On the Feedback Solutions of Differential Oligopoly Games with Hyperbolic Demand Curve and Capacity Accumulation

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July 24, 2013

Abstract

To safeguard analytical tractability and the concavity of objective functions, the vast majority of models belonging to oligopoly theory relies on the restrictive assumption of linear demand functions. Here we lay out the analytical solution of a differential Cournot game with hyperbolic inverse demand, where firms accumulate capacity over time *à la* Ramsey. The subgame perfect equilibrium is characterised via the Hamilton-Jacobi-Bellman equations solved in closed form both on infinite and on finite horizon setups. To illustrate the applicability of our model and its implications, we analyse the feasibility of horizontal mergers in both static and dynamic settings, and find appropriate conditions for their profitability under both circumstances. Static profitability of a merger implies dynamic profitability of the same merger. It appears that such a demand structure makes mergers more likely to occur than they would on the basis of the standard linear inverse demand.

JEL Classification: C73, L13

Keywords: capacity, differential game, Markov-perfect equilibrium, Hamilton-Jacobi-Bellman equation, horizontal mergers

1 Introduction

Most of the existing literature on oligopoly theory (either static or dynamic) assumes linear inverse demand functions, as this, in addition to simplifying calculations, also ensures both concavity and uniqueness of the equilibrium, which, in general, wouldn't be warranted in presence of convex demand systems (see [16] and [10], *inter alia*). However, the use of linear demand function is in sharp contrast with the standard microeconomic approach to consumer behaviour, where the widespread adoption of Cobb-Douglas preferences (or their log-linear affine transformation) yields hyperbolic inverse demand functions. The same applies to the so-called quasi-linear utility function, concave in consumption and linear in money, that again yields a convex demand system. Indeed, both preference structures share the common property of producing isoelastic demand functions.[1]

Furthermore, such demand functions have been employed in rent-seeking games, where important contributions have been provided by [7] and [32], which showed the strict relations between the rent-seeking games and the Cournot oligopoly models. The isoelastic demand function has been widely analyzed in static settings (duopoly, [24], triopoly, [25]), devoting a particular attention to the stability of equilibrium and the possible chaotic implications (see also [26] and [27]).

In fact, this is sometimes openly referred to in the field of industrial organization, where researchers mention the opportunity of dealing with non-linear demand curves, and then promptly leave it aside for the sake of tractability.[2] Additionally, the econometric approach to demand theory has produced the highest efforts in building up a robust approach to the estimation of non-linear individual and market inverse demand functions, yielding a large empirical evidence in this direction.[3]

A relevant market demand structure was proposed by Anderson and Engers at the beginning of the 90s ([1]), which deserves some discussion: calling Q the aggregate quantity of a homogeneous good and $p(Q)$ the inverse demand function, they assumed the form $p(Q) = (a - Q)^{\frac{1}{\alpha}}$ to generalize the linear case ($a > 0$ is the market reservation price). Such inverse demand leads to different scenarios depending on parameter α: convex if $\alpha > 1$, concave if $\alpha < 1$, linear if $\alpha = 1$, as is outlined by the following Figures.

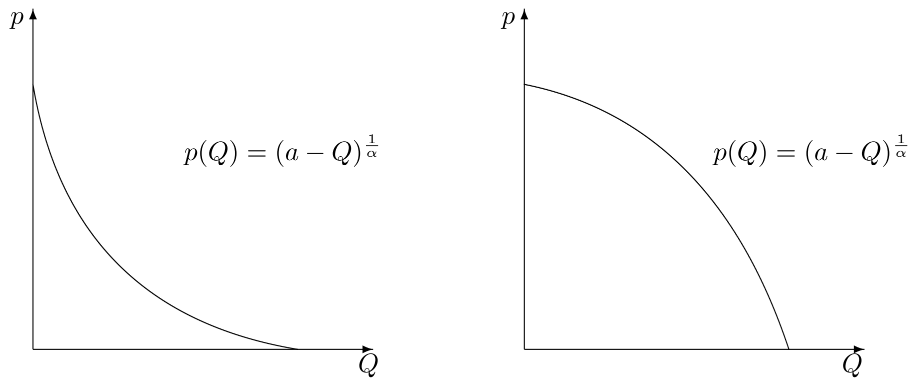

Figure 1. Anderson and Engers inverse demand functions for $\alpha > 1$ and $\alpha < 1$

[1] For a thorough illustration of these issues in consumer theory, see the classical textbooks: [9] and [35], *inter alia*.

[2] A noteworthy example in this respect is [30] (pp. 53-54), using quasi-linear utility function to define the concept of consumer surplus.

[3] See [17], [33] and [34], *inter alia*.

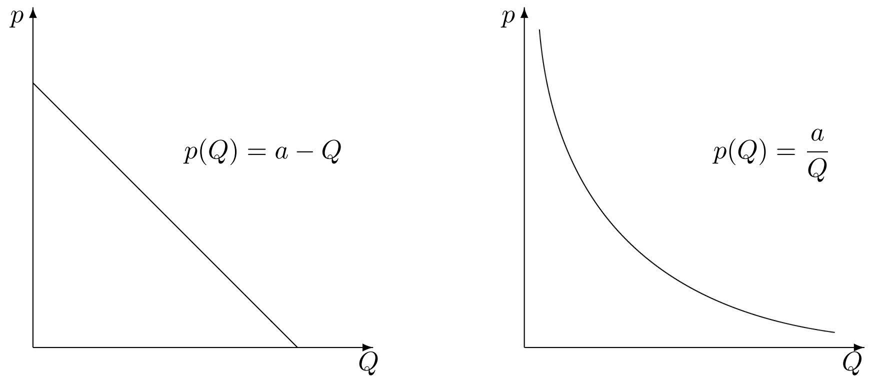

Figure 2. Inverse demand functions in the linear and in the hyperbolic cases

As can be simply seen, only the hyperbolic inverse demand function has an important technical prerogative: it admits an unbounded strategy space (or control space, in the differential game setup) for the quantity variables, which are feasible if they are positive. In other words, for every positive level of production, the market price of the good, and consequently the net revenue, is positive.

