Population Growth and Technological Change: a Pure Welfarist Approach

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Abstract

In this paper we characterize the optimal steady state and dynamics of an intertemporal economy in presence of endogenous fertility. For doing this we propose a new version of Critical Level Utilitarianism (CLU) a là Blackorby et al (1995), that we call Relative CLU, which is shown to be axiomatically founded and allows for time-varying critical level. In particular, we show that under such social preferences both the short and the long run effects of technological shocks on the optimal population growth are non-trivial, depending on the characteristics of both preferences and production technology. However, we show that the negative relationship between fertility and technological development emerges under a broad class of preferences, which is typically found in empirical research.

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1. Introduction

The issue of population growth has been extensively analyzed in economics. After the pioneering studies of scholars such as Malthus and Ricardo, the classical approach has experienced a decline in its popularity since the empirical evidence seemed inconsistent with its fundamental implications, according to which economic growth would lead individuals to reproduce at a faster rate, and, this, in turn, would cause decreasing per capita food supply and decreasing living standards towards subsistence levels. However, the same empirical evidence has contributed to a renewed interest in the topic of population economics since the 1960s due to two apparently contrasting phenomena: on the one hand, the worrying prospects of developed countries who have been undergoing a long-lasting demographic transition involving longer average expected lifetimes and reduced fertility rates, and, on the other hand, the still sustained population growth rates of the underdeveloped and developing countries.

Consequently, such phenomena have inspired several works concerning, for example, the issue of optimal population growth rate (Samuelson 1975, Deardorff 1976 and, more recently, Jaeger and Kuhle 2009, de la Croix et al 2012), the relationship between endogenous fertility and the welfare state (in particular, as for social security, see, for example, Cigno and Rosati 1992, Zhang and Nishimura 1992 and 1993, Cremer et al 2006, Yew and Zhang 2009, Meier and Wrede 2010; see also Spataro and Fanti 2011 for debt optimality and Spataro and Renström 2012 for taxation analysis in presence of endogenous fertility). Another strand of literature has been focusing on the

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long-run relationship between endogenous fertility, technological progress and economic growth (see, for example, Galor and Weil 2000, Fernandez-Villaverde 2001, Hansen and Prescott 2002, Greenwood and Seshadri 2002, Kalemli-Ozcan 2002, Lagerlof 2003, Doepke 2004, Soares 2005, Tamura 2006, Jones and Tertilt 2006, Jones et al 2010[1]).

A promising branch of the above mentioned literature has been putting much effort in providing a coherently founded toolbox to population economics. In fact, it is well known that the evaluation of alternative public policies often implies the comparison of states of the world with different population and this task can be problematic when welfarist criteria are to be used, that is, criteria based on the (sum of) well-being (utilities) of the individuals who are alive in different states of the world[2].

Despite the relevance of this problem, in our opinion the theoretical foundations of social evaluation with variable populations have received little attention in the literature. Typically, welfarist principles are adopted such as classical utilitarianism, where the objective is the sum of the utilities over the population, or average utilitarianism, where the objective is the welfare of a representative individual, or some mixture of both.

However, these criteria pose several problems. For example, classical utilitarianism cannot avoid the repugnant conclusion (RC henceforth; see Parfit 1976, 1984, Blackorby et al. 2002), whereby any state in which each member of the population enjoys a life above neutrality is declared inferior to a state in which each member of a larger population lives a life with lower utility (Blackorby et al. 1995, 2002). Note that in models of economic growth such a conclusion takes the form of an upper corner solution for the population growth rate (society reproduces at its physical maximum rate).

There are several ways to deal with the RC (indeed, some authors believe the latter is not really a problem. See for example Tännsjö 2002). In fact, several economists have decided to circumvent the problem by resorting to particular preferences (see for example Barro-Becker 1988, Becker-Barro 1989, and, although in different frameworks, Ng 1986, Hurka 2000, and, Galor and Weil 2000). However, typically such solutions are not, or at least only partially, axiomatically founded, that is, hardly compatible with some postulates that we believe social orderings should abide in order to be ethically acceptable.

A possible way out of this problem is represented by Critical Level Utilitarianism (CLU) which is axiomatically founded (see Blackorby et al. 1995). The critical level is defined as a utility value (α) of an extra person that, if added to the (unaffected) population, would make society as well off as without that person.

CLU has been criticised by several authors: Parfit (1984) and Shiell (2008) argue that CLU cannot avoid the RC because, as long as average utility is higher than α, it would be always socially preferable to get larger populations with lower utility levels, sufficiently close to α; and this conclusion, according to these authors, would be "repugnant" if the critical level is set too low. Moreover, Broome (1992) points out that, in case the critical level is too high, the addition of a person whose life is worth living (i.e. with a positive utility level) would be prevented. The latter conclusion would replicate the one stemming from the application of the average utility criterion. Finally, Ng (1986) and Arrhenius (2000) pointed out that CLU involves counterintuitive social orderings in case the average utility is lower than α (i.e. the so called "sadistic conclusion"[3]).

It is worth noting that in most cases the tradition of population ethics has dealt with the RC in static environments (see, for example, Shiell 2008), which is not ideal, since population issues naturally entail strong dynamic implications.

The first attempt to characterize the problem in a dynamic framework is provided by Renström and Spataro (2011) where a full characterization of an intertemporal economy entailed with CLU

[1] See Ehrlich and Lui (1997), Galor (2005) and Doepke (2008) for in-depth reviews on these topics.

[2] Among works using non utilitarian principles, see, for example, Golosov, Jones and Tertilt (2007).

[3] The sadistic conclusion implies that adding a person with negative utility to a population may be preferred to an alternative where all individuals have positive utilities, although below the critical level (see Arrhenius 2000).

preferences is carried out. In that work it is shown that the RC is avoided only if the critical level belongs to a positive open interval, thus somehow answering to some of the criticisms mentioned above concerning the choice of α.

Following the latter paper, in the present work we want to make a step forward in the analysis of the relationship between economic growth, CLU preferences and technological change. More precisely, we propose a new population criterion, "Relative Critical Level Utilitarianism" (RCLU), where the critical level is allowed to depend on past utility levels .

The main reason underlying our work is that we believe that a fixed critical level is, to all extent, too rigid. After all, the decision to have children is taken by parents, and the latter are likely to choose the number of their offspring on the basis of their own standard of living. Hence, in this work we assume that the current critical level depends on the stage of development that a society has been experiencing so far, such that parents will condition the choice of fertility, comparing their own children's welfare in relation to their own welfare.

If this is not the case, that is if the critical level is assumed to be fixed by the society once for all, several drawbacks may arise. For example, CLU models can provide little justification to the high fertility rates that several African or Asian countries are currently experiencing, unless one is prone to believe that such behaviours are completely irrational and, thus, hardly treatable by economic models.

Since we derive our model's welfare function by restricting the set through population postulates, our contribution represents also a choice theoretic foundation to existing models (such as Kremer 1993 and Hansen and Prescott 2002) that have exogenously assumed a relationship between population growth rates and individuals' standard of living[4].

Moreover, we also aim at building a model that allows for non-trivial dynamics of population growth, as suggested by several empirical studies (see, for example, Galor 2005 and Jones and Tertilt 2006, Bar and Leukina 2010)[5].

In fact, in the present paper we show that under RCLU both an interior optimum and rich dynamics for population growth emerge, depending both on preferences and on technology.

Finally, we avoid endogenizing technological progress because we want to keep our analysis as close as possible to the Cass-Koopmans-Ramsey approach. We leave the extensions to endogenous accumulation of technology for future research**.**

The paper is organized as follows: in section 2 we lay out the model setup, in section 3 we provide the solution and discuss the existence and the stability of the long run equilibrium. In section 4 we discuss the effects of technological shocks on both the short run dynamics and long run equilibrium. In section 5 we explore the role of preferences in driving the results. Finally, an Appendix contains the axiomatic foundation of RCLU.

2. The model

2.1 Preferences

We concentrate on a single dynasty (household) or a policymaker choosing consumption and population growth over time, so as to maximize:

$$W(u_{t-1}, N_t, u_{t+1}, N_{t+1}, \ldots) = \sum_{s=0}^{\infty} \beta^s N_{t+s} \left[u_{t+s} - \alpha(u_{t-1+s})\right] \tag{1}$$

[4] For a review of previous related literature on endogenous preferences, see Macunovich (1998).

[5] In fact, as Jones and Tertilt state "[…] despite the extensive empirical work, there is still no conclusive evidence on the exact relationship between income and fertility. Quoting Heckman and Walker (1990), «most economists would agree with Ward and Butz (1980) that current and future wages and income […] are likely determinants of fertility. There is little agreement beyond this»". (Jones and Tertilt 2006, p. 12).

where N is the population (family) size, u the instantaneous utility function and $\beta \in (0,1)$ the intergenerational discount factor and $\alpha(u_{t-1+s})$ is the critical level utility. In Appendix A we show that this is the only formulation satisfying: Independence of the Long Dead, Stationarity, Independence of Distant Future Generations, Independence of Utilities of Unconcerned Individuals, Anonymity, Strong Pareto and Relative Critical Level Dependence.

Note that, differently from previous literature, we allow such a critical value to be a positive function of previous generation's utility (only if $\alpha(u_{t-1+s})$ is a constant this social ordering would coincide with the CLU). We call our population criterion "Relative CLU" (RCLU).

Throughout our paper we make the simplifying assumption that the critical level function is linear, that is.

$$\alpha(u_{t-1+s}) = \alpha u_{t-1+s}$$

with $u' > 0, u'' < 0$ and $\alpha < 1$. Moreover, for analytical tractability, we will work in continuous time. We denote the population growth rate as n_t, i.e.

$$\frac{\dot{N}_t}{N_t} = n_t \tag{2}$$

with $n_t \in [\underline{n}, \overline{n}]$. The continuous time version of eq. (1) can be written as (see Appendix B.1):

$$U = \int_0^\infty e^{-\rho t} N_t u(c_t)[1 - \alpha(n_t - \rho)]dt\,. \tag{3}$$

The integral is finite only if $(\rho - \overline{n}) > 0$, which we assume throughout the paper; moreover, given that $\alpha < 1$, the latter assumption implies $1 - \alpha(n - \rho) > 0$.

2.2. Technology

Assuming a constant returns-to-scale (CRS) production technology, $F=(K_t,L_t)= A_t \tilde{F}(K_t,L_t)$, with A_t the parameter representing total factor productivity, and a capital depreciation rate of zero, the capital accumulation equation is:

$$\dot{K}_t = F(K_t, N_t) - c_t N_t \tag{4}$$

3. Solution

The current-value Hamiltonian of the household's problem is the following:

$$H = N_t u(c_t)[1 - \alpha(n_t - \rho)] + q_t[F(K_t, N_t) - c_t N_t] + \lambda_t n_t N_t + v_t(\overline{n} - n_t) + \vartheta_t(n_t - \underline{n}) \tag{5}$$

The term $\lambda_t n_t N_t$ in the Hamiltonian associated with eq. (2) captures the fact that at each instant of time the population size is given (and thus is a state variable) and can only be controlled by the choice of n (which is a control variable). The law of motion for the population size is provided by (2). Hence, λ_t can be interpreted as the shadow value of population.

