n$, $\sum_{j=1}^n \lambda_j(x) = 1$. Therefore, the quotas may vary on time and agents may follow that guru who forecasts better than the others the actual exchange rate when they do not believe on the possibility of strong leaps of the variation of the exchange rate.

Finally, there is a market maker who operates adjusting the price according to the aggregate excess demand of euros:

$$E(t+1) = E(t) + g_3 \left(\sum_{j=1}^n \lambda_j \left(k \left(\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}, E(t) \right) \right) \gamma k_j \left(\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}, E(t) \right) \right), \tag{3}$$

where $\gamma > 0$ is a constant that transform the dollars in euros and represent the reactivity of agents, and $g_3 : \mathbb{R} \to \mathbb{R}$ is a regular function such that $g_3(0) = 0$ and, for every $s \in \mathbb{R}$, $g_3'(s) > 0$. For every $j \in \{1, \dots, n\}$, the quantity

$$k_j \left(\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}, E(t) \right)$$

represents the bias of the guru j, while:

$$\gamma k_j \left(\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}, E(t) \right)$$

is the excess of demand of any single agent who adheres to the guru j. Finally,

$$\sum_{j=1}^n \lambda_j \left(k \left(\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}, E(t) \right) \right) \gamma k_j \left(\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}, E(t) \right)$$

is the aggregate excess of demand of euros. Given the effect on the balance of trade, the exchange rate affects the GDP of both countries which afterwards have a feedback effect on the exchange rate through its fundamental value.

2.2 Absorbtion Model

From the income-expenditure model, we know that that a generic demand of goods is given by the sum of expenditures (consumptions, investments, government expenditure and net export):

$$D = \bar{C} + I + G + X - \bar{Z} = C + mpcY + I + G + X - mpiY = A + mpcY + X - mpiY \quad (4)$$

where A is the sum of the autonomous components of demand $C+I+G$, mpc is the marginal propensity to consume, and mpi is the marginal propensity to import. In all the following reasonings, A is assumed to be positive and $0 < mpi < mpc < 1$.

Let $Y_1(t)$ be the income of country 1 (let's say Europe) at time t in its own currency (euros), $Y_2(t)$ the income of country 2 (USA) at time t expressed in dollars. As stated above, the exports of one country are the imports of the other country, so that in each period the demand/expenditure of European citizens in euros is:

$$D_1(t) = A_1 + mpc_1 Y_1(t) + \frac{mpi_2 Y_2(t)}{E(t)} - mpi_1 Y_1(t) \quad (5)$$

and that of US citizens in dollars is:

$$D_2(t) = A_2 + mpc_2 Y_2(t) + mpi_1 Y_1(t) E(t) - mpi_2 Y_2(t). \quad (6)$$

Moreover, for every $i \in \{1, 2\}$, the GDP of country i at time $t+1$ depends on its previous value and the excess of demand at time t:

$$Y_i(t+1) = Y_i(t) + g_i(D_i(t) - Y_i(t)) \quad (7)$$

where g_i is a C^1 function defined on $\mathbb{R}$ such that the first derivative is positive and $g(0) = 0$[3]. Please note that, with a fixed exchange rate ($E_t = E$) we would achieve the well-known absorbtion model studied in a normal course of International Economics. Moreover, the model is close to that developed by Laursen and Metzler (1950).

Given equation 3 and the GDP in both countries (eq. 7), we have to study the following discrete dynamical system:

$$\begin{cases} Y_1(t+1) = Y_1(t) + g_1\left(A_1 + mpc_1 Y_1(t) + \frac{mpi_2 Y_2(t)}{E(t)} - mpi_1 Y_1(t) - Y_1(t)\right) \\ Y_2(t+1) = Y_2(t) + g_2\Big(A_2 + mpc_2 Y_2(t) + mpi_1 Y_1(t) E(t) - mpi_2 Y_2(t) - Y_2(t)\Big) \\ E(t+1) = E(t) + g_3\left(\sum_{j=1}^{n} \lambda_j \left(k\left(\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}, E(t)\right)\right) \gamma k_j \left(\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}, E(t)\right)\right) \end{cases} \quad (8)$$

where the functions Y_1, Y_2 and E are assumed to be positive.