With these considerations in mind, it appears desirable to investigate the bearings of hyperbolic demand systems on the performance of firms operating in oligopolistic markets, using thus a setup with solid microfoundations corroborated by robust empirical evidence, even though this is a costly approach in terms of analytical tractability.

With specific reference to differential games, the use of linear inverse demand functions (jointly with either linear or quadratic cost functions) allows for the closed-form solution of the feedback equilibrium through the Bellman equation of the representative firm, as the model takes a linear-quadratic form and therefore one can stipulate that the corresponding candidate value function is also linear-quadratic. However, there is no particular reason to believe that a linear function describes correctly virtually any market demand in the real world, and therefore it is of primary interest to design, if possible, closed-form solutions of market games with non-linear demand functions. To the best of our knowledge, the only existing examples of differential oligopoly games with non-linear market demand are in [5], [13] and [20]. The first one uses a non-linear demand *à la* Anderson and Engers ([1]) and also investigates horizontal mergers, whereas the second one carries out a profitability assessment of small horizontal mergers subject to a sticky price dynamics. The third one employs a hyperbolic demand with sticky prices (as in [31] and [15]) as well, but leaves the merger issue out of the picture. Other non linear-quadratic structures are investigated by [18] and [19].[4]

The aim of this paper is to illustrate a way out of the aforementioned problem, offered by dynamic game theory. We are going to illustrate a dynamic Cournot model where firms (i) accumulate capacity *à la* Ramsey (1928), (ii) bear an instantaneous cost of holding any given capacity, and (iii) discount future profits at a constant rate. The main results are threefold.

[4]For excellent and largely complementary surveys of differential oligopoly games, see [11] and [21].

1. First, we determine the feedback information structure of the differential game both over infinite and finite horizon, solving in closed form the related Hamilton-Jacobi-Bellman systems of equations when the accumulation dynamics is linear and the accumulation cost is a polynomial function of a generic degree of the firm's capital endowment. When the accumulation follows a linear growth dynamics, the resulting feedback equilibrium coincides with the open-loop equilibrium, hence it is indeed subgame perfect.
2. Secondly, we investigate the standard Ramsey Ak model where the inverse demand is a hyperbolic curve. By applying the above results, we are able to completely characterize the equilibrium structure. Some results on feasibility of states and controls and on the comparison between the firms' profits in the static and in the dynamic frameworks are also featured.
3. Finally, we use it to investigate the profit (or, private) incentive towards horizontal mergers, to find that taking a dynamic perspective widens the range of privately feasible mergers. In particular, the form of the demand curve is crucial to allow for profitable mergers to take place in all setups:[5] That is, the presence of discounting, depreciation and a cost associated to holding capacity increases the firms' willingness to merge horizontally, for any admissible merger size. Any merger, of course, has undesirable consequences on consumer surplus and ultimately for welfare (at least in this model, where the efficiency defense is not operating).

The remainder of the paper is structured as follows. Section 2 features the basic concepts on the static game, and subsequently the complete calculation and the related properties of the feedback information structure of the differential game with capacity accumulation over infinite and finite horizons. In Section 3 two applications are taken into account and characterized: the Cournot-Ramsey differential game and the analysis of profitability of horizontal mergers. Section 4 incorporates our conclusions and further possible developments.

2 The oligopoly game with hyperbolic inverse demand

2.1 A summary of the static game

Consider a market where N single-product firms supply individual quantities $q_i > 0$, $i = 1, \dots, N$. The good is homogeneous, and the market inverse demand function is

$$p(q_1, \dots, q_N) = \frac{a}{q_1 + \cdots + q_N} = \frac{a}{Q}. \tag{1}$$

(1) is the outcome of the constrained maximum problem of a representative consumer endowed with a log-linear utility function

$$U(Q) = \log[Q] + m, \tag{2}$$

[5]To the best of our knowledge, scant attention has been devoted to the implications of dynamic competition on merger incentives, with the exceptions of Dockner and Gaunersdorfer in 2001 ([11]) and Benchekroun in 2003 ([2]), using a price dynamics *à la* Simaan and Takayama ([31]) and Cellini and Lambertini ([5]) adopting a Ramsey-type capital accumulation dynamics. All of these contributions, however, assume linear demand functions.

where m is a numeraire good whose price is normalised to one. The budget constraint establishes that the consumer's nominal income Y must be large enough to cover the expenditure, so that $Y \geq p(Q)Q + m$. The representative consumer is supposed to solve the following:

$$\max_{Q} \; L(Q) = U(Q) + \mu\left(Y - p(Q)Q - m\right). \tag{3}$$

Solving the above problem, one obtains indeed the hyperbolic inverse demand function (1), where $a = \mu^{-1} > 0$.

On the supply side, production entails a total cost $C_i = cq_i$, where $c > 0$ is a constant parameter measuring marginal production cost. Market competition takes place *à la* Cournot-Nash; therefore, firm i chooses q_i so as to maximise profits $\pi_i = (p(Q) - c)\, q_i$. This entails that the following first order conditions must be satisfied (given their form, it is not necessary to assume interior solutions):

$$\frac{\partial \pi_i}{\partial q_i} = \frac{aQ_{-i}}{(q_i + Q_{-i})^2} - c = 0 \tag{4}$$

where $Q_{-i} \equiv \sum_{j \neq i} q_j$. The associated second order condition:

$$\frac{\partial^2 \pi_i}{\partial q_i^2} = -\frac{2aQ_{-i}}{(q_i + Q_{-i})^3} < 0 \tag{5}$$

is always met when $q_i > 0$ for all $i = 1, \ldots, N$. Then, imposing the symmetry condition on all outputs, i.e. $q_i = q_j = q$, one obtains the individual Cournot-Nash equilibrium output $q^{CN} = \dfrac{a\,(N-1)}{N^2 c}$, yielding profits $\pi^{CN} = \dfrac{a}{N^2}$. If the N firms were operating under perfect competition, then $p^* = c$ and therefore $q^* = \dfrac{a}{Nc}$.