The first order conditions of the problem imply:

$$\frac{\partial H}{\partial c_t} = N_t u'_t\,[1-\alpha(n_t-\rho)] - N_t q_t = 0 \Rightarrow u'_t\,[1-\alpha(n_t-\rho)] = q_t \tag{6}$$

$$\frac{\partial H}{\partial n_t} = N_t(-\alpha u_t + \lambda_t) - \nu_t + \vartheta_t = 0 \Rightarrow \alpha u_t = \lambda_t + \frac{(\vartheta_t - \nu_t)}{N_t} \tag{7}$$

$$\frac{\partial H}{\partial N_t} = \rho\lambda_t - \dot{\lambda}_t \Rightarrow \dot{\lambda}_t = (\rho - n_t)\lambda_t - u_t[1-\alpha(n\ -\rho)] - q_t[F_{N_t} - c_t] \tag{8}$$

$$\frac{\partial H}{\partial K_t} = \rho q_t - \dot{q}_t \Rightarrow \dot{q}_t = q_t[\rho - F_{K_t}], \tag{9}$$

eq. (4) and the transversality conditions

$$\lim_{t\to\infty} e^{-\rho t} q_t K_t = 0, \ \lim_{t\to\infty} e^{-\rho t}\lambda_t N_t = 0. \tag{10}$$

In what follows we assume interiority of the solution for n,[6] such that both ν and ϑ in eq. (7) are zero and along the transition path $\dot{n} \neq 0$ (indeed we will show in Remark 1 the necessary condition whereby the RC, i.e. $n = \bar{n}$, is avoided).

Let us define the capital intensity $k \equiv K/N$, such that, by exploiting CRS in the production function and assuming that $L=N$ we can write: $F(K,L) = Nf(k)$, $F_N(K,N) = f(k) - f'(k)k$. Moreover, by substituting for eqs. (6) and (7) into (8) we get:

$$\dot{\lambda}_t = -u_t - u'_t\,[1-\alpha(n_t-\rho)][F_{N_t} - c_t]. \tag{11}$$

We obtain three dynamic equations that, together with the transversality conditions, fully characterize our dynamic system (we omit time subscripts for the sake of notation):

$$\dot{c} = -\frac{u}{\alpha u'} + \frac{G}{\alpha}[c - F_N] \tag{12}$$

$$\dot{k} = f(k) - c - nk \tag{13}$$

$$\dot{n} = \frac{G}{\alpha}\left[\frac{u''}{u'}\dot{c} - (\rho - F_K)\right] = G\left\{\frac{u''}{u'}\left[-\frac{u}{\alpha u'} + \frac{G}{\alpha}(c - F_N)\right] - (\rho - F_K)\right\} \tag{14}$$

where $G \equiv 1 + \alpha(\rho - n) > 0$. Eq. (12) is obtained by taking the time derivative of eq. (7) and combining with eq. (11), and eq. (14) by taking the time derivative of (6) and combining with eq. (9). Finally, eq. (13) is eq. (4) in per capita terms.

[6] For corner n the economy would behave as in the standard Cass-Koopmans-Ramsey model with exogenous constant population growth.

3.1. The steady-state equilibrium

From eqs. (14), (13) and (12) respectively, the steady-state solution is characterized by the following equations:

$$f'^{*} = \rho \tag{15}$$

$$n^{*} = \frac{f^{*} - c^{*}}{k^{*}} \tag{16}$$

$$c^{*} = \frac{1}{G^{*}} \frac{u^{*}}{u'^{*}} + F_N^{*} \,. \tag{17}$$

Some comments on the solution are worth doing. Eq. (15) pins down a unique positive value of steady-state capital intensity k^*, as is standard in the Cass-Koopmans-Ramsey model with exogenous fertility. Eq. (16) states that at the optimum both fertility, n^* and consumption, c^* must satisfy the resources available for the economy. Finally, eq. (17) states that at the optimum both consumption and fertility should be chosen in such a way that the addition to social welfare of increasing the population at the margin, u, should equal the marginal value (in utility units) of what a newborn takes out of society, $Gu'[c-F_N]$. In fact, the net cost of a new individual in terms of resources is the difference between what she consumes, c, and what she brings, F_N (the marginal value of labour). If the social value of one more person is the same as the cost, society (or planner) is indifferent in altering the population size and consumption is at its optimal value.

By using eqs. (15) and (16) and exploiting CRS we get[7]:

$$\rho - n = \frac{c - F_N}{k} \,. \tag{18}$$

Moreover, by defining $\eta(c) \equiv u/(u'c) > 0$ as the inverse of the elasticity of utility, with $\eta > 1$ for $c > 0$ (under concavity of the utility function) and by using eqs. (15) and (17) we get $c = \frac{GF_N}{G - \eta(c)}$, where $G - \eta(c)$ is strictly positive. Finally, plugging (18) into the definition of G the following expression, providing the implicit solution for steady-state consumption, descends:

$$\frac{c - F_N}{k} = \frac{\eta(c) - 1}{2\alpha} + \frac{1}{2\alpha}\sqrt{\left(\eta(c) - 1\right)^2 + 4\frac{\alpha\eta(c)F_N}{k}} \,. \tag{19}$$

Notice that the solutions for k and c (i.e. eqs. 15 and 19) give also the solution for the population growth rate, by eq. (18). Now we can give the following Remark, clarifying the role of RCLU in avoiding the RC:

Remark 1: The Repugnant Conclusion cannot be avoided if α=0.

Proof: Recall that the Repugnant Conclusion implies that $n = \bar{n}$. Suppose now that α=0. Integrate (11) between t_0 and T to obtain

[7] For sake of notation from now on we omit the "star" superscript for steady-state values whenever it causes no ambiguity to the reader.

$$\lambda_T e^{-\int_{t_0}^{T}(\rho-n_s)ds} - \lambda_{t_0} = -\int_{t_0}^{T} e^{-\int_{t_0}^{t}(\rho-n_s)ds} \left[u_t - u'_t(c_t - F_{N_t})\right]dt$$

Then, as $T \to \infty$, by exploiting the transversality condition we have

$$\lambda_{t_0} = \int_{t_0}^{\infty} e^{-\int_{t_0}^{t}(\rho-n_s)ds} u'_t c_t \left[\frac{u_t}{u'_t c_t} - 1 + \frac{F_{N_t}}{c_t}\right]dt$$

As $\frac{u_t}{u'_t c_t} - 1 > 0$ under concavity of utility function, it descends, for any $\tau \geq t_0$, that $\lambda_\tau > 0$ which, by eq. (7) implies $v_\tau > 0$ and, thus, $n_\tau = \bar{n}$. □

3.2. Existence and uniqueness of the steady state

Given that equation (19) yields only an implicit expression for steady-state consumption, the Proposition below provides the conditions under which both existence and uniqueness of the solution are guaranteed:

Proposition 1: If $\eta'(c) \leq 0$ then necessary and sufficient for the existence of a unique steady state is that $\alpha > 0$. If $\eta'(c) > 0$ and $\alpha \leq k/F_N$, then necessary and sufficient for the existence of a unique steady state is that $0 < \eta'(c) \leq \alpha/k$, for all c, (the latter inequality being strict if $\eta(c)$ is linear and $\alpha = k/F_N$).

Proof: See Appendix B.2. □

In other words, the latter condition states that in order for there to be a unique steady state, the inverse of the elasticity of the utility function, $\eta(c)$, which measures the gain in terms of social welfare stemming from consumption activity of an extra person, should not be too reactive to consumption changes.

When $\eta'(c) > 0$ we have stated the result for $\alpha \leq k/F_N$. While similar results are obtained for $\alpha > k/F_N$, we choose not to focus on that particular case.[8] The reason is that $\alpha \leq k/F_N$ is a sufficient condition for stability (which we will come to in the next section).

3.3. Stability of the equilibrium

We now show the conditions under which the equilibrium trajectory is saddle-path stable. Let us linearize the system given by eqs. (12)-(14) around the steady state. We get the following expression for the Jacobian matrix:

[8] In short, the result is if $\eta'(c) > 0$ and $\alpha > k/F_N$, then a concave or linear $\eta(c)$ yields a unique steady state, while for a convex $\eta(c)$, $\eta'(c) < \alpha/k$, is necessary and sufficient for existence and uniqueness.

$$J=\begin{bmatrix} \Omega & \frac{G}{\alpha}f''k & -(\rho-n)k \\ -1 & \rho-n & -k \\ \left(\frac{\Omega}{\rho-n}-1\right)\frac{\Omega}{k} & \frac{\Omega G f''}{\alpha(\rho-n)} & -\Omega+(\rho-n) \end{bmatrix}$$

where $\Omega\equiv\frac{1}{\alpha}\left(-1+\frac{u''u}{u'^2}+G\right)=\frac{1}{\alpha}[-\eta(c)\sigma(c)+\alpha(\rho-n)]$ and $\sigma(c)=-\frac{u''c}{u'}>0$ is the inverse of the intertemporal elasticity of substitution of consumption.

Hence, the characteristic equation associated with the Jacobian matrix above is:

$$(\rho-n-\lambda)\left[\lambda^2-\lambda(\rho-n)+\frac{Gf''k}{\alpha}\left(1+\frac{\Omega}{\rho-n}\right)\right]=0 \tag{20}$$

the roots to which are:

$$\lambda_1=\rho-n\,,\ \lambda_{2,3}=\frac{(\rho-n)}{2}\pm\sqrt{\frac{(\rho-n)^2}{4}-\frac{G}{\alpha}f''k\left(1+\frac{\Omega}{\rho-n}\right)}\,. \tag{21}$$

Given that $\lambda_1>0$, then necessary and sufficient condition for saddle-path stability is that $\lambda_2>0$ and $\lambda_3<0$. Sufficient for the latter conditions is that $\rho-n+\Omega>0$.

However, since this condition is not easily interpretable, we provide more transparent sufficient conditions below through the following Proposition:

Proposition 2: Sufficient for stability of the steady state is

Either $\eta'(c)\leq 0$
Or $0<\eta'(c)<\alpha/k$ and $\alpha\leq k/F_N$.

Proof: S*ee Appendix B.3.* □

Hence, sufficient conditions for existence and uniqueness of the equilibrium provided in Proposition 1 also imply stability of the steady-state equilibrium.

Note that, since at the steady state the following relationship holds, $\alpha\Omega=GF_N/c-c\eta'(c)$, it follows that if $\eta'(c)\leq 0$, then $\Omega>0$. On the other hand, since if $\eta'(c)=\alpha/k=1/F_N$ then $\Omega<0$, it descends that when $0<\eta'(c)<\alpha/k$, Ω can take either sign. We will exploit and analyse in deeper detail this property in the next sections. Finally, under the conditions provided in Proposition 2, inequality $\rho-n+\Omega>0$ holds true.

3.4. Transitional dynamics

We now focus on the properties of the transitional dynamics of the model. We can summarize our findings through the following Proposition:

Proposition 3: Along the transition path, if k_0<(>)k*, then both consumption and capital intensity increase (decrease). As for n, if k_0<(>)k*, it increases (decreases) if Ω>0, while it decreases (increases) if Ω<0.