3 Dynamic analysis

Consider the functions $r_1 : \mathbb{R}_{++} \to \mathbb{R}_{++}$ defined, for every $s \in \mathbb{R}_{++}$, as

$$r_1(s) = \frac{A_1(1 + mpi_2 - mpc_2) mpi_1 s + A_2 mpi_1 mpi_2}{(1 + mpi_1 - mpc_1)(1 + mpi_2 - mpc_2)s - mpi_1 mpi_2 s},$$

[3] In Westerhoff (2012) this is linear while for Naimzada and Pireddu (2013a) is sigmoidal.

and $r_2 : \mathbb{R}_{++} \to \mathbb{R}_{++}$ defined, for every $s \in \mathbb{R}_{++}$, as

$$r_2(s) = \frac{A_2(1 + mpi_1 - mpc_1)mpi_2 + A_1 mpi_1 mpi_2 s}{(1 + mpi_1 - mpc_1)(1 + mpi_2 - mpc_2) - mpi_1 mpi_2}.$$

The following proposition is our main result. Its proof is in the appendix.

Proposition 1. *The set of steady states of the system* (8) *is given by*

$$S = \left\{ \left(\frac{r_1(E)}{mpi_1}, \frac{r_2(E)}{mpi_2}, E \right) \in \mathbb{R}^3_{++} : E \in \Gamma \right\}$$

where

$$\Gamma = \left\{ E \in \mathbb{R}_{++} : \sum_{j=1}^{n} \lambda_j \left(k \left(\frac{r_2(E)}{r_1(E)}, E \right) \right) k_j \left(\frac{r_2(E)}{r_1(E)}, E \right) = 0 \right\}. \quad (9)$$

In particular,

$$(Y_1^*, Y_2^*, E^*) = \left(\frac{A_1}{1 - mpc_1}, \frac{A_2}{1 - mpc_2}, \frac{A_2\, mpi_2\, (1 - mpc_1)}{A_1\, mpi_1\, (1 - mpc_2)} \right) \quad (10)$$

belongs to S and is the unique steady if, for every $(z_1, z_2) \in \mathbb{R}^2_{++}$,

$$\text{sign}(k_1(z_1, z_2)) = \ldots = \text{sign}(k_n(z_1, z_2)).$$

Moreover, defining [4]

$$R = g_3'(0)\gamma \sum_{j=1}^{n} \lambda_j(\mathbf{0}) \frac{\partial k_j}{\partial z_1} (E^*, E^*), \quad (11)$$

we have that (10) *is unstable if one of the following set of conditions holds true:*

A1) $R < 0$;

A2) $\frac{2}{1-mpc_1+mpi_1} < g_1'(0) < \frac{2}{1-mpc_1}$, $g_2'(0) < \frac{2}{1-mpc_2+mpi_2}$, $R > 0$;

A3) $g_1'(0) < \frac{2}{1-mpc_1+mpi_1}$, $\frac{2}{1-mpc_2+mpi_2} < g_2'(0) < \frac{2}{1-mpc_2}$, $R > 0$;

A4) $g_1'(0) > \frac{4}{1-mpc_1}$, $g_2'(0) > \frac{4}{1-mpc_2}$, $R > 2\left(1 + \frac{2mpi_1}{1-mpc_1}\right)\left(1 + \frac{2mpi_2}{1-mpc_2}\right)$.

Finally, assuming $mpc_1 = mpc_2$, $mpi_1 = mpi_2$, $g_1'(0) = g_2'(0)$, and $R > 0$ we have that

B1) if $g_1'(0) \geq \frac{2}{1-mpc_1}$, *then* (10) *is unstable;*

B2) if $g_1'(0) < \frac{2}{1-mpc_1}$ *and* $\frac{R}{2} > \frac{1+mpc_1-2mpi_1}{1+mpc_1}$, *then* (10) *is unstable;*

B3) if $g_1'(0) < \frac{2}{1-mpc_1}$ *and* $\frac{R}{2} < \frac{1+mpc_1-2mpi_1}{1+mpc_1}$, *then* (10) *is asymptotically stable.*