It is apparent that the above solutions (i.e., both the Cournot-Nash equilibrium and the perfectly competitive equilibrium) are well-defined and feasible for all $c > 0$.

In the remainder of the paper, we will turn our attention to a differential game where the demand structure is the same as here. We will separately investigate the infinite horizon and finite horizon cases.

2.2 Feedback solutions of the differential game on an infinite horizon

We are going to consider a market existing over $t \in [0\,, +\infty)$, and which is served by N firms producing a homogeneous good. Let $q_i(t) \in (0,\; +\infty)$ define the quantity sold by firm i at time t. Firms compete *à la* Cournot, the demand function at time t being:

$$p\,(t) = \frac{a}{Q\,(t)}, \qquad\qquad Q\,(t) = \sum_{i=1}^{N} q_i\,(t). \tag{6}$$

In order to produce, firms bear linear instantaneous costs $C_i\,(t) = cq_i\,(t)$, where $C > 0$. Moreover, they must accumulate capacity or physical capital $k_i(t) \in [0,\; +\infty)$ over time. If we denote with $y_i(t)$ the output produced by firm i at time t, we assume that k_i affects the production of y_i in the sense that $\dfrac{\partial y_i}{\partial k_i} > 0$. Capital accumulates as a result of intertemporal relocation of unsold output $y_i(t) - q_i(t)$.[6] This can be interpreted in two ways. The first consists in viewing this setup as a corn-corn model, where unsold output is reintroduced in the production process. This interpretation is admittedly unrealistic (although systematically mentioned in the whole macroeconomic debate stemming from Ramsey [28]) as it implies that the final good and the capital input are homogeneous. The second consists in thinking, a bit more realistically, of a two-sector economy where there exists an industry producing the capital input which can be traded against the final good at a price equal to one (for further discussion, see [5]), this channel being used to finance investment projects as an alternative to borrowing from the banking sector (which otherwise should be explicitly modelled). In a nutshell, the Ramsey dynamics is a blackbox hiding a market where firms sell part of their production at a fixed price for fundraising aimed at financing their growth.

Unlike the standard macroeconomic approach to growth models in a Ramsey fashion, here we will allow for the presence of an instantaneous cost of holding installed capacity. This cost will be $\Gamma_i(k_i(t))$, possibly asymmetric across firms. In the remainder, we will refer to $\dfrac{\partial \Gamma_i}{\partial k_i}$ as a measure of the *opportunity cost* of a unit of capacity. We will employ $k_i(t)$ as the i-th state variable subject to the following dynamic constraints:

$$\begin{cases} \dfrac{dk_i(t)}{dt} \equiv \dot{k}_i(t) = G_i(k_i(t)) - q_i(t) \\ k_i(t_0) = k_{i0} > 0 \end{cases}, \tag{7}$$

where $G_i(k_i(t))$ is a $C^2(\mathbb{R}_+)$ function affecting the growth dynamics of capital. The i-th firm's strategic variable is $q_i(t)$, while the i-th firm's state variable is $k_i(t)$.

A standard assumption in dynamic economic models concerns the kind of discounting: since the game takes place in a unique market, all firms discount profits at a same constant rate $\rho \geq 0$, possibly depending on monetary factors such as the expected rate of currency depreciation.

The problem of firm i is to choose the output level $q_i(t)$ so as to maximise its own discounted flow of profits (from now on, we will omit time arguments whenever possible):

$$\max_{q_i \in \mathbb{R}_+} J_i(k_{i0}, t_0) \equiv \int_{t_0}^{\infty} \left[(p(Q(\tau),\ \tau) - c)\, q_i(\tau) - \Gamma_i(k_i(\tau)) \right] e^{-\rho(\tau - t_0)} d\tau =$$

$$= \int_{t_0}^{\infty} \left[\left(\frac{a}{\sum_{i=1}^{N} q_i(\tau)} - c \right) q_i(\tau) - \Gamma_i(k_i(\tau)) \right] e^{-\rho(\tau - t_0)} d\tau. \tag{8}$$

We are going to adopt the dynamic programming approach, relying on the Hamilton-Jacobi-Bellman equations[7], generally considered as the most powerful tool in dynamic Game Theory and in a large number of economic models (both in discrete and continuous time).

The issue to be tackled is the determination of $V_i(k_{i0}, t_0)$, i.e. the optimal value of the i-th objective functional (8), over the set of all feasible paths $q_i : [t_0, \infty) \longrightarrow \mathbb{R}$. When the involved time horizon is infinite, the optimal value function does not depend on the initial time, but only on the initial state. In both cases, each optimal value

[6] Of course, capacity decumulates whenever $y_i(t) - q_i(t) \leq 0$.

[7] In the 50s Richard Bellman extended the Hamilton-Jacobi theory, which was developed in the 19th century to describe and solve problems in Classical Mechanics.

function can be achieved as the solution of a Hamilton-Jacobi-Bellman (HJB) equation (for the complete derivation, see [12], Chapter 3).

Call $k_{i0} = k$. Denoting with $V_i(k)$ the i-th optimal value function for (8), the Hamilton-Jacobi-Bellman (HJB) system of equations reads as follows:

$$\rho V_i(k) = \max_{q_i} \left\{ \left(\frac{a}{\sum_{i=1}^{N} q_i} - c \right) q_i - \Gamma_i(k) + \frac{\partial V_i}{\partial k}(G_i(k) - q_i) \right\}, \tag{9}$$

for all $i = 1, \dots, N$. Note that because J_i only depends on the i-th capital, in (9) the first order partial derivatives of V_i with respect to the remaining state variables do not appear.