Proof: *See Appendix B.4.* □

A comment on the results obtained so far is worth making: in our model both per-capita consumption and capital behave as in the standard Cass-Koopmans-Ramsey model. However, the rate of growth of population can take either dynamic path, depending on the sign of Ω.

In order to provide an economic interpretation of this property of the model, we resort to steady-state relations by differentiating eq. (17), for a given level of k, which yields:

$$\frac{dc}{dn} = \frac{c - F_N}{\Omega}.$$

The latter equation represents the marginal rate of substitution of society's indifference curves between consumption and fertility, and the sign of Ω determines the intrinsic nature of the relationship between these two goods. In fact, if Ω>0, the goods are complements, while if Ω<0 they are substitutes. Moreover, if Ω is strongly negative (that is, the system is unstable), then the indifference curves are flat, such that a corner solution for n ($\underline{n}$) is likely to occur: in other words, in this case society prefers to devote as much resources to consumption as possible and minimize those dedicated to raising children.

4. The effects of the technological change

In this paragraph we carry out comparative statics and dynamics analyses in order to assess the effects of changes in total factor productivity, A, on both the long run and the transitional path of the economic system. Preliminarily, we discuss the sign of the Jacobian matrix of our model.

The determinant of the Jacobian matrix, J, is equal to:

$$D(J) = \frac{Gf''k}{\alpha}\left(\rho - n + \Omega\right). \tag{22}$$

Given that under stability $\left(\rho - n + \Omega\right)$>0, and that f''<0, G>0, it descends that the determinant of the Jacobian matrix, is negative.

4.1 The effect of A on steady-state consumption

Using Cramer's rule, we can write:

$$\frac{dc}{dA} = \frac{D(J_c)}{Det(J)}$$

where J_c is the Jacobian matrix in which the first column is substituted for by the derivatives of eqs. (12) to (14) (with negative sign) with respect to A.

Since $\frac{dc}{dA}=\frac{D(J_c)}{Det(J)}>0$, we can summarize our findings through the following Proposition:

Proposition 4: A positive technological shock increases the long run per-capita consumption.
Proof*: See Appendix B.5* □

4.2 The effect of A on steady-state capital intensity

Let us now turn to the effects of a technological shock on the capital intensity. In formula, we can write the following:

$$\frac{dk}{dA}=\frac{D(J_k)}{Det(J)}$$

where J_k is the Jacobian matrix in which the second column is substituted out by the derivatives of eqs. (12) to (14) (with negative sign) with respect to *A*.

Since $\frac{dk}{dA}=\frac{D(J_k)}{Det(J)}>0$, we can provide the following Proposition:

Proposition 5: A positive technological shock increases the long run per-capita capital intensity
Proof: *See Appendix B.6.* □

4.3 The effect of A on steady-state fertility rate

Preliminarily, let us focus on the determinant of J_n, which is the Jacobian matrix in which the third column is substituted out by the derivatives of eqs. (12) to (14) (with negative sign) with respect to *A*. It turns out to be:

$$D(J_n)=-\frac{G}{\alpha^2 A}\left[\alpha\Omega\left(F_K(\rho-n)-f''f\right)+Gf''f\right]. \tag{23}$$

Recalling that $\gamma=\frac{\log d(k)}{\log d(TMRS)}=-\frac{f'}{f''}\frac{f-f'k}{f}\frac{1}{k}=-\frac{f'}{f''k}\left(1-\frac{f'k}{f}\right)$ is the elasticity of substitution of inputs, then eq. (23) can be rewritten as (see Appendix B.7):

$$D(J_n)=-\frac{G}{\alpha^2 A}\left\{c\eta'(c)\left[f''f-F_K(\rho-n)\right]+\eta f''f(1-\gamma)\right\} \tag{24}$$

such that we can provide the following sufficient conditions for the sign of dn/dA.

Proposition 6:

1) Sufficient for $dn/dA<0$ is:

   $\eta'(c)\geq 0$ and $\gamma\leq 1$ (with one inequality strict) *or*

$\Omega \leq 0$

2) Sufficient for $dn/dA>0$ is:

   $\eta'(c) \leq 0$ and $\gamma \geq 1$ (with one inequality strict)

3) Sufficient for $dn/dA=0$ is:

   $\eta'(c)=0$ and $\gamma = 1$.

Proof: *See Appendix B.7* □

In the light of the analysis carried out so far, we can interpret our findings as follows. A technological improvement renders capital more productive, making it convenient for the economy to invest higher amounts of resources in capital accumulation and in production of goods. Moreover, such an increased level of productivity allows for higher per-capita consumption. As for fertility, things are more complex. In fact, when total productivity increases, by eq. (18) two different forces drive the change in the population growth rate: on the one hand, the change in the ratio *c/k* and, on the other hand, the change in the ratio F_N/k. The former effect depends on the relative elasticity of consumption and capital, while the latter is a function of the elasticity of substitution of inputs[9]. Which effect will prevail depends on the exact shape of the functions that are used for preferences and production technology, that is Ω and γ.

To start with, we can provide the following corollary:

Corollary 1: If the production function has unitary factor elasticity of substitution, for example Cobb-Douglas, then the sign of *dn/dA* depends only on individuals' preferences, i.e., on the sign of $\eta'(c)$. More precisely, a positive technological shock decreases (increases) the fertility rate if and only if $\eta'(c)>(<)0$.

Proof: *If, $\gamma=1$, F_N/k remains constant, for any level of A, and the result descends by observation of eq. (24).* □

We will analyse the shape of $\eta'(c)$ and of Ω for a broad class of individuals' preferences in section 5, after completing the analysis of the transitional dynamics.

4.4. Completing the transitional dynamics

In this section we complete the characterization of the transitional dynamics of the system generated by a positive technological shock. We will assume that such a shock affects the system at some time *t*=0, when it is at its steady state and the shock comes as a surprise.

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[9] In order to show this property, recall that $\gamma = -\frac{f'}{f''k}\left(1-\frac{f'k}{f}\right)$. Moreover, at the steady state $F_N = f - \rho k$. If we define *f=Ag(k)*, we get $\frac{d(F_N/k)}{dA} = \frac{d(f/k)}{dA} = \frac{g}{k} - \frac{d(g/k)}{dk}\frac{dk}{dA} = \frac{f}{Ak}(1-\gamma)$.

As for consumption and capital intensity, things are clear: whenever A increases, both steady-state consumption and capital intensity are increased. Moreover, by recalling Proposition 4, it can be shown that the initial jump in c is:

$$\frac{dc_0}{dA} = -C\frac{dk}{dA} + \frac{dc}{dA}, \text{ where } C = \frac{(\rho - n - \lambda_3)(\rho - n)}{\Omega + (\rho - n)} > 0 . \tag{25}$$

(see Appendix B.4). Since both $\frac{dk}{dA}$ and $\frac{dc}{dA}$ are positive, the sign of (25) is ambiguous. In any case, no matter the sign of the initial jump of consumption, after the positive technological shock consumption and capital intensity increase steadily towards their new steady-state values.

As for the dynamics of the rate of growth of population, things are more complex, in that, as we have shown in Propositions 3 and 6, both the steady-state change and the dynamic path followed by n after the technological shock are ambiguous. However, we can show that the initial jump is always negative.

We can summarize our results on the initial jump of c and n through the following:

Proposition 7: When a positive technological shock occurs:

$$\frac{dc_0}{dA} \gtrless 0 \text{ if } \Omega \lessgtr \overline{\Omega}, \text{ with } \overline{\Omega} = \frac{-\lambda_3 \alpha f\left(\frac{G}{\alpha} + \rho - n\right) + (\rho - n)G(f - f'k) + (\rho - n)^2 \alpha f}{Gf'k} > 0 .$$

As for the initial jump of the fertility rate:

$$\frac{dn_0}{dA} < 0 .$$

Proof: *See Appendix B.8* □

Let us now assume that $\gamma = 1$ for the sake of simplicity. We can summarize the behaviour of the system after a technological shock as follows: after an increase of total factor productivity, while capital intensity steadily increases towards to new higher steady-state value, as for the rate of growth of population we can have three possible cases (recall that $\alpha\Omega = GF_N / c - c\eta'(c)$ in steady state):

a) $\eta'(c)>0$ (in this case $dn/dA<0$) and $\Omega < 0$. After the technological shock n falls at an intermediate level between the old and the new (lower) steady-state value and then, according to Proposition 3, it decreases towards its new steady-state value.
b) $\eta'(c)>0$ (in this case $dn/dA<0$) and $\Omega>0$. According to Proposition 3, when $\Omega>0$ n increases towards its new lower steady-state value. Hence after the technological shock the fertility rate experiences an undershooting reaction, since it instantaneously falls below the new steady-state value.
c) $\eta'(c)<0$ (in this case $dn/dA>0$ and necessarily $\Omega>0$). The dynamics is the same as the previous case, although the new steady-state value of n is higher.

Finally, as for per-capita consumption, in case a) there is a positive jump while in cases b) and c) the jump is positive (negative) if $\Omega < (>) \overline{\Omega}$. Then, in any case, it steadily increases towards its new higher steady-state value.

For illustrative purposes, in Figure 1 we depict our findings on the dynamic behavior of *n* and *c*, after an unforeseen positive technological shock occurring in t_0, for the case in which the inputs elasticity of substitution, γ, is equal to 1.

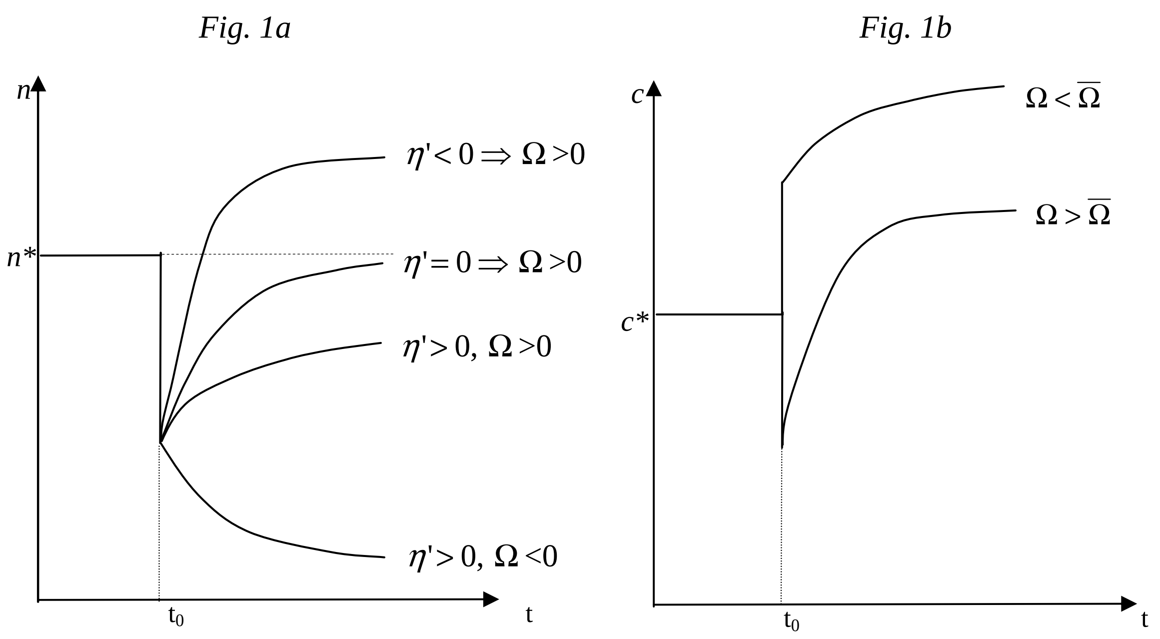

Figure 1: Qualitative dynamics of the population growth rate and of consumption per-capita after a positive technological shock, for $\gamma = 1$. Note that not necessarily the initial jump of *n* is quantitatively the same, regardless the sign of *η'*.