Given the mild assumptions on the functions involved in the model, it is straightforward that multiple equilibria may arise for many specifications of them. Among the steady states there is always (10) which has some interesting economic properties. First of all, differently from Laursen and Metzler (1950), the steady state values of the Incomes do not depend

[4] We denote by $\mathbf{0}$ the element in $\mathbb{R}^n$ whose components are all zero.

neither on the variation of the Exchange Rate nor on the marginal propensity to import, but only from their marginal propensity to consume. This means that, at that equilibrium, the flexible exchange rate can isolate the two economic systems, as supposed by Friedman, and that an increase of Income in country 1 does not lead to an increase of Income in country 2, as suggested by the Absorption Model.

Moreover, the exchange rate in (10) is the value that guarantees the equilibrium of the Balance of Payment[5] At that steady state so if the gurus observe that the exchange rate is equal the fundamental, then they believe that in the future the exchange rate will not change so that, differently from Naimzada and Ricchiuti (2009) and in De Grauwe and Rovira (2012), they don't make systematic mistakes. It is worth noting that in particular (10) is unique if all gurus act as fundamentalists (contrarians).

From a stability point of view, we find several sufficient conditions for either stability or instability. These conditions compare the parameters R, $g_1'(0)$ and $g_2'(0)$ to suitable functions of marginal propensities. Note that R takes into account the reactivity of agents (via γ), gurus (via k) and exchange market (via $g_3'(0)$), while the parameters $g_1'(0)$ and $g_2'(0)$ represents the reactivity of good markets. In particular, under the assumption that there exists $j^* \in \{1, \ldots, n\}$ such that

$$\frac{\partial k_{j^*}}{\partial z_1}(E^*, E^*) \neq 0, \tag{12}$$

we get instability when all the gurus are contrarians (see A1). Instability can also be obtain when all the gurus are fundamentalists, provided the reactivity of gurus, agents and markets are high enough (see A4). Finally, under the assumption that the two countries are equal and all gurus are fundamentalists, we get stability when the good market reactivity is less than twice the well-known oversimplified multiplier and all the other reactivities are low enough.

4 Conclusions

We develop a heterogenous agents model in which the fundamental value of the exchange rate is endogenous, it depends on the balance of payments and through it gives and receives feed-backs to the real markets. In this way, we overcome the exogenous determination of fundamentals, one of the most important shortcomings of the literature of the heterogenous agents model, as explained in De Grauwe and Rovira (2012). Other authors have recently worked on this topic. Specifically, our work is strictly related to that of Westerhoff (2012) and Naimzada and Pireddu (2013a, 2013b). However, differently from these papers, we propose a model for the exchange rate determination, linking the exchange market with goods market of two countries as in the Absorption model. Moreover, our excess of demand are quite general and are valid for a large set of functions.

We have, analytically, shown the conditions for steady stats, finding a steady state which is unique when all agents/gurus act as fundamentalists (contrarians). Moreover, general stability conditions, related among the others with the reactivity of the markets and that of different agents involved, have been discussed.

In the future we would like to use this framework to analyze the case in which the financial account is opened, even if this entails the definition of the money market.

[5] This is the only steady state having that property.

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A Proof of Proposition 1

Defining $y_1(t) = mpi_1 Y_1(t)$ and $y_2(t) = mpi_2 Y_2(t)$, we can write system (8) as follows

$$\begin{cases} y_1(t+1) = y_1(t) + mpi_1 g_1 \left(A_1 + \frac{mpc_1}{mpi_1} y_1(t) + \frac{y_2(t)}{E(t)} - y_1(t) - \frac{1}{mpi_1} y_1(t) \right) \\ y_2(t+1) = y_2(t) + mpi_2 g_2 \left(A_2 + \frac{mpc_2}{mpi_2} y_2(t) + y_1(t)E(t) - y_2(t) - \frac{1}{mpi_2} y_2(t) \right) \\ E(t+1) = E(t) + g_3 \left(\gamma \sum_{j=1}^n \lambda_j \left(k \left(\frac{y_2(t)}{y_1(t)}, E(t) \right) \right) k_j \left(\frac{y_2(t)}{y_1(t)}, E(t) \right) \right) \end{cases}$$