To proceed with the analytical solution of the feedback problem, we are going to determine the symmetric Nash equilibrium of the game. Suitable symmetry assumptions are commonly employed both in Static and in Differential Game Theory, in that the players are identical firms, having an identical initial capital endowment and identical payoff structures[8]. This leads us to introduce suitable symmetry conditions from the beginning of the procedure: one is $q_i = q_j$ for all $i \neq j$, saying that the equilibrium quantity must be symmetric across all firms. The assumption of symmetry across capitals states that, from the standpoint of a generic firm i, the rivals' capacities (and therefore also their weights in the value function) must be equal when the respective growth dynamics and cost structures are equal.

Proposition 1. *Assuming symmetry across all variables, the HJB equation of the problem is given by:*

$$\frac{\partial V}{\partial k} = \rho \frac{V(k)}{G(k)} + \frac{\Gamma(k) - \frac{a}{N^2}}{G(k)}, \tag{10}$$

Proof. Maximizing the r.h.s. of (9) with respect to q_i yields:

$$\frac{aQ_{-i}}{(q_i + Q_{-i})^2} - c - \frac{\partial V_i}{\partial k} = 0, \tag{11}$$

then, by assuming symmetry on the relevant variables and functions, i.e. $q_1 = \ldots = q_N = q$, $V_1 = \ldots = V_N = V$, $\Gamma_1(\cdot) = \ldots = \Gamma_N(\cdot) = \Gamma(\cdot)$, $G_1(\cdot) = \ldots = G_N(\cdot) = G(\cdot)$, we have that (11) yields the following expression for the optimal strategy q^*:

$$q^* = \frac{a(N-1)}{N^2 \left(\frac{\partial V}{\partial k} + c \right)}, \tag{12}$$

which must be plugged into (9) to achieve:

$$\rho V(k) = \left(\frac{N \left(\frac{\partial V}{\partial k} + c \right)}{N-1} - c \right) \frac{a(N-1)}{N^2 \left(\frac{\partial V}{\partial k} + c \right)} - \Gamma(k) + \frac{\partial V}{\partial k} \left(G(k) - \frac{a(N-1)}{N^2 \left(\frac{\partial V}{\partial k} + c \right)} \right) \iff$$

[8] Note that in such a problem some asymmetric elements might be admitted when they do not change the HJB dramatically, for example growth rates or marginal cost parameters, but that goes beyond the scope of our paper

$$\iff \rho V(k) = \frac{a\left(N\frac{\partial V}{\partial k}+c\right)}{N^2\left(\frac{\partial V}{\partial k}+c\right)} - \Gamma(k) + \frac{\partial V}{\partial k}G(k) - \frac{\partial V}{\partial k}\left(\frac{a(N-1)}{N^2\left(\frac{\partial V}{\partial k}+c\right)}\right) \iff$$

$$\iff \rho V(k) + \Gamma(k) - \frac{\partial V}{\partial k}G(k) = \frac{a}{N^2\left(\frac{\partial V}{\partial k}+c\right)}\left(N\frac{\partial V}{\partial k} + c - (N-1)\frac{\partial V}{\partial k}\right) \iff$$

$$\iff \rho V(k) + \Gamma(k) - \frac{\partial V}{\partial k}G(k) = \frac{a}{N^2\left(\frac{\partial V}{\partial k}+c\right)}\left(\frac{\partial V}{\partial k}+c\right) \iff$$

$$\iff \frac{\partial V}{\partial k} = \rho\frac{V(k)}{G(k)} + \frac{\Gamma(k) - \frac{a}{N^2}}{G(k)}.$$

□

Corollary 2. *(10) admits the following family of solutions in any interval properly contained in the set $\{k \in \mathbb{R}_+ \mid G(k) \neq 0\}$:*

$$V^*(k) = \left(\widetilde{C} + \int^k \left(\frac{\Gamma(s) - \frac{a}{N^2}}{G(s)}\right) e^{-\int^s \frac{\rho}{G(\tau)}d\tau} ds\right) e^{\int^k \frac{\rho}{G(s)}ds}, \tag{13}$$

where $\widetilde{C}$ is a constant depending on the initial conditions of (10).

Expression (13) is useful to characterize the standard cases. In particular, when the capital's production function is linear and it does not involve fixed costs in absence of capital, it suggests us the following result:

Proposition 3. *If $G(k)$ is linear, $G(0) = 0$ and $\Gamma(k)$ is an m-th degree polynomial in k, then one solution of (10) is an m-th degree polynomial in k as well.*

Proof. By assumption, call $G(k) = \alpha k$ and $\Gamma(k) = \sum_{l=0}^m \beta_l k^l$. Replacing such functions in (13) yields:

$$V^*(k) = \left(\widetilde{C} + \int^k \left(\frac{\sum_{l=0}^m \beta_l s^l - \frac{a}{N^2}}{\alpha s}\right) e^{-\int^s \frac{\rho}{\alpha\tau}d\tau} ds\right) e^{\int^k \frac{\rho}{\alpha s}ds} =$$

$$= \left(\widetilde{C} + \int^k \left(\frac{\sum_{l=0}^m \beta_l s^l - \frac{a}{N^2}}{\alpha s}\right) s^{-\frac{\rho}{\alpha}} ds\right) k^{\frac{\rho}{\alpha}} =$$

$$= \widetilde{C}k^{\frac{\rho}{\alpha}} + \left[\int^k \left(\frac{1}{\alpha}\sum_{l=0}^m \beta_l s^{l-\frac{\rho}{\alpha}-1} - \frac{a}{\alpha N^2}s^{-\frac{\rho}{\alpha}-1}\right) ds\right] k^{\frac{\rho}{\alpha}} =$$

$$= \widetilde{C}k^{\frac{\rho}{\alpha}} + \sum_{l=0}^m \frac{\beta_l}{\alpha l - \rho}k^l + \frac{a}{\rho N^2},$$

hence the solution corresponding to the choice $\widetilde{C} = 0$ is an m-th degree polynomial in k. □

By plugging the solution into (12), it follows that:

Corollary 4. *If the assumptions of Proposition 3 hold, the optimal feedback strategy is given by:*

$$q^*(k) = \frac{a(N-1)}{N^2\left(\sum_{l=1}^{m} \frac{l\beta_l}{\alpha l - \rho} k^{l-1} + c\right)}. \tag{14}$$

From expression (14) it is immediate to deduce that firms'optimal strategy is a decreasing function of the initial capital, and it is feasible (positive) irrespective of the parameters, and consequently of the industry under examination.