5. Examples of preference classes

In this section we explore the properties of $\eta'(c)$, which is crucial for determining the reaction both of fertility and consumption to a productivity shock**.** We focus on the shape of Ω in section 6.

5.1. The HARA utility function

Let us take a large class of preferences, HARA, normalized in such a way that $u \geq 0$ and $u(0) = 0$:

$$u(c) = \frac{M}{\varepsilon - 1}\left[(a + \varepsilon c)^{\frac{\varepsilon - 1}{\varepsilon}} - a^{\frac{\varepsilon - 1}{\varepsilon}}\right]$$

where $(a + \varepsilon c) > 0$, $a \geq 0$, *M>0.* Recall that such a function encompasses the iso-elastic utility ($a = 0$ and $\varepsilon > 1$) and, for $a > 0$, encompasses the normalized logarithmic utility as $\varepsilon \to 1$, negative exponential as $\varepsilon \to 0$ (and $M = a^{\frac{1}{\varepsilon}}$), quadratic utility if $\varepsilon = -1$ and linear utility ($a = 0$

and $\varepsilon \to \infty$).

For this class of preferences $\eta(c) \equiv \frac{u}{u'c} = \frac{\varepsilon}{\varepsilon - 1} + \frac{a}{\varepsilon - 1} \frac{1 - \left(1 + \frac{\varepsilon}{a} c\right)^{\frac{1}{\varepsilon}}}{c}$ and the following Lemma descends:

Lemma 1: Under HARA preferences and *a*>0:

$\eta'(c) > 0$ for , $c \in [0, \infty)$

$\eta'(c) = \frac{1}{2a}$ if $\varepsilon = \frac{1}{2}$,

$$\lim_{c \to \infty} \eta'(c) = \begin{cases} 0 \text{ if } \varepsilon > \frac{1}{2} \\ \\ \infty \text{ if } \varepsilon < \frac{1}{2} \end{cases}$$

if $\varepsilon = -1$ (quadratic utility), then $\lim_{c \to a} \eta'(c) = \infty$.

Proof: *See Appendix B.9* □

Hence, under a broad class of utility functions, $\eta'(c) > 0$.

We now turn to some special cases of HARA where we obtain closed form solutions.

5.1.1 CES utility function

Under CES utility (obtained from HARA by setting $a = 0$ and $\varepsilon = \frac{1}{\sigma}$), $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ (up to a multiplicative constant), with $\sigma \in (0,1)$, we have $\eta(c) = \frac{1}{1-\sigma}$, $\eta'(c) = 0$ and, thus, $\Omega > 0$. Then we obtain the following closed form solution: $\rho - n = \frac{c - F_N}{k} = \frac{\sigma}{2\alpha(1-\sigma)}\left(1 + \sqrt{1 + 4\frac{\alpha F_N}{(1-\sigma)\sigma^2 k}}\right)$.

In the light of the results above, we can give the following two Corollaries:

Corollary 2: If the utility function is CES, then the sign of *dn/dA* depends only on the sign of $(1-\gamma)$. More precisely, a technological shock decreases (increases) the steady-state fertility rate if and only if the factor elasticity of substitution is lower (higher) than unity.

Proof: *under CES utility,* $\eta'(c) = 0$ *and, by eq. (24), sign(dn/dA)=-sign* $(1-\gamma)$.

Corollary 3: If utility is CES (or Cobb-Douglas), and the production function has γ =1, then *dn/dA* is always zero.

5.1.2 HARA utility function with $\varepsilon = \dfrac{1}{2}$

This case is worth doing in that is allows to get a closed form solution. More precisely, when $\varepsilon = \dfrac{1}{2}$, $u(c) = 2\left[-\left(a + \dfrac{1}{2}c\right)^{-1} + a^{-1}\right]$. Recall that such a utility function displays the following properties $\eta(c) = \dfrac{2a + c}{2a}$, $\eta'(c) = \dfrac{1}{2a} > 0$. Finally, the following closed form solutions hold:

$$\rho - n = \frac{c - F_N}{k} = \frac{F_N + \sqrt{F_N{}^2 + (2a + F_N)(2a\alpha - k)\dfrac{F_N}{k}}}{2a\alpha - k} \quad \text{with } k < 2\alpha a\text{, i.e. } \eta'(c) < \frac{\alpha}{k}.$$

5.2. Non HARA preferences

Having shown that for a large class of preferences $\eta'(c) \geq 0$, we will now provide an example of a utility function for which $\eta'(c)$ is negative (obviously this will be a non-HARA function).

Consider the following utility function:

$u(c) = c + \dfrac{c^{1-\sigma}}{1-\sigma}$ with $\sigma \in (0,1)$. Notice that $u(0) = 0$; it can be shown that $\eta(c) = \dfrac{1 + \dfrac{c^{-\sigma}}{1-\sigma}}{1 + c^{-\sigma}} > 0$, $\eta'(c) = -\dfrac{\sigma}{1-\sigma}\dfrac{c^{-1-\sigma}}{\left(1 + c^{-\sigma}\right)^2} < 0$ and $\Omega > 0$. In this situation, although we do not have a closed form solution, by Proposition 6, dn/dA can be positive (for example, if $\gamma \geq 1$).

6. The shape of Ω

Up to now we have shown that the dynamics of the model, and in particular of population growth, triggered by a productivity shock, can be very rich, depending both on technology and on preferences.

However, when $\Omega < 0$ and the factor elasticity of substitution is equal to 1 (which seems plausible to assume, in that, otherwise the share of income of either capital or labour would shrink to zero in the long run under highly-frequent technological shocks), then the effects of a positive shock of total productivity are univocally determined, in that the latter causes an increase of the steady state values of both capital intensity and consumption and a decrease of the steady-state value of the population growth rate. Moreover, as for the dynamics of the economic system we have shown that when $\Omega < 0$, $\dfrac{dn_0}{dA} < 0, \dfrac{dc_0}{dA} > 0$ and thereafter *n* (*c*) decreases (increases) towards the new steady-state value.

Note that that the dynamics arising in the latter case seems in line with the empirical evidence concerning the (negative) relationship between technological progress and per-capita income, on the one hand, and population growth, on the other hand, which, according to some scholars, has been characterizing the most developed countries over the last two centuries (see, for example, Galor 2005 and Jones and Tertilt 2006).

Hence, in this section we pin down some sufficient conditions for characterizing the sign of Ω. More precisely, we provide the conditions insuring that that beyond a certain value of total factor productivity, i.e. for $A > \overline{A}$, then both the dynamics of the system and the initial jumps of c and n are the ones mentioned above, in that $\Omega < 0$.

Proposition 8: Suppose that $0<\eta'(c)<\alpha/k$ and $1/4 \leq \alpha F_N/k \leq 1$; for any preference class where $\sigma(c)$ takes on values between $\frac{\eta(c)-1}{\eta(c)}$ and 1. Now, if $\frac{\sigma' c}{\sigma} > 1-\sigma$ (which is fulfilled for HARA preferences if $\varepsilon < 1$ and $a>0$), then there exists a unique level of A, $\overline{A}$, such that, at the steady state, $\Omega(\overline{A}) = 0$ with $\Omega'(\overline{A}) < 0$.

Proof: *See Appendix B.10.* □

Hence, we can conclude that, when $\gamma = 1$, under HARA preferences, with $\varepsilon < 1$ and $a>0$, beyond a certain level of technology, a positive technological shock will univocally increase consumption and capital intensity, and decrease population growth, both immediately and in the long run.

7. Conclusions

In this paper we develop a growth model with endogenous fertility which can avoid the repugnant conclusion. We propose an extended version of critical level utilitarianism (CLU) which we name Relative CLU. It consists of conditioning the critical level used by parents for deciding to give birth to an extra child on their own standard of living. We show that such social welfare criterion for variable population size is axiomatically founded.

Later, we provide a full characterization of the existence, stability and dynamic properties of the equilibrium. In particular, we analyze the relationship between the model variables and technological shocks and we show that the set of responses of the population growth rate to the latter shocks are quite rich, depending on both preferences and technology. For example, we show that when preferences are of the HARA form (but not CES) and the elasticity of substitution of production factors is equal to one, a positive shock of total factor productivity increases the steady-state values of both per-capita consumption and capital intensity and reduces the steady-state fertility rate. As for the latter, the sign of the initial jump and of the growth rate along the transition path is not necessarily the same as the sign of the change of the steady-state value. Moreover, we pin down the conditions under which the model can replicate the negative empirical relationship between technological change and per-capita income, on the one hand, and population growth, on the other hand, unveiled in some recent works for the developed countries over the last two centuries. These conditions state that such a negative relationship applies beyond a certain level of technological development.

Finally, we show that the model allows for even richer long-run effects of technological shocks on population growth (that is, positive shocks in total factor productivity can increase population growth rate in the long run) when preferences are of non-HARA form and/or the elasticity of substitution of production factors is different from one.

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Appendix

Appendix A. *Relative Critical-Level Utilitarianism*

We propose a population criterion in the spirit of CLU, but where the judgment (the critical level of utility for life worth living) is relative to the existing generation's level of wellbeing. According to such a criterion, a society at low level of utility sets a lower threshold of utility for the next generation, and a society with high living standard sets a higher level. So if parents had a good life, they require their children to have a good life as well, and vice versa. We call such a criterion Relative Critical Level Utilitarianism (RCLU). It is reasonable that societies 10000 years ago had an entirely different target level of utility for life worth living than societies today. Also, societies following the more flexible population criterion will find it easier to adapt to fluctuations in the external environment. For example, if food resources become scarce for a period and utility falls below an absolute critical level, society does not stop reproducing under RCLU, but may still view life worth living if the children have a utility related to parent's utility. Societies following (absolute) CLU may become extinct as they stop reproducing when utility for a number of generations fall below the absolute threshold level.

Following the approach by Koopmans (1960), we begin by assuming a general welfare function defined over population and utility alternatives. We then introduce one by one the postulates (axioms) to arrive at the final welfare representation. The approach has the advantage that the exposition is simpler and far more accessible, and one can clearly see how the welfare function is developed.