Defining now

$$\begin{aligned} & a_1 = A_1 > 0, \quad a_2 = A_2 > 0 \\ & b_1 = -\frac{mpc_1}{mpi_1} + 1 + \frac{1}{mpi_1} > 1, \quad b_2 = -\frac{mpc_2}{mpi_2} + 1 + \frac{1}{mpi_2} > 1 \\ & f_1 = mpi_1 g_1, \quad f_2 = mpi_2 g_2, \quad f_3 = g_3 \end{aligned} \tag{13}$$

we get the following system

$$\begin{cases} y_1(t+1) = y_1(t) + f_1 \left(a_1 - b_1 y_1(t) + \frac{y_2(t)}{E(t)} \right) \\ y_2(t+1) = y_2(t) + f_2 \Big(a_2 - b_2 y_2(t) + y_1(t)E(t) \Big) \\ E(t+1) = E(t) + f_3 \left(\gamma \sum_{j=1}^n \lambda_j \left(k \left(\frac{y_2(t)}{y_1(t)}, E(t) \right) \right) k_j \left(\frac{y_2(t)}{y_1(t)}, E(t) \right) \right) \end{cases} \tag{14}$$

Denoting by $\widehat{S}$ the set of steady states of (14), we have that the function $\Phi : \widehat{S} \to S$ mapping $(y_1, y_2, E) \in \widehat{S}$ into $(y_1/mpi_1, y_2/mpi_2, E) \in S$ is well defined and is a bijection. Moreover, $(y_1, y_2, E) \in S$ is asymptotically stable if and only if $\Phi(y_1, y_2, E) \in S$ is asymptotically stable, and $(y_1, y_2, E) \in S$ is unstable if and only if $\Phi(y_1, y_2, E) \in S$ is unstable. As a consequence, we can prove Proposition 1 about system (8) working with system (14) and properly using the bijection Φ and equalities (13).

Note first that, for every $s \in \mathbb{R}_{++}$,

$$r_1(s) = \frac{a_1 b_2 s + a_2}{(b_1 b_2 - 1)s} \quad \text{and} \quad r_2(s) = \frac{a_2 b_1 + a_1 s}{b_1 b_2 - 1}.$$

Since the system

$$\begin{cases} a_1 - b_1 y_1 + \frac{y_2}{E} = 0 \\ a_2 - b_2 y_2 + y_1 E = 0 \\ y_1, y_2, E > 0 \end{cases} \tag{15}$$

has a set of solutions given by

$$\{(r_1(E), r_2(E), E) \in \mathbb{R}^3_{++} : E > 0\},$$

we have that

$$\widehat{S} = \{(r_1(E), r_2(E), E) \in \mathbb{R}^3_{++} : E \in \Gamma\},$$

where Γ is defined in (9), and that immediately implies the first statement of Proposition 1. Note also that

$$E^* = \frac{a_2(b_1-1)}{a_1(b_2-1)} \in \Gamma.$$

Then

$$(y_1^*, y_2^*, E^*) = \left(\frac{a_1}{b_1-1}, \frac{a_2}{b_2-1}, \frac{a_2(b_1-1)}{a_1(b_2-1)} \right), \tag{16}$$

belongs to $\widehat{S}$ and since $F(y_1^*, y_2^*, E^*)$ is equal to (10) and belongs to S, we get the second statement of Proposition 1.

Assume now that, for every $(z_1, z_2) \in \mathbb{R}^2_{++}$, we have that

$$\operatorname{sign}(k_1((z_1,z_2))) = \ldots = \operatorname{sign}(k_n((z_1,z_2))),$$

and prove that $\widehat{S} = \{(y_1^*, y_2^*, E^*)\}$, so that $S = \{F(y_1^*, y_2^*, E^*)\}$. Indeed,

$$\sum_{j=1}^{n} \lambda_j \left(k \left(\frac{r_2(E)}{r_1(E)}, E \right) \right) k_j \left(\frac{r_2(E)}{r_1(E)}, E \right) = 0$$

if and only if

$$\frac{r_2(E)}{r_1(E)} - E = 0,$$

that is,

$$\frac{a_2(b_1-1)E - a_1(b_2-1)E^2}{a_2 + a_1 b_2 E} = 0$$

and that equation has E^* has unique solution. That proves the third part of Proposition 1.