2.3 Feedback solutions of the differential game on a finite horizon

Before investigating the finite horizon case, we should recall that in such cases sometimes the objective functional to be maximized is J_i plus a scrap value. The scrap value depends on the terminal value of the state, and can be seen as a kind of terminal prize or a final product of the accumulation of the state (see several examples in [12]). Even if we are not going to consider any scrap value to avoid further technical complications, it can suggest a further interpretation for a corn-corn model. Provided that capital and output are proportional, the scrap value can represent a part of unsold output, which can be employed in a subsequent game starting after time T, as the initial condition for the new capital to be accumulated.

On a finite horizon $[t,\ T]$, where $0 \le t < T < \infty$, the HJB system of equations of our problem takes the following form[9]:

$$-\frac{\partial V_i}{\partial t} + \rho V_i(k) = \max_{q_i} \left\{ \left(\frac{a}{\sum_{i=1}^{N} q_i} - c \right) q_i - \Gamma_i(k) + \frac{\partial V_i}{\partial k}(G_i(k) - q_i) \right\}, \tag{15}$$

where V_i depends on both k and initial time t. Differently from the infinite horizon case, we must additionally take into account the transversality conditions on all V_i:

$$\lim_{t \longrightarrow T} V_i(k,t) = 0. \tag{16}$$

Proposition 5. *If $G(k)$ is linear, $G(0) = 0$ and $\Gamma(k)$ is an m-th degree polynomial in k, then the system (15) admits the following solution:*

$$V(k,t) = \frac{-\beta_0 + \frac{a}{N^2}}{\rho}[1 - e^{\rho(t-T)}] + \sum_{l=1}^{m} \frac{\beta_l}{\alpha l - \rho}[1 - e^{(\rho - \alpha l)(t-T)}]k^l. \tag{17}$$

Proof. As in Proposition 3, call $G(k) = \alpha k$ and $\Gamma(k) = \sum_{l=0}^{m} \beta_l k^l$. The maximization of the r.h.s. of (15) yields:

$$\frac{aQ_{-i}}{(q_i + Q_{-i})^2} - c - \frac{\partial V}{\partial k} = 0, \tag{18}$$

[9]The difference with respect to equation (9) in the infinite horizon case consists in the presence of the first partial order derivative with respect to time.

then, by assuming symmetry on the relevant variables and functions, i.e. $q_1 = \ldots = q_N = q$, $V_1 = \ldots = V_N = V$, $\Gamma_1(\cdot) = \ldots = \Gamma_N(\cdot) = \Gamma(\cdot)$, $G_1(\cdot) = \ldots = G_N(\cdot) = G(\cdot)$, we have that (18) yields the following expression for the optimal strategy q^*:

$$q^* = \frac{a(N-1)}{N^2\left(\dfrac{\partial V}{\partial k} + c\right)}, \tag{19}$$

which must be plugged into (15) to achieve (the steps are analogous to those in Proposition 3, so we omit them):

$$-\frac{\partial V(k,t)}{\partial t} + \rho V(k,t) + \sum_{l=0}^{m} \beta_l k^l - \alpha k \frac{\partial V(k,t)}{\partial k} - \frac{a}{N^2} = 0. \tag{20}$$

We guess a function of the following kind for $V(k,t)$:

$$V(k,t) = \sum_{l=0}^{m} A_l(t) k^l, \tag{21}$$

where $A_l(t) \in C^1([t,T])$ and the transversality conditions are $A_l(T) = 0$ for all $l = 0, 1, \ldots, m$. Plugging (21) into (20), we obtain:

$$-\sum_{l=0}^{m} \dot{A}_l(t) k^l + \rho \sum_{l=0}^{m} A_l(t) k^l - \frac{a}{N^2} + \sum_{l=0}^{m} \beta_l k^l - \alpha k \sum_{l=0}^{m} l A_l(t) k^{l-1} = 0,$$

subsequently, all the coefficients of the powers of k are supposed to vanish, giving rise to the following dynamic system:

$$\begin{cases} -\dot{A}_0(t) + \rho A_0(t) - \dfrac{a}{N^2} + \beta_0 = 0 \\ -\dot{A}_1(t) + \rho A_1(t) - \alpha A_1(t) + \beta_1 = 0 \\ \ldots \\ -\dot{A}_m(t) + \rho A_m(t) - m\alpha A_m(t) + \beta_m = 0 \end{cases}.$$

By employing the transversality conditions, we achieve the following unique solutions:

$$A_0(t) = \frac{-\beta_0 + \dfrac{a}{N^2}}{\rho}[1 - e^{\rho(t-T)}],$$

$$A_l(t) = \frac{\beta_l}{\alpha l - \rho}[1 - e^{(\rho - \alpha l)(t-T)}],$$

for all $l = 1, \ldots, m$. Finally, substituting the found solutions in (21), we obtain the optimal value function in closed form:

$$V(k,t) = \frac{-\beta_0 + \dfrac{a}{N^2}}{\rho}[1 - e^{\rho(t-T)}] + \sum_{l=1}^{m} \frac{\beta_l}{\alpha l - \rho}[1 - e^{(\rho - \alpha l)(t-T)}] k^l. \tag{22}$$

□

Corollary 6. *If the assumptions of Proposition 5 hold, the optimal feedback strategy is given by:*

$$q^*(k,t) = \frac{a(N-1)}{N^2\left(\sum_{l=1}^m \frac{l\beta_l}{\alpha l-\rho}[1-e^{(\rho-\alpha l)(t-T)}]k^{l-1}+c\right)}. \tag{23}$$

From (23) we can note that $q^*(k,T) = q^{CN}$, meaning that the optimal strategy tends to the Cournot-Nash equilibrium at finite time irrespective of parameters and of the initial capital endowment. A direct comparison between the equilibrium quantities will be carried out in the next Section.