For any time t we define the alternatives, X_t, as the utility vector $u_t=\{u_t^1,\ldots,u_t^N\}$ of generation t and the size of the next generation, N_{t+1}, i.e. $X_t=\{u_t, N_{t+1}\}$. We assume the population criterion is represented by a general welfare function.

$$W_t = W_t\left(\ldots, X_{t-2,} X_{t-1}, X_t, X_{t+1}, \ldots\right) \tag{A.1}$$

We assume from the outset that the welfare function is twice differentiable.

We begin with a number of independence and dependence postulates (axioms). It is reasonable to assume that the choice of utility vectors and population sizes in the sufficiently distant future is independent of the outcomes of the earlier generations. That is, the preferences over u_t and N_{t+1} for large s should be independent of u_{t-1-s} and N_{t-s}. This is the axiom of Independence of the Long Dead as in Blackorby et al. (1995). To allow the possibility that population judgments depend on the utility level of the parental generation, we must allow for X_{t+1} to depend on X_t.

Independence of the Long Dead
Preferences over X_t are independent of X_{t-1-s} for $s\geq1$.

Parental Dependence[10]
Preferences over X_t depend on X_{t-1}.

First, independence of the long dead implies that any indifference relation between u_{t+1} and N_{t+2} (at time t+1) must be independent of the value of X_{t-1}. That is the preferences over X_{t+1} must be represented by some $\tilde{W}_{t+1}$:

[10] If generations do not overlap this is Dependence of the Recently Dead.

$$\tilde{W}_{t+1} = \tilde{W}_{t+1}\left(X_t, X_{t+1}, X_{t+2}...\right) \tag{A.2}$$

and any preferences over current and future X must be represented by some V_t, i.e. the aggregator function in Koopmans (1960):

$$W_t = V_t\left(X_{t-1}, X_t, \tilde{W}_{t+1}\left(X_t, X_{t+1}, X_{t+2}...\right)\right) \tag{A.3}$$

Note that we cannot have $W_t = V_t\left(X_{t-1}, \tilde{W}_{t+1}\left(X_t, X_{t+1}, X_{t+2}...\right)\right)$ as then the preferences over X_t would be independent of X_{t-1}, and thus violate Parental Welfare Dependence. Consequently, Independence of the Long Dead and Parental Welfare Dependence imply a weakly separable welfare function of the form (A.3).

Notice that $\tilde{W}_{t+1}$ is potentially different from W_t, however, if the function is different, then the preferences over X_{t+1} as of time t+1 (given a particular history of X_t), would be different from the preferences over X_{t+1} as of time t-s, $s \geq 1$, which would generate time-inconsistency. We shall therefore assume time-independence (or time consistency):

Stationarity (Time Independence/Time Consistency)
Denote a sequence of all future alternatives as $X_\tau^f = \{X_{\tau+1}, X_{\tau+2}, \ldots\}$. Then, for any history $X_t = \overline{X}$ and future values X_{t+1}^f the preferences over X_{t+1} should be the same as over X_t, given history $X_{t-1} = \overline{X}$ and future values X_t^f, if $X_t^f = X_{t+1}^f$.

Stationarity implies the functions

$$W_t = V_t\left(\overline{X}, X_t, \tilde{W}_{t+1}\left(X_t, X_t^f\right)\right) \tag{A.4}$$
$$\tilde{W}_{t+1} = V_{t+1}\left(\overline{X}, X_{t+1}, \hat{W}_{t+2}\left(X_{t+1}, X_{t+1}^f\right)\right) \tag{A.4'}$$

must represent the same preferences over X_{t+1}, which can only be the case if $V_t = V_{t+1}$ and $\tilde{W}_{t+1} = \hat{W}_{t+2} = W_t = W$. Consequently, from (A.3), we have

$$W\left(X_{t-1}, X_t, X_{t+1}...\right) = V\left(X_{t-1}, X_t, W\left(X_t, X_{t+1}, X_{t+2}...\right)\right) \tag{A.5}$$

That is a recursively separable social welfare function.

This welfare function, however, implies that in general the preferences over, say, u_t and N_{t+1}, depend on the values of u and N of *all* future generations. We limit such dependence as follows.

Independence of Distant Future Generations
Preferences over X_t are independent of X_{t+s}, for all $s \geq 2$.

We note that we cannot have independence of X_{t+1}, without sacrificing dependence on X_{t-1}. Thus, a necessary consequence of Parental Dependence and Independence of the Long Dead, is dependence of at least the next (immediate) generation.

The only possible way of limiting the dependence is to increase the degree of separability to

$$W\left(X_{t-1}, X_t, X_{t+1}...\right) = V\left(\phi\left(X_{t-1}, X_t,\right), W\left(X_t, X_{t+1}, X_{t+2}...\right)\right) \tag{A.6}$$

One more recursion gives

$$W(X_{t-1}, X_t, X_{t+1}...) = V(\phi(X_{t-1}, X_t,), V(X_t, X_{t+1}, W(X_{t+1}, .X_{t+2}, ...))) \quad \text{(A.7)}$$

To save on notation, denote

$$\phi^t = \phi(X_{t-1}, X_t) \quad \text{(A.8)}$$
$$\phi^{t+1} = \phi(X_t, X_{t+1}) \quad \text{(A.8')}$$

Any indifference relation between N_{t+1} and u_t^i is given by

$$\left.\frac{\partial N_{t+1}}{\partial u_t^i}\right|_{dW=0} = \frac{\dfrac{\partial V(\phi^t, V(\phi^{t+1}, W))}{\partial \phi^t}\dfrac{\partial \phi^t}{\partial u_t^i} + \dfrac{\partial V(\phi^t, V(\phi^{t+1}, W))}{\partial V(\phi^{t+1}, W)}\dfrac{\partial V(\phi^{t+1}, W)}{\partial \phi^{t+1}}\dfrac{\partial \phi^{t+1}}{\partial u_t^i}}{\dfrac{\partial V(\phi^t, V(\phi^{t+1}, W))}{\partial \phi^t}\dfrac{\partial \phi^t}{\partial N_{t+1}} + \dfrac{\partial V(\phi^t, V(\phi^{t+1}, W))}{\partial V(\phi^{t+1}, W)}\dfrac{\partial V(\phi^{t+1}, W)}{\partial \phi^{t+1}}\dfrac{\partial \phi^{t+1}}{\partial N_{t+1}}} \quad \text{(A.9)}$$

For this indifference relation to be independent of *W*, that is of X_t^f, the ratio of the partial derivatives must be constant:

$$\frac{\dfrac{\partial V(\phi^t, V(\phi^{t+1}, W))}{\partial V(\phi^{t+1}, W)}\dfrac{\partial V(\phi^{t+1}, W)}{\partial \phi^{t+1}}}{\dfrac{\partial V(\phi^t, V(\phi^{t+1}, W))}{\partial \phi^t}} = \beta \quad \text{(A.10)}$$

This implies that the function *V* must be linear:

$$W(X_{t-1}, X_t, X_{t+1}...) = \phi(X_{t-1}, X_t,) + \beta[\phi(X_t, X_{t+1}) + \beta W(X_{t+1}, .X_{t+2}, ...)] \quad \text{(A.11)}$$

Or by recursion

$$W(X_{t-1}, X_t, X_{t+1}...) = \sum_{s=0}^{\infty} \beta^s \phi(X_{t-1+s}, X_{t+s}) \quad \text{(A.12)}$$

That is, a discounted sum of generational welfare functions.

Having established the consequences of Independence of the Long Dead, Independence of Distant Future Generations, and Parental Dependence, we now turn to the specifics of the population choice.

We now write the pairs $X_{t-1} = \{u_{t-1}, N_t\}$ and $X_t = \{u_t, N_{t+1}\}$ explicitly. Recall that u_t is a vector of individual utilities $\{u_t^1, \ldots, u_t^N\}$, i.e. $\phi_t = \phi(u_{t-1}, N_t, u_t, N_{t+1})$. Note that N_{t+1} can be dropped at time t as we are only considering welfarist criteria (population only matters to the extent it brings individual utilities), then

$$\phi_t = \phi(u_{t-1}, N_t, u_t) \quad \text{(A.13)}$$

We will now define our population criterion for each generation.

Relative Critical Level

A society is indifferent adding individuals to an existing population, everything else equal, if the utility of those added equals a critical level function $\alpha(u_{t-1})$, depending on the utility of consumption of someone in the previous generation. If individuals differ within a generation, this level of consumption must be identified. We label this as reference-level consumption, c_{t-1}^{r}.

The critical level cannot depend on the population size of the previous generation (only the reference consumption level), as can be seen in (A.13) (it is ruled out by Independence of the Long Dead, as N_{t-1} was a decision taken by the N_{t-2} generation. The critical level must also satisfy anonymity (to be introduced later), implying it must be invariant with respect to any permutation (renaming the indexes).

A permissible relative critical level is the utility of a fraction, $0<\delta<1$, of the consumption level of the top r^{th} individual of cohort t-1:

$$\alpha(u_{t-1}^{r}) = u(\delta c_{t-1}^{r}) \tag{A.14}$$

This level is invariant to adding individuals of critical level consumption, if delta is not too large. To see this, order consumption (by renaming individuals) as $c_{t-1}^{1} \geq c_{t-1}^{2} \geq \cdots \geq c_{t-1}^{r} \geq \cdots \geq c_{t-1}^{N_{t-1}}$.

Adding individuals of the critical level (applied to their generation), will not change the level of c_{t-1}^{r} as long as $c_{t-1}^{r} \geq \delta c_{t-2}^{r}$. Furthermore, the critical level is invariant with respect to renaming individuals (as the ordering of consumption is independent of the 'names').

It should be noted that average consumption as reference consumption does not have the property of being independent of adding individuals of the critical level (as it would be declining). The same is also true for median consumption. However, if $c_{t-1}^{i} = c_{t-1}$ for all I (as under a first-best allocation) then the average, the median, and the top r^{th} levels are the same.

Consequently, (A.11) is:

$$W(X_{t-1}, X_t, \ldots) = \phi\left(u_{t-1}^{r}, N_t, u_t\right) + \beta\phi\left(u_t^{r}, N_{t+1}, u_{t+1}\right) + \beta^2 W(X_{t+1}, .X_{t+2}, \ldots) \tag{A.15}$$

For given population size, the social preferences over individual utilities may depend on population size and past utilities. We shall impose that the social preferences over utilities within a cohort should be independent of utilities of unconcerned individuals (but not necessarily their existence), that is if the utility vector $\{u_t\}$ is preferred to another utility vector $\{u_t'\}$, this should be the case regardless utilities of other individuals. We require

Independence of Utilities of Unconcerned Individuals

Preferences over $\{u_t^i\}$ and $\{u_t^j\}$ are independent of $\{u_t^h\}$ for $h \neq i,j$, and of u_{t-1}^{r}.

We also require[11]:

Anonymity

Preferences over $\{u_t^i\}$ are independent of identity i.

[11] These are birth-date dependent statements, as in Blackorby et al (1997).

Strong Pareto Principle
Welfare is increasing in each u_t^i.