Let us move on to prove the last part of the proposition. We analyse then the stability of the state (y_1^*, y_2^*, E^*) defined in (16) in order to deduce information about the stability of $F(y_1^*, y_2^*, E^*)$, that is, (10). Consider the three following functions whose domain is $\mathbb{R}^3_{++}$,

$$F_1(y_1, y_2, E) = y_1 + f_1 \left(a_1 - b_1 y_1 + \frac{y_2}{E} \right)$$

$$F_2(y_1, y_2, E) = y_2 + f_2 \Big(a_2 - b_2 y_2 + y_1 E \Big)$$

$$F_3(y_1, y_2, E) = E + f_3 \left(\gamma \sum_{j=1}^{n} \lambda_j \left(k \left(\frac{y_2}{y_1}, E \right) \right) k_j \left(\frac{y_2}{y_1}, E \right) \right)$$

Then we have that:

$$\begin{aligned}
\tfrac{\partial F_1}{\partial y_1}\left(y_1^*, y_2^*, E^*\right) = &\quad 1 - b_1 f_1'(0) \\
\tfrac{\partial F_1}{\partial y_2}\left(y_1^*, y_2^*, E^*\right) = &\quad \tfrac{a_1(b_2-1)}{a_2(b_1-1)} f_1'(0) \\
\tfrac{\partial F_1}{\partial E}\left(y_1^*, y_2^*, E^*\right) = &\quad -\tfrac{a_1^2(b_2-1)}{a_2(b_1-1)^2} f_1'(0) \\
\tfrac{\partial F_2}{\partial y_1}\left(y_1^*, y_2^*, E^*\right) = &\quad \tfrac{a_2(b_1-1)}{a_1(b_2-1)} f_2'(0) \\
\tfrac{\partial F_2}{\partial y_2}\left(y_1^*, y_2^*, E^*\right) = &\quad 1 - b_2 f_2'(0) \\
\tfrac{\partial F_2}{\partial E}\left(y_1^*, y_2^*, E^*\right) = &\quad \tfrac{a_1}{(b_1-1)} f_2'(0) \\
\tfrac{\partial}{\partial y_1} F_3\left(y_1^*, y_2^*, E^*\right) = &\quad -f_3'(0)\tfrac{a_2(b_1-1)^2}{a_1^2(b_2-1)}\gamma \textstyle\sum_{j=1}^n \lambda_j(\mathbf{0})\tfrac{\partial k_j}{\partial z_1}\left(\tfrac{a_2(b_1-1)}{a_1(b_2-1)}, \tfrac{a_2(b_1-1)}{a_1(b_2-1)}\right) \\
\tfrac{\partial F_3}{\partial y_2}\left(y_1^*, y_2^*, E^*\right) = &\quad f_3'(0)\tfrac{(b_1-1)}{a_1}\gamma \textstyle\sum_{j=1}^n \lambda_j(\mathbf{0})\tfrac{\partial k_j}{\partial z_1}\left(\tfrac{a_2(b_1-1)}{a_1(b_2-1)}, \tfrac{a_2(b_1-1)}{a_1(b_2-1)}\right) \\
\tfrac{\partial F_3}{\partial E}\left(y_1^*, y_2^*, E^*\right) = &\quad 1 + f_3'(0)\gamma \textstyle\sum_{j=1}^n \lambda_j(\mathbf{0})\tfrac{\partial k_j}{\partial z_2}\left(\tfrac{a_2(b_1-1)}{a_1(b_2-1)}, \tfrac{a_2(b_1-1)}{a_1(b_2-1)}\right)
\end{aligned}$$