3 Applications

3.1 The Cournot-Ramsey game

In this well-known example, in order to produce, firms must accumulate capacity or physical capital $k_i(t)$ over time. We chose to consider the kinematic equations for capital accumulation as in Ramsey ([28]), i.e. the following dynamic constraints:

$$\begin{cases} \dot{k}_i(t) = Ak_i(t) - q_i(t) - \delta k_i(t) \\ k_i(0) = k_{i0} > 0 \end{cases}, \tag{24}$$

where $Ak_i(t) = y_i(t)$ denotes the output produced by firm i at time t and $\delta > 0$ denotes the decay rate of capital, equal across firms. I.e., this is the familiar Ak version of the Ramsey model, where $A = \frac{\partial y_i}{\partial k_i} > 0$ represents the constant growth rate at which output is produced as capital gets accumulated, essentially a measure of output productivity. The related cost will be $\Gamma_i(t) = bk_i(t)$, with $b \geq 0$, representing the aforementioned opportunity cost of a unit of capacity.

Because $G_i(k_i(t)) = \alpha k_i(t) = (A-\delta)k_i(t)$, we can apply all the results collected in the previous Section [10]. In particular, if we posit the following:

$$\beta_0 = 0, \qquad \beta_1 = b, \qquad \beta_2 = \ldots = \beta_m = 0, \qquad \alpha = A - \delta,$$

then the application of formulas (13), (14), (22) and (23) respectively entail:

- **Infinite horizon:**

$$V^*(k) = \frac{b}{A-\delta-\rho}k + \frac{a}{\rho N^2},$$

$$q^*(k) = \frac{a(N-1)(A-\delta-\rho)}{N^2[b+c(A-\delta-\rho)]}.$$

[10] Since the differential game at hand is a linear state one, i.e. one where, if we call $\mathcal{H}_i(\cdot)$ the i-th firm's Hamiltonian function, we have that: $\frac{\partial^2\mathcal{H}_i(\cdot)}{\partial q_i \partial k_j} = \frac{\partial^2\mathcal{H}_i(\cdot)}{\partial k_j^2} = 0$ for all $i,j = 1,\ldots,N$, (for more on linear state games, see [12](ch. 7), *inter alia*), the open-loop equilibrium is subgame perfect as it coincides with the feedback equilibrium $q^*(k,t)$ yielded by the HJB equation. Moreover, this would hold true also in the more general case where $y_i(t) = G_i(k_i(t))$, with $G_i'(k_i(t)) > 0$ and $G_i''(k_i(t)) \leq 0$. That is, state-linearity is not necessary to yield subgame perfection in a Cournot-Ramsey game. For more on this issue, see [4] and [6].

- **Finite horizon:**[11]

$$V^*(k,t) = \frac{a}{\rho N^2}[1 - e^{\rho(t-T)}] + \frac{b[1 - e^{(\rho-A+\delta)(t-T)}]}{A - \delta - \rho}k,$$

$$q^*(k,t) = \frac{a(N-1)(A-\delta-\rho)}{N^2[b(1 - e^{(\rho-A+\delta)(t-T)}) + c(A-\delta-\rho)]}.$$

In order to ensure the feasibility (i.e., the positivity) of such strategies, we need suitable parametric assumptions. In the next three Propositions, we will choose A as the crucial parameter, so as to refer all conditions to the level of output productivity.

Proposition 7. *In the infinite horizon case, if one of the following conditions:*

1. $A > \rho + \delta$,

2. $\delta < A < -\frac{b}{c} + \rho + \delta$,

holds, then $q^*(k)$ is feasible.

Proof. In particular, in both cases we have to assume $A > \delta$ to ensure accumulation of capital (otherwise $\dot{k}_i(t)$ would be negative at all t). If $A > \rho + \delta$, the numerator and the denominator of $q^*(k)$ are both positive, implying its feasibility. On the other hand, if $\delta < A < -\frac{b}{c} + \rho + \delta$, the numerator and the denominator are both negative, so feasibility is ensured as well. □

Moreover, here $q^*(k)$ is constant whereas $k^*(t)$ grows unbounded in that $A > \delta$. We can easily compare the optimal output with the one in the Cournot-Nash static setup:

Proposition 8. *In the infinite horizon case, we have that:*

1. *if* $A > -\frac{b}{c} + \rho + \delta$, *then* $q^{CN} > q^*(k)$;

2. $A < -\frac{b}{c} + \rho + \delta$, *then* $q^*(k) > q^{CN}$.

Proof. It suffices to evaluate the difference between optimal quantities:

$$q^*(k) - q^{CN} = \frac{a(N-1)(A-\delta-\rho)}{N^2[b + c(A-\delta-\rho)]} - \frac{a(N-1)}{N^2 c} = -\frac{ab(N-1)}{N^2 c[b + c(A-\delta-\rho)]}.$$

□

In the finite horizon case, the situation is different and we need to establish a time interval over which $q^*(k,t)$ is feasible. However, note that at $t = T$ the optimal strategy coincides with the Cournot-Nash optimal strategy: $q(k,T) = q^{CN}$.

[11] As an ancillary observation, it is worth noting that here, since the feedback optimal strategy coincides with the open-loop one and the strategic contributions cannot be distinguished under symmetry, the co-state variable at initial time appearing in the open-loop formulation of the game, which we omit here for brevity, can be appropriately considered as a shadow price of an additional unit of capacity, while, in general, this is true only of the partial derivative of the value function at initial state (for more on this aspect, see [3]).