Independence of the utilities of unconcerned individuals first requires (A.15) to be weakly separable in $\{u_t\}$, i.e.

$$W(X_{t-1},...)=\phi\big(\Pi(u_{t-1}^r,N_t),N_t,\Psi(u_t,N_t)\big)+\beta\phi\big(\Pi(u_t^r,N_{t+1}),N_{t+1},\Psi(u_{t+1},N_{t+1})\big)+\beta^2 W(X_{t+1},...)$$

for some functions Π and Ψ, that is the vector u_t enters as a function $\Psi(u_t)$, such that $\phi(u_{t-1}, N_t, \Psi(u_t), N_{t+1})$. Second, it requires $\Psi(u_t)$ to be an additive function. To see this, differentiate the welfare function with respect to u^j_t and u^i_t to obtain:

$$\frac{\dfrac{\partial W(X_{t-1},...)}{\partial u_t^j}}{\dfrac{\partial W(X_{t-1},...)}{\partial u_t^i}}=\frac{\dfrac{\partial \Psi}{\partial u_t^j}}{\dfrac{\partial \Psi}{\partial u_t^i}}$$

which can be independent of u^h, $h \neq i,j$, only if it is additive: $\Psi_t=\sum_{i=1}^{N_t}\psi(u_t^i,N_t)$. Anonymity requires the function ψ to be independent of i, and must be a strictly increasing function due to the Strong Pareto Principle. Differentiating with respect to u^r_t and u^i_t gives:

$$\frac{\dfrac{\partial W(X_{t-1},...)}{\partial u_t^r}}{\dfrac{\partial W(X_{t-1},...)}{\partial u_t^i}}=\frac{\dfrac{\partial \phi^t}{\partial \Psi}\dfrac{\partial \psi(u_t^r,N_t)}{\partial u_t^r}+\beta\dfrac{\partial \phi^{t+1}}{\partial \Pi}\dfrac{\partial \Pi(u_t^r,N_{t+1})}{\partial u_t^r}}{\dfrac{\partial \phi^t}{\partial \Psi}\dfrac{\partial \psi(u_t^i,N_t)}{\partial u_t^i}}$$

Which can be independent of u^h, $h \neq i,r$, only if it is ϕ is linear. Consequently

$$W(X_{t-1},...)=\Pi(u_{t-1}^r,N_t)+\sum_{i=1}^{N_t}\psi(u_t^i,N_t)+\beta\left(\Pi(u_t^r,N_{t+1})+\sum_{i=1}^{N_t}\psi(u_{t+1}^i,N_{t+1})\right)+\beta^2 W(X_{t+1},...) \tag{A.16}$$

All the reasoning up until now has been for fixed populations. We now turn to the population criterion itself.

Relative Critical Level Utilitarianism

A society is indifferent adding individuals to an existing population, everything else equal, if the utility of those added equals a critical level function $\alpha(u_{t-1}^r)$.

Suppose we add m number of individuals, with utilities at the critical level, $\alpha(u_{t-1}^r)$, in (A.16), then

$$W(X_{t-1},...)=\Pi(u_{t-1}^r,N_t+m)+\sum_{i=1}^{N_t}\psi(u_t^i,N_t+m)+m\psi(\alpha(u_{t-1}^r),N_t+m)+\beta(\Pi+\Psi)+\beta^2 W(X_{t+1},...) \tag{A.17}$$

Notice that Π and Ψ in period t+1 are unaffected by the population size N_t (a consequence of our definition of reference utility). Then (A.17), by definition of critical level utility, must be constant in m, i.e.

$$\frac{dW}{dm} = \frac{\partial \Pi(u_{t-1}^r, N_t + m)}{\partial N_t} + \sum_{i=1}^{N_t} \frac{\partial \psi(u_t^i, N_t + m)}{\partial N_t} + \psi(\alpha(u_{t-1}^r), N_t + m) + m \frac{\partial \psi(\alpha(u_{t-1}^r), N_t + m)}{\partial N_t} = 0$$

Since $\alpha(u_{t-1}^r)$ cannot depend on the utilities of the other individuals, the derivative of ψ with respect to N_t must be zero, consequently

$$\psi(\alpha(u_{t-1}^r)) = -\frac{\partial \Pi(u_{t-1}^r, N_t + m)}{\partial N_t}$$

Then, $\Pi(u_{t-1}^r, N_t) = -\psi(\alpha(u_{t-1}^r))N_t$ which substituted into (A.17) gives

Proposition A.1 (Relative Critical Level Utilitarianism)

A social welfare function (representing a social welfare ordering over population and consumption choice) that satisfies Independence of the Long Dead, Stationarity, Independence of Distant Future Generations, Independence of Utilities of Unconcerned Individuals, Anonymity, Pareto Principle and Relative Critical Level, must take the form:

$$W(u_{t-1}, N_t, u_t, N_{t+1}, \ldots) = \sum_{s=0}^{\infty} \beta^s \sum_{i=1}^{N_{t+s}} \left[\psi(u_{t+s}^i) - \psi(\alpha(u_{t-1+s}^r))\right] \qquad \text{(A.18)}$$

where $\psi'(u_{t+s}^i) - \psi'(\alpha(u_{t-1+s}))\alpha'(u_{t-1+s}) > 0$ *for any s.*

Note that if the Relative Critical Level does not depend on the past reference utility, then α is constant, and the population principle reduces to Generalised Critical Level Utilitarianism as in Blackorby et al. (1995). If the critical level is zero, it reduces to Generalised Classical Utilitarianism.

When considering a first best allocation within each generation, i.e. $u_t^1 = u_t^2 = \ldots = u_t^N$ the function ψ is redundant, and we have

$$W(u_{t-1}, N_t, u_t, N_{t+1}, \ldots) = \sum_{s=0}^{\infty} \beta^s N_{t+s} \left[u_{t+s} - \alpha(u_{t-1+s})\right] \qquad \text{(A.19)}$$

Throughout our paper we make the simplifying assumption that the critical level function is linear.

Appendix B: Proofs

Appendix B.1: The form of eq. (3)

By starting from eq. (1) and collecting utility terms of the same date, the welfare function W can be written as:

$$W = \sum_{t=0}^{\infty} \beta^t N_t u(c_t)(1 - \alpha\beta(1 + n_t)) - \alpha N_0 u(c_{-1})$$

By ignoring c_{-1} as it is irrelevant for the planning horizon, and defining $\beta = \frac{1}{1+\rho}$ we get: $\sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t N_t u(c_t)\left(1-\alpha\frac{1+n_t}{1+\rho}\right)$. In continuous time, by approximating $\frac{1+n_t}{1+\rho} \approx -(\rho - n_t)$ the latter expression can be written as follows:

$$U = \int_0^{\infty} e^{-\rho t} N_t u(c_t)[1 - \alpha(n_t - \rho)]dt. \qquad \square$$

Appendix B.2: Proof of Proposition 1

A c, say c^*, solving (18) is a steady-state solution. Then given c*, one can find the steady-state n as well (as the left-hand side is also equal to ρ-n).

Notice that, given any ρ, we can find a steady-state capital stock, say k^*, and an associated F_N/k. The left-hand side of (18) is negative at c=0 and linearly increasing in c (with coefficient 1/k). The right-hand side (RHS) is increasing in η(c) and is positive at c=0 (as either η(0)>1, if Inada conditions hold, or η(0)=1, otherwise).

If η(c) is constant or decreasing in c, $\eta'(c) \leq 0$, (so the RHS is constant or decreasing in c) , then there is always a unique c, say c^*, solving the equation above. Consequently, if $\eta'(c) \leq 0$, then there always exists a unique steady state.

If $\eta'(c) > 0$, RHS cannot grow at a too high rate, to ensure existence of c^*. We restrict to the case when $\alpha \leq k/F_N$, yielding RHS convex in η (or linear if $\alpha = k/F_N$).

(i) If η is convex or linear in c, then RHS is convex in c (or possibly linear in c, if η is linear and if $\alpha = k/F_N$). Sufficient for there being a solution is that the slope of RHS is not greater than 1/k for any c. Sufficient for this is that in the limit as $c \to \infty$, the slope of RHS is not greater than 1/k. Necessary for a unique steady state is that the slope never is greater than 1/k (otherwise there would be a second intersection for some large c). Consequently necessary and sufficient for there being a unique steady state is that $\lim_{c\to\infty}\frac{dRHS}{dc} \leq \frac{1}{k}$.

Taking the derivative of RHS we get:

$$\frac{dRHS}{dc} = \frac{\eta'(c)}{2\alpha} + \frac{1}{\alpha}\frac{(\eta(c)-1) + 2\frac{\alpha F_N}{k}}{\sqrt{(\eta(c)-1)^2 + 4\frac{\alpha\eta(c)F_N}{k}}}\eta'(c) \text{ or}$$

$$\frac{dRHS}{dc}=\left[\frac{1}{2\alpha}+\frac{1}{2\alpha}\frac{1+2\frac{\alpha F_N}{k(\eta(c)-1)}}{\sqrt{1+4\frac{\alpha\eta(c)F_N}{k(\eta(c)-1)^2}}}\right]\eta'(c).$$

Then as c goes to infinity (since η is increasing in c) we have

$$\lim_{c\to\infty}\frac{dRHS}{dc}=\frac{\lim_{c\to\infty}\eta'(c)}{\alpha}.$$

Thus, if η(c) is increasing and convex (or linear), then necessary and sufficient for existence of a unique steady state is that $\eta'(c)\leq\alpha/k$ (with strict inequality if η and RHS are linear).

(ii) If η(c) is concave, then RHS can be either convex or concave for regions of c. Then sufficient for a unique steady state is that the slope of the RHS is lower than 1/k for all c. Combining (i) and (ii) we have that necessary and sufficient for existence of a unique steady state is that $\eta'(c)\leq\alpha/k$ (with strict inequality if η(c) is linear and $\alpha = k/F_N$). The condition is also necessary. Suppose it is not, then we can pick a convex η(c) and have a contradiction. □

Appendix B.3: Proof of Proposition 2

From eq. (21) we get that necessary and sufficient for the stability of the equilibrium is $1+\frac{\Omega}{(\rho-n)}>0$ or, by using the definition of Ω, $2+\frac{Gk}{\alpha}\frac{u''}{u'}>0$. Moreover, by using the definition of G and of σ we get $2-\left(\frac{1}{\alpha}+\rho-n\right)\sigma\frac{k}{c}>0$. Next, by using eq. (19) we get $2\frac{c}{k}-\sigma\left(\frac{1}{\alpha}+\frac{c-F_N}{k}\right)>0$. Moreover, by using the definition of $c\eta'$ it descends

$$2\frac{c-F_N}{k}+2\frac{F_N}{k}-\left(\frac{\eta-1+c\eta'}{\eta}\right)\left(\frac{1}{\alpha}+\frac{c-F_N}{k}\right)>0 \tag{B.1}$$

If $\eta'<\alpha/k$, then sufficient for stability is

$$2\frac{c-F_N}{k}+2\frac{F_N}{k}\geq\left(\frac{\eta-1}{\eta}+\frac{\alpha}{\eta}\frac{c}{k}\right)\left(\frac{1}{\alpha}+\frac{c-F_N}{k}\right)\text{ or}$$

$$0\geq\frac{\alpha}{\eta}\left(\frac{c-F_N}{k}\right)^2+\left(\frac{\alpha}{\eta}\frac{F_N}{k}-1\right)\frac{c-F_N}{k}+\frac{\eta-1}{\alpha\eta}-\frac{2\eta-1}{\eta}\frac{F_N}{k},$$