Note that

$$R = f_3'(0)\gamma \sum_{j=1}^n \lambda_j(\mathbf{0})\frac{\partial k_j}{\partial z_1}\left(\frac{a_2(b_1-1)}{a_1(b_2-1)}, \frac{a_2(b_1-1)}{a_1(b_2-1)}\right),$$

where R is defined in (11), and, for every $j \in \{1, \ldots, n\}$ and $s \in \mathbb{R}$,

$$\frac{\partial k_j}{\partial z_1}(s,s) = -\frac{\partial k_j}{\partial z_2}(s,s).$$

Then we have that

$$\frac{\partial F_3}{\partial y_1}\left(y_1^*, y_2^*, E^*\right) = -\frac{a_2(b_1-1)^2}{a_1^2(b_2-1)} R$$

$$\frac{\partial F_3}{\partial y_2}\left(y_1^*, y_2^*, E^*\right) = \frac{(b_1-1)}{a_1} R$$

$$\frac{\partial F_3}{\partial E}\left(y_1^*, y_2^*, E^*\right) = R$$

In order to study the stability of (16), we study the stability of the polinomial

$$P(s) = \det\left(sI - D(F_1, F_2, F_3)\left(y_1^*, y_2^*, E^*\right)\right) =$$

$$\det \begin{bmatrix} s - 1 + b_1 f_1'(0) & -\frac{a_1(b_2-1)}{a_2(b_1-1)} f_1'(0) & \frac{a_1^2(b_2-1)}{a_2(b_1-1)^2} f_1'(0) \\ -\frac{a_2(b_1-1)}{a_1(b_2-1)} f_2'(0) & s - 1 + b_2 f_2'(0) & -\frac{a_1}{(b_1-1)} f_2'(0) \\ \frac{a_2(b_1-1)^2}{a_1^2(b_2-1)} R & -\frac{(b_1-1)}{a_1} R & s - 1 + R \end{bmatrix}$$

that is, we study whether all of its roots lie inside the unit circle or there is at least one root lying outside the unit circle. A computation shows that

$$P(s) = (s-1)^3 + \rho_1(s-1)^2 + \rho_2(s-1) + \rho_3$$

where

$$\rho_1 = R + b_2 f_2'(0) + b_1 f_1'(0),$$

$$\rho_2 = Rb_2 f_2'(0) + Rb_1 f_1'(0) + b_1 b_2 f_1'(0) f_2'(0) - f_1'(0)R - f_1'(0) f_2'(0) - f_2'(0)R,$$

$$\rho_3 = R f_1'(0) f_2'(0)(b_1 b_2 - b_1 - b_2 + 1).$$

A simple condition which is sufficient for instability is $P(1) < 0$, that is, $\rho_3 < 0$. Since each factor in $f_1'(0)f_2'(0)(b_1b_2 - b_1 - b_2 + 1)$ is positive, we have that $P(1) < 0$ if and only if $R < 0$. That proves Statement A1 of Proposition 1.

An other simple condition for instability is $P(-1) > 0$, that is, $-8 + 4\rho_1 - 2\rho_2 + \rho_3 > 0$. Note that,

$$-8 + 4\rho_1 - 2\rho_2 + \rho_3 > 0$$

if and only if

$$R\Big[4 - 2b_2 f_2'(0) - 2b_1 f_1'(0) + 2f_2'(0) + 2f_2'(0) + f_1'(0)f_2'(0)(b_1b_2 - b_1 - b_2 + 1)\Big] > \tag{17}$$

$$8 - 4b_2 f_2'(0) - 4b_1 f_1'(0) + 2f_1'(0)f_2'(0)b_1b_2 - 2f_1'(0)f_2'(0)$$

Since

$$4 - 2b_2 f_2'(0) - 2b_1 f_1'(0) + 2f_2'(0) + 2f_2'(0) + f_1'(0)f_2'(0)(b_1b_2 - b_1 - b_2 + 1) =$$

$$\Big(2 - f_1'(0)(b_1 - 1)\Big)\Big(2 - f_2'(0)(b_2 - 1)\Big),$$

and $f_1'(0), f_2'(0) > 0$, we have that (17) is implied by

$$R\Big(2 - f_1'(0)(b_1 - 1)\Big)\Big(2 - f_2'(0)(b_2 - 1)\Big) > 8 - 4b_2 f_2'(0) - 4b_1 f_1'(0) + 2f_1'(0)f_2'(0)b_1b_2$$

that is

$$R\Big(2 - f_1'(0)(b_1 - 1)\Big)\Big(2 - f_2'(0)(b_2 - 1)\Big) > 2\Big(2 - f_1'(0)b_1\Big)\Big(2 - f_2'(0)b_2\Big). \tag{18}$$