Proposition 9. *In the finite horizon case we have that:*

1. *If* $\delta < A < -\frac{b}{c} + \rho + \delta$*,* $q^*(k,t)$ *is feasible for each* $t \in [0,\ T)$*.*

2. *If* $A > -\frac{b}{c} + \rho + \delta$*,* $q^*(k,t)$ *is feasible for each* $t \in (\widetilde{t},\ T)$*, where*

$$\widetilde{t} = T + \frac{1}{\rho - A + \delta} \ln \left[\frac{c}{b}(A - \rho - \delta) + 1 \right].$$

Proof. We are going to consider the two different cases:

If $A < \rho + \delta$, then the numerator is negative, hence we have to ensure that the denominator is negative too:

$$b(1 - e^{(\rho - A + \delta)(t-T)}) + c(A - \delta - \rho) < 0 \iff \ldots \iff$$

$$\iff e^{(\rho - A + \delta)(t-T)} > \frac{c}{b}(A - \delta - \rho) + 1.$$

If the r.h.s. is negative, i.e. $A < -\frac{b}{c} + \rho + \delta$, $q^*(k,t)$ is positive for all $t \in [0,\ T)$. If the r.h.s. is positive, then:

$$t - T > \frac{1}{\rho - A + \delta} \ln \left[\frac{c}{b}(A - \rho - \delta) + 1 \right],$$

hence if we call $\widetilde{t} = T + \frac{1}{\rho - A + \delta} \ln \left[\frac{c}{b}(A - \rho - \delta) + 1 \right]$, $q^*(k,t) > 0$ over $(\widetilde{t}, T)$.

Subsequently, consider $A > \rho + \delta$, we have to prove the positivity of the denominator of $q^*(k,t)$:

$$b(1 - e^{(\rho - A + \delta)(t-T)}) + c(A - \delta - \rho) > 0 \iff \ldots \iff$$

$$\iff t - T > \frac{1}{\rho - A + \delta} \ln \left[\frac{c}{b}(A - \rho - \delta) + 1 \right],$$

then $q^*(k,t) > 0$ over $(\widetilde{t}, T)$, meaning that the relevant condition for the restriction of the time interval is $A > -\frac{b}{c} + \rho + \delta$. □

The next Figures sketch the behaviour of the optimal strategy in finite horizon, showing the difference between its possible domains $(\widetilde{t},\ T]$ and $[0,\ T]$ in compliance with Proposition 9. As can be seen, the strategy is defined over the whole interval $[0, T]$ only if A is sufficiently small, i.e. if the expansion of capital does not grow too fast.

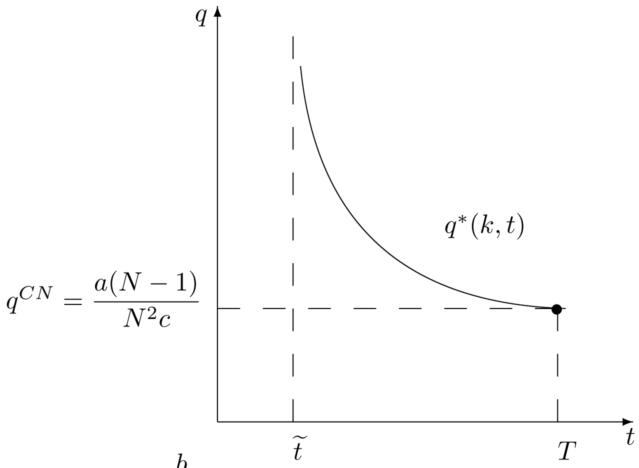

Figure 3. If $A > -\frac{b}{c} + \rho + \delta$, $q^*(k,t)$ is decreasing and it reaches q^{CN} at T.

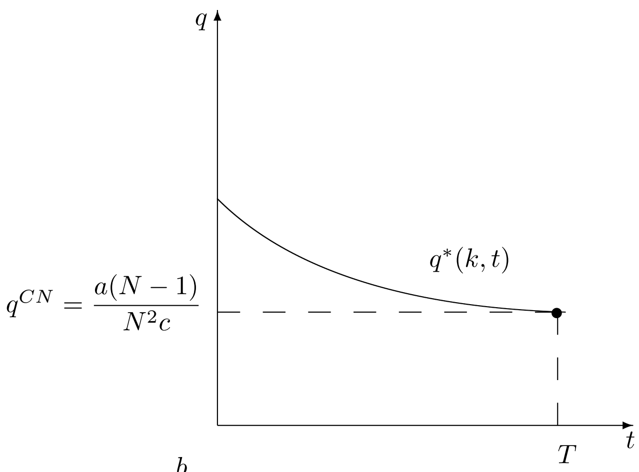

Figure 4. If $A < -\frac{b}{c} + \rho + \delta$, $q^*(k,t)$ is decreasing and it reaches q^{CN} at T.

Now we are going to focus our attention on the dynamic behaviour of the optimal capacity. If we substitute $q^*(k,t)$ in the dynamic constraint (24), we can also achieve the expression of the optimal state $k^*(t)$:

$$\begin{cases} \dot{k}_i(t) = (A-\delta)k_i(t) - \dfrac{a(N-1)(A-\delta-\rho)}{N^2[b(1-e^{(\rho-A+\delta)(t-T)}) + c(A-\delta-\rho)]} \\ k_i(0) = k_{i0} \end{cases} ,$$

whose unique solution is given by:

$$k^*(t) = \left(k_0 - \frac{a(N-1)(A-\delta-\rho)}{N^2}\int_0^t \frac{e^{-(A-\delta)s}}{b(1-e^{(\rho-A+\delta)(s-T)})+c(A-\delta-\rho)}ds\right)e^{(A-\delta)t}. \tag{25}$$

Now the joint feasibility of $q^*(t)$ and $k^*(t)$ can be evaluated:

Proposition 10. *Under the same assumptions as in Proposition 9, if $q^*(t)$ is feasible and if*

$$k_0 > \frac{a(N-1)(A-\delta-\rho)}{N^2}\int_0^t \frac{e^{-(A-\delta)s}}{b(1-e^{(\rho-A+\delta)(s-T)})+c(A-\delta-\rho)}ds$$

for all $t \in (0,\ T]$, then $k^*(t)$ is feasible as well.