Using the quadratic equation (that gives eq. (18)): $\left(\frac{c-F_N}{k}\right)^2=\frac{\eta-1}{\alpha}\frac{c-F_N}{k}+\frac{\eta}{\alpha}\frac{F_N}{k}$, we get the sufficient condition for stability as

$$0 \geq \left(\alpha \frac{F_N}{k} - 1 \right)\left(\frac{c - F_N}{k} - \frac{\eta - 1}{\alpha} \right)$$

The last term is positive (by the solution to the quadratic equation). Consequently, a sufficient condition for stability is $\alpha \leq k / F_N$.
If $\eta'(c) \leq 0$, it's simpler. Using eq. (19) we get as sufficient condition:

$$2\frac{c - F_N}{k} + 2\frac{F_N}{k} \geq \left(\frac{\eta - 1}{\eta} \right)\left(\frac{1}{\alpha} + \frac{c - F_N}{k} \right) \text{ or}$$

$$\frac{\eta + 1}{\eta}\frac{c - F_N}{k} + 2\frac{F_N}{k} \geq \left(\frac{\eta - 1}{\eta} \right)\frac{1}{\alpha} \quad \text{or by using (B.1)}$$

$$\frac{(\eta - 1)^2}{2\eta\alpha} + \frac{\eta + 1}{2\eta\alpha}\sqrt{(\eta - 1)^2 + 4\frac{\alpha\eta F_N}{k}} + 2\frac{F_N}{k} \geq 0 \text{, which is always true.} \qquad \square$$

Appendix B.4: Proof of Proposition 3

Let us rewrite the dynamical system as follows:

$$\begin{bmatrix} \dot{c} \\ \dot{k} \\ \dot{n} \end{bmatrix} = J \begin{bmatrix} c - c^* \\ k - k^* \\ n - n^* \end{bmatrix} \tag{B.2}$$

to which the general solution is

$$\begin{bmatrix} c - c^* \\ k - k^* \\ n - n^* \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ e^{\lambda_3 t} \end{bmatrix}. \tag{B.3}$$

Since $\lambda_1 > 0$, $\lambda_2 > 0$, it descends that the coefficients $A=B=D=E=G=H$ are equal to zero, otherwise the solution would be explosive and would violate transversality conditions.

Hence, by differentiating the system (B.3) with respect to time and equating it to the corresponding equations of system (B.2) we get:

$$\begin{cases} \lambda_3 C = J_{11}C + J_{12}F + J_{13}I \\ \lambda_3 F = J_{21}C + J_{22}F + J_{23}I \\ \lambda_3 I = J_{31}C + J_{32}F + J_{33}I \end{cases}$$

Since F can be easily obtained from eq. (B.3) by exploiting the starting condition, such that $F = k_0 - k^*$, the system boils down to the following two equations[12]:

$$\begin{cases}(J_{11} - \lambda_3)C + J_{13}I = -J_{12}[k_0 - k^*] \\ J_{21}C + J_{23}I = (\lambda_3 - J_{22})[k_0 - k^*]\end{cases}$$

The solutions for C and I are:

$$C = \frac{-J_{12}J_{23} - (\lambda_3 - J_{22})J_{13}}{(J_{11} - \lambda_3)J_{23} - J_{13}J_{21}} \text{ and } I = \frac{(J_{11} - \lambda_3)(\lambda_3 - J_{22}) + J_{12}J_{21}}{(J_{11} - \lambda_3)J_{23} - J_{13}J_{21}}.$$

Finally, by substituting from the Jacobian matrix, we get that

$$C = -\frac{\frac{G}{\alpha}f''k - (\rho - n - \lambda_3)(\rho - n)}{(\Omega - \lambda_3) + (\rho - n)}, \text{ and } I = -\frac{(\Omega - \lambda_3)(\lambda_3 - \rho + n) - \frac{G}{\alpha}f''k}{(\Omega - \lambda_3)k + (\rho - n)k}.$$

Moreover, by substituting for the expression for $\frac{G}{\alpha}f''k$ stemming from (20) and simplifying, it follows that:

$$C = \frac{(\rho - n - \lambda_3)(\rho - n)}{\Omega + \rho - n} \text{ and } I = \Omega\frac{(\rho - n - \lambda_3)}{k(\Omega + \rho - n)}. \quad \text{(B.4)}$$

Since the denominators of C and I are positive under stability and $\lambda_3<0$, it descends that $C>0$. On the contrary, the sign of I is ambiguous and more precisely, $sign(I)=sign(\Omega)$.

Finally, the linearized solution of the dynamic system is

$$\begin{cases}c - c^* = Ce^{\lambda_3 t}[k_0 - k^*] \\ k - k^* = e^{\lambda_3 t}[k_0 - k^*] \\ n - n^* = Ie^{\lambda_3 t}[k_0 - k^*]\end{cases} \quad \text{(B.5)}$$

with C>0 and sign(I)=sign(Ω). □

Appendix B.5: Proof of Proposition 4

Recall that the determinant of the Jacobian matrix is $D(J) = \frac{Gf''k}{\alpha}(\rho - n + \Omega)$, which is negative under stability of the equilibrium. Since

[12] One of the three equations is redundant as it can be obtained by using the other two.

$$J_c = \begin{bmatrix} \frac{G}{\alpha}\frac{\partial F_N}{\partial A} & \frac{G}{\alpha} f''k & -(\rho-n)k \\ -\frac{\partial f}{\partial A} & \rho - n & -k \\ \frac{G}{\alpha}\left[\frac{1}{k}\left(\frac{\Omega}{(\rho-n)}-1\right)\frac{\partial F_N}{\partial A} - \frac{\partial F_K}{\partial A}\right] & \frac{\Omega G f''}{\alpha(\rho-n)} & -\Omega + (\rho-n) \end{bmatrix}$$

it follows that:

$$Det(J_c) = \frac{Gk}{\alpha^2}\left\{\frac{\partial F_K}{\partial A}\left[Gf''k - \alpha(\rho-n)^2\right] + f''\left[G\frac{\partial F_N}{\partial A} + \frac{\partial f}{\partial A}\alpha(\rho-n)\right]\right\}. \qquad \text{(B.6)}$$

If we define $f = Ag(k)$ we obtain $\frac{\partial F_N}{\partial A} = \partial\frac{Ag - Ag'k}{\partial A} = \frac{F_N}{A} > 0$ and $\frac{\partial F_K}{\partial A} = \partial\frac{Ag'}{\partial A} = \frac{F_K}{A} > 0$. Hence,

$$\frac{dc}{dA} = \frac{D(J_c)}{Det(J)} > 0. \qquad \square$$

Appendix B.6: Proof of Proposition 5

$$J_k = \begin{bmatrix} \Omega & \frac{G}{\alpha}\frac{\partial F_N}{\partial A} & -(\rho-n)k \\ -1 & -\frac{\partial f}{\partial A} & -k \\ \left(\frac{\Omega}{\rho-n}-1\right)\frac{\Omega}{k} & \frac{G}{\alpha}\left[\frac{1}{k}\left(\frac{\Omega}{(\rho-n)}-1\right)\frac{\partial F_N}{\partial A} - \frac{\partial F_K}{\partial A}\right] & -\Omega + (\rho-n) \end{bmatrix}$$

By exploiting the steady-state relationships $F_K \equiv f' = \rho$ and $F_N = f - f'k$, we can write the following

$D(J_k) = -\frac{G}{\alpha}k(\rho - n + \Omega)\frac{\partial F_K}{\partial A}$ <0, under stability of the equilibrium. (B.7)

$\square$

Appendix B.7. Proof of Proposition 6

First,

$$J_n = \begin{bmatrix} \Omega & \frac{G}{\alpha} f''k & \frac{G}{\alpha}\frac{\partial F_N}{\partial A} \\ -1 & \rho - n & -\frac{\partial f}{\partial A} \\ \frac{\Omega-(\rho-n)}{(\rho-n)k}\Omega & \frac{Gf''\Omega}{\alpha(\rho-n)} & \frac{G}{\alpha}\left[\frac{1}{k}\left(\frac{\Omega}{(\rho-n)}-1\right)\frac{\partial F_N}{\partial A}-\frac{\partial F_K}{\partial A}\right] \end{bmatrix}$$

The Determinant is

$$D(J_n) = -\frac{G}{\alpha^2}\left[-\alpha\Omega\left(-\frac{\partial F_K}{\partial A}(\rho-n)+f''\frac{\partial f}{\partial A}\right)+Gf''\left(\frac{\partial F_N}{\partial A}+k\frac{\partial F_K}{\partial A}\right)\right]$$

By recalling that $\frac{\partial F_K}{\partial A}=\frac{F_K}{A}$, $\frac{\partial f}{\partial A}=\frac{f}{A}$, $\frac{\partial F_N}{\partial A}=\frac{F_N}{A}$ and exploiting CRS, we obtain:

$$D(J_n) = -\frac{G}{\alpha^2 A}\left[\alpha\Omega\left(F_K(\rho-n)-f''f\right)+Gf''f\right]. \qquad \text{(B.8)}$$

Moreover, by exploiting the steady-state relationships $\alpha\Omega=\left(\frac{GF_N}{c}-c\eta'(c)\right)$ and $-\alpha\Omega=-G+\eta+c\eta'(c)$ we get

$$-\frac{G}{\alpha^2 A}\left[\left(c\eta'(c)+\eta\right)f''f+\left(\frac{GF_N}{c}-c\eta'(c)\right)F_K(\rho-n)\right]$$

where $\left(c\eta'(c)+\eta\right)=1+\eta(c)\sigma(c)>0$. Finally, by using the steady-state relationship $Gk(\rho-n)=\frac{u}{u'}\equiv\eta c$ and collecting terms we get

$$D(J_n) = -\frac{G}{\alpha^2 A}\left\{c\eta'(c)\left[f''f-F_K(\rho-n)\right]+\eta f''f(1-\gamma)\right\} \qquad \text{(B.9)}$$

which provides the results. □

Appendix B.8. Proof of Proposition 7

As for consumption, by exploiting eq. (25) we have:

$$Det(J)\frac{dc_0}{dA} = Det(J_c) - C\cdot Det(J_k).$$

Moreover, using eqs. (B.4), (B.6) and (B.7) and simplifying we get:

$$Det(J)\frac{dc_0}{dA}=\frac{Gf'k}{\alpha A}\left[\frac{f''f}{f'}\left(\frac{G}{\alpha}+\rho-n\right)-\lambda_3(\rho-n)\right].$$

By exploiting the expression for f'' from eq. (20) it follows that:

$$Det(J)\frac{dc_0}{dA}=\frac{Gf'k\lambda_3}{\alpha A}\left[\frac{-\lambda_3\frac{\alpha f}{Gkf'}\left(\frac{G}{\alpha}+\rho-n\right)+(\rho-n)\left(\frac{f}{f'k}-1\right)+(\rho-n)^2\frac{\alpha f}{Gkf'}-\Omega}{1+\frac{\Omega}{(\rho-n)}}\right].$$

Hence, we get:

$$Det(J)\frac{dc_0}{dA}\underset{<}{\overset{>}{-}}0\Leftrightarrow\Omega\underset{<}{\overset{>}{-}}\overline{\Omega}.$$

where $\overline{\Omega}=\dfrac{-\lambda_3\alpha f\left(\frac{G}{\alpha}+\rho-n\right)+(\rho-n)G(f-f'k)+(\rho-n)^2\alpha f}{Gf'k}>0$. (B.10)

Finally, since $Det(J)$ <0, we can conclude that $\frac{dc_0}{dA}\underset{<}{\overset{>}{-}}0$ if $\Omega\underset{>}{\overset{<}{-}}\overline{\Omega}$.