It can be verified that (18) holds true if one of the following set of conditions is satisfied:

- $\frac{2}{b_1} < f_1'(0) < \frac{2}{b_1 - 1}$, $f_2'(0) < \frac{2}{b_2}$, and $R > 0$;
- $f_1'(0) < \frac{2}{b_1}$, $\frac{2}{b_2} < f_2'(0) < \frac{2}{b_2 - 1}$, and $R > 0$;
- $f_1'(0) > \frac{4}{b_1 - 1}$, $f_2'(0) > \frac{4}{b_2 - 1}$, $R > 2\frac{(b_1+1)(b_2+1)}{(b_1-1)(b_2-1)}$.

Those conditions are exactly the ones in the Statements $A2$, $A3$ and $A4$ of Proposition 1.

In order to prove the last part of the proposition assume $mpc_1 = mpc_2$, $mpi_1 = mpi_2$, $g_1'(0) = g_2'(0)$, and $R > 0$. That implies $b_1 = b_2$ and $f_1'(0) = f_2'(0)$ and

$$P(s) = (s-1)\left[(s-1)^2 + 2bf(s-1) + f^2(b^2 - 1)\right] + R\left[(s-1)^2 + 2f(b-1)(s-1) + f^2(b-1)^2\right]$$

where we set $b = b_1$ and $f = f_1'(0)$. Let $P(s) = (s-1)K(s) + RJ(s)$, where

$$K(s) = (s-1)^2 + 2bf(s-1) + f^2(b^2 - 1),$$

and

$$J(s) = (s-1)^2 + 2f(b-1)(s-1) + f^2(b-1)^2.$$

The equation $K(s) = 0$ has three distinct solutions given by

$$s_1 = 1 - f(b+1), \quad s_2 = 1 - f(b-1), \quad 1,$$

with $s_1 < s_2 < 1$, and since $J(s) = (s-1) + f(b-1)]^2 = (s - s_2)^2$, we also have that $P(s_2) = 0$, $P'(s_2) < 0$, $P(s_1) > 0$ and, for every $s \in [1, +\infty)$, $P(s) > 0$. As a consequence P has three distinct real roots $s_1^* < s_2^* < s_3^*$ such that

$$s_1^* < s_1, \quad s_2^* = s_2, \quad s_3^* \in (s_2, 1).$$

As a consequence, if $f \geq \frac{2}{b+1}$, then $s_1^* < s_1 \leq -1$ and (16) is unstable: that proves Statement B1 of Proposition 1.

Assuming instead $f < \frac{2}{b+1}$, we get $s_1 > -1$ and then, $s_1^* < -1$ if and only if $P(-1) > 0$, while $s_1^* \in (-1, 1)$ if and only if $P(-1) < 0$.

A computation shows that

$$P(-1) = -8 + 8fb - 2f^2b^2 + 2f^2 + R(2 - f(b-1))^2,$$

and then $P(-1) > 0$ if and only if

$$R(2 - f(b-1))^2 > 8 - 8fb + 2f^2b^2 - 2f^2 = 2(2 - f(b+1))(2 - f(b-1))$$

if and only if

$$\frac{R}{2} > \frac{2 - f(b+1)}{2 - f(b-1)}.$$

Analogously we have that $P(-1) < 0$ if and only if

$$\frac{R}{2} < \frac{2 - f(b+1)}{2 - f(b-1)}.$$

In the first case we get instability while in the second case we get asymptotic stability: by a substitution we obtain conditions of Statements B2 and B3 of Proposition 1.