As for fertility, recall that from (B.5)

$n-n^*=Ie^{\lambda_3 t}\left[k_0-k^*\right]$, where sign($I$)=sign($\Omega$).

Now, suppose that at some instant t=0 a positive technological shock occurs. We have that

$$\frac{dn_0}{dA}=-I\frac{dk^*}{dA}+\frac{dn^*}{dA}\Rightarrow sign\left(Det(J)\frac{dn_0}{dA}\right)=sign\left[Det(J_n)-I\cdot Det(J_k)\right]. \quad \text{(B.11)}$$

Exploiting (B.5). (B.7) and (B.8) we can write:

$$Det(J)\frac{dn_0}{dA}=-\frac{Gf'}{\alpha A}\left[f''\frac{f}{f'}\left(\frac{G}{\alpha}-\Omega\right)+\lambda_3\Omega\right].$$

Notice that the term in square brackets is negative if $\Omega\geq 0$; furthermore, since $Det(J)$ <0, we then have $\frac{dn_0}{dA}<0$ if $\Omega\geq 0$.

To show that the latter result holds also for negative Ω, use eq. (20) to substitute for f'' to obtain:

$$Det(J)\frac{dn_0}{dA}=-\frac{Gf'\lambda_3}{\alpha A}\left[\frac{\rho-n}{\rho-n+\Omega}(\rho-n-\lambda_3)\frac{\alpha f}{Gkf'}\left(\frac{G}{\alpha}-\Omega\right)+\Omega\right].$$

If Ω <0, then $\frac{\rho-n}{\rho-n+\Omega}$ >1 and the bracketed term above is greater than:

$$(\rho-n-\lambda_3)\frac{\alpha f}{Gkf'}\left(\frac{G}{\alpha}-\Omega\right)+\Omega>(\rho-n-\lambda_3)\frac{f}{kf'}+\Omega>\rho-n-\lambda_3+\Omega>\rho-n+\Omega>0,$$

where first inequality follows from Ω being negative, second stems from concavity of production function, and third from stability.

Hence, the bracketed term is positive, and so, $Det(J)\frac{dn_0}{dA}>0$ which implies $\frac{dn_0}{dA}<0$, also if $\Omega<0$.

□

Appendix B.9. Proof of Lemma 1

Under HARA preferences, with *a>0*, we get that $\eta(c)\equiv\frac{u}{u'c}=\frac{\varepsilon}{\varepsilon-1}+\frac{a}{\varepsilon-1}\frac{1-\left(1+\frac{\varepsilon}{a}c\right)^{\frac{1}{\varepsilon}}}{c}$ and $\eta'(c)=\frac{\psi(c)}{c^2}$, where $\psi(c)=\frac{a}{\varepsilon-1}\left[\left(1+\frac{\varepsilon}{a}c\right)^{\frac{1-\varepsilon}{\varepsilon}}\left(1+\frac{\varepsilon-1}{a}c\right)-1\right]$ and $\psi'(c)=\frac{c}{a}\left(1+\frac{\varepsilon}{a}c\right)^{\frac{1-2\varepsilon}{\varepsilon}}>0$.

Since $\lim_{c\to0}\eta'(c)=\lim_{c\to0}\frac{\psi(c)}{c^2}=\frac{0}{0}$, by using l'Hôpital's rule we get that $\lim_{c\to0}\eta'(c)=\lim_{c\to0}\frac{\psi'(c)}{2c}=\frac{1}{2a}$. Finally, since $\psi'(c)>0$ for $c>0$, it descends that $\eta'(c)>0$ for $c\in[0,\infty[$.

Moreover, since $\lim_{c\to\infty}\eta'(c)=\frac{\infty}{\infty}$ by using l'Hôpital's rule we get that,

$$\lim_{c\to\infty}\eta'(c)=\lim_{c\to\infty}\frac{\psi'(c)}{2c}=\frac{1}{2a}\left(1+\frac{\varepsilon}{a}c\right)^{\frac{1-2\varepsilon}{\varepsilon}}=\begin{cases}0 \text{ if } \varepsilon>\frac{1}{2}\\ \infty \text{ if } \varepsilon<\frac{1}{2}\end{cases}.$$

Next, if $\varepsilon=\frac{1}{2}$ then $\eta'(c)=\frac{1}{2a}$, while, for $\varepsilon=-1$ (normalized quadratic utility), it descends $\eta'(c)=\frac{a}{2}\left(\frac{1}{a-c}\right)^2$, such that $\lim_{c\to a}\eta'(c)=\infty$.

Finally, if $\varepsilon=1$, then $u=\ln(c+1)$, $\eta'=\frac{c-\ln(c+1)}{2c}$, such that $\lim_{c\to0}\eta'(c)=\frac{1}{2}\frac{1}{c+1}=\frac{1}{2}$ and $\lim_{c\to\infty}\eta'(c)=0$. □

Appendix B.10. Proof of Proposition 8

Recall that $\alpha\Omega=[-\eta(c)\sigma(c)+\alpha(\rho-n)]$ and, by exploiting eqs. (18) and (19) it descends

$$2\alpha\Omega = \eta - 1 + \sqrt{(\eta-1)^2 + 4\frac{\alpha\eta F_N}{k}} - 2\eta\sigma \tag{B.12}$$

Hence, when $\sigma = \frac{\eta-1}{\eta}$ (which is lower than 1), then Ω>0, while for $\sigma = 1$, $\Omega < 0$. Hence, Ω changes sign at least once within $\left(\frac{\eta-1}{\eta},1\right)$ such an interval for σ. In fact, if Ω=0 then

$$\sigma = \frac{\eta-1}{2\eta} + \frac{1}{2\eta}\sqrt{(\eta-1)^2 + 4\frac{\alpha\eta F_N}{k}} < 1 \;\; if \;\; \frac{\alpha F_N}{k} < 1. \tag{B.13}$$

Recall that exploiting (19) and the relationship $\eta\sigma - (\eta-1) = \eta' c$, stability condition 0<$\eta'(c)c$<$c\alpha/k$; implies the following inequality to hold:

$$\sigma \leq \frac{\eta-1}{\eta} + \frac{\alpha F_N}{\eta k} + \frac{\eta-1}{\eta 2} + \frac{1}{2\eta}\sqrt{(\eta-1)^2 + 4\frac{\alpha\eta F_N}{k}} \tag{B.14}$$

In order to guarantee that the inversion of the sign of Ω (i.e. condition B.9) does not violate stability condition (i.e. B.10), it is sufficient to insure that the RHS of (B.14) is bigger than one, that is, after some manipulation.

$$0 < -(\eta-1) + 8\left(\frac{\alpha F_N}{k} - 1\right)\left(\frac{\alpha F_N}{k} - \frac{1}{4}\right). \tag{B.15}$$

Since we assumed that $\left(\frac{\alpha F_N}{k} - 1\right) < 0$, then sufficient condition for (B.15) to hold if that $\frac{\alpha F_N}{k} > \frac{1}{4}$.

Moreover, under the assumption that $\gamma = 1$, the ratio $\frac{F_N}{k}$ is constant with respect to A. In order to guarantee that Ω changes sign only once, we will provide sufficient conditions whereby $\left.\frac{d2\alpha\Omega}{dA}\right|_{\Omega=0} < 0$. By differentiating (B.12) w.r.t A, we have:

$$\frac{d2\alpha\Omega}{dA} = \left[\eta' + \eta'\frac{1}{2}\frac{2(\eta-1) + 4\frac{\alpha F_N}{k}}{\sqrt{(\eta-1)^2 + 4\frac{\alpha\eta F_N}{k}}} - 2\eta'\sigma - 2\eta\sigma'\right]\frac{dc}{dA} =$$

$$\left[1 + \frac{1}{2}\frac{2(\eta-1) + 4\frac{\alpha F_N}{k}}{\sqrt{(\eta-1)^2 + 4\frac{\alpha\eta F_N}{k}}} - 2\sigma - 2\frac{\eta\sigma'}{\eta'}\right]\eta'\frac{dc}{dA} =$$

$$= 2\frac{\eta\sigma}{c}\left[\left(1 - \sigma + \frac{\alpha\Omega}{\eta^2\sigma} + \frac{\alpha^2\Omega^2}{\eta^2\sigma}\right)\frac{\eta\sigma - (\eta-1)}{2\eta\sigma - (\eta-1) + 2\alpha\Omega} - \eta\frac{\sigma' c}{\sigma}\right]\frac{dc}{dA} \tag{B.16}$$

Moreover, by evaluating the (B.16) in Ω=0 and exploiting the relationship: $\eta\sigma-(\eta-1)=\eta' c$, one gets that

$$\frac{d2\alpha\Omega}{dA}=2\frac{\eta\sigma}{c}\left[(1-\sigma)\frac{\eta' c}{\eta' c+\eta\sigma}-\frac{\sigma' c}{\sigma}\right]\frac{dc}{dA}. \tag{B.17}$$

Since $\frac{\eta' c}{\eta' c+\eta\sigma}<1$, then it turns out that sufficient condition to have $\frac{d2\alpha\Omega}{dA}<0$ is that

$$(1-\sigma)-\frac{\sigma' c}{\sigma}\leq 0. \tag{B.18}$$

As for HARA preferences, with *a>0*, one can show that $\sigma=\frac{c}{a+\varepsilon c}$, $\sigma'=\frac{a}{(a+\varepsilon c)^2}$. Hence, recalling that $\eta\sigma-(\eta-1)=\eta' c$, we have that for such a class of preferences, $\eta'>0$ implies that $\eta\sigma-(\eta-1)=\eta' c>0 \Rightarrow \sigma>\frac{\eta-1}{\eta}, \forall c>0$. Moreover,

$$\sigma=1 \;\Rightarrow \frac{c}{a+\varepsilon c}=1 \Rightarrow c=\frac{a}{1-\varepsilon}>0 \text{ if } \varepsilon<1. \tag{B.19}$$

Finally, as for eq. (B.18), it turns out that

$$(1-\sigma)-\frac{\sigma' c}{\sigma}=\frac{\varepsilon-1}{a+\varepsilon c}\leq 0 \Leftrightarrow \varepsilon\leq 1. \tag{B.20}$$

However, by (B.19), $\varepsilon<1$. □