Proof. It immediately follows from the positivity of $q^*(t)$ and from the expression (25). □

The next Proposition provides the exact expression of $k^*(t)$:

Proposition 11. *The optimal state of the Cournot-Ramsey game is given by the following function:*

$$k^*(t) = \Bigg[k_0 - \frac{a(N-1)(A-\delta-\rho)}{N^2}\left(\frac{e^{-(A-\delta-\rho)T}}{\rho}(e^{-\rho t}-1)+\right.$$

$$\left.\left.+\frac{b+c(A-\delta-\rho)}{\rho e^{-2(A-\delta-\rho)T}}\left(\log\left(\frac{e^{-(A-\delta-\rho)T}-(b+c(A-\delta-\rho))e^{-(A-\delta-\rho)t}}{e^{-(A-\delta-\rho)T}-(b+c(A-\delta-\rho))}\right)\right)\right)\right]e^{(A-\delta)t}. \tag{26}$$

Proof. The explicit calculation of (25) needs the calculation of the related integral:

$$I(t) = \int_0^t \frac{e^{-(A-\delta)s}}{C_1 - C_2 e^{(\rho-A+\delta)s}}ds,$$

where $C_1 = b + c(A-\delta-\rho)$ and $C_2 = e^{-(\rho-A-\delta)T}$. We have that:

$$I(t) = \int_0^t \frac{(-C_2)e^{-(A-\delta-\rho)s}+C_1-C_1}{(-C_2)e^{\rho s}(C_1-C_2e^{(\rho-A+\delta)s})}ds = \frac{1}{C_2}\left[\frac{e^{-\rho s}}{\rho}\right]_0^t + \frac{C_1}{C_2}\int_0^t \frac{1}{e^{\rho s}(C_1-C_2e^{(\rho-A-\delta)s})}ds =$$

$$= \frac{1}{\rho C_2}\left[e^{-\rho t}-1\right] + \frac{C_1}{C_2}\int_0^t \frac{1}{e^{\rho s}(C_1-C_2e^{(\rho-A-\delta)s})}ds.$$

Then, applying the change of variable $x = e^{-\rho s}$, leading to the change of differential $ds = -\frac{dx}{\rho x}$, we obtain:

$$I(t) = \frac{1}{\rho C_2}\left[e^{-\rho t}-1\right] - \frac{C_1}{\rho C_2}\int_1^{e^{-\rho t}} \frac{dx}{C_1 - C_2 x^{\frac{A-\delta-\rho}{\rho}}}ds =$$

$$= \frac{1}{\rho C_2}\left[e^{-\rho t}-1\right] + \frac{1}{\rho C_2}\int_{e^{-\rho t}}^1 \frac{dx}{1-\frac{C_2}{C_1}x^{\frac{A-\delta-\rho}{\rho}}}ds =$$

$$= \frac{1}{\rho C_2}\left[e^{-\rho t}-1\right] + \frac{1}{\rho C_2}\int_{e^{-\rho t}}^{1}\sum_{k=0}^{\infty}\left(\frac{C_2}{C_1}x^{\frac{A-\delta-\rho}{\rho}}\right)^k ds =$$

$$= \frac{1}{\rho C_2}\left[e^{-\rho t}-1\right] + \frac{1}{\rho C_2}\left[\sum_{k=0}^{\infty}\frac{\left(\frac{C_2}{C_1}\right)^k\left(x^{\frac{A-\delta-\rho}{\rho}}\right)^{k+1}}{k+1}\right]_{e^{-\rho t}}^{1} =$$

$$= \frac{1}{\rho C_2}\left[e^{-\rho t}-1\right] + \frac{1}{\rho C_2}\left[\frac{C_1}{C_2}\sum_{l=1}^{\infty}\frac{\left(\frac{C_2}{C_1}\right)^l}{l} - \frac{C_1}{C_2}\sum_{l=1}^{\infty}\frac{\left(\frac{C_2}{C_1}e^{-(A-\delta-\rho)t}\right)^l}{l}\right] =$$

$$= \frac{1}{\rho C_2}\left[e^{-\rho t}-1\right] + \frac{C_1}{\rho C_2^2}\left[-\log\left(1-\frac{C_1}{C_2}\right) + \log\left(1-\frac{C_1}{C_2}e^{-(A-\delta-\rho)t}\right)\right] =$$

$$= \frac{1}{\rho C_2}\left[e^{-\rho t}-1\right] + \frac{C_1}{\rho C_2^2}\left[\log\left(\frac{C_2 - C_1 e^{-(A-\delta-\rho)t}}{C_2 - C_1}\right)\right]. \qquad (27)$$

Finally, plugging (27) into (25) yields the complete expression (26)of the optimal capital:

$$k^*(t) = \left[k_0 - \frac{a(N-1)(A-\delta-\rho)}{N^2}I(t)\right]e^{(A-\delta)t} =$$

$$= \left[k_0 - \frac{a(N-1)(A-\delta-\rho)}{N^2}\left(\frac{e^{-(A-\delta-\rho)T}}{\rho}(e^{-\rho t}-1)+\right.\right.$$

$$\left.\left. + \frac{b+c(A-\delta-\rho)}{\rho e^{-2(A-\delta-\rho)T}}\left(\log\left(\frac{e^{-(A-\delta-\rho)T} - (b+c(A-\delta-\rho))e^{-(A-\delta-\rho)t}}{e^{-(A-\delta-\rho)T} - (b+c(A-\delta-\rho))}\right)\right)\right)\right]e^{(A-\delta)t}.$$

□

Given (26), we can evaluate $k^*(T)$, i.e. the terminal value of capital at the end of the time interval[12]:

$$k^*(T) = \left[k_0 - \frac{a(N-1)(A-\delta-\rho)}{N^2}I(T)\right]e^{(A-\delta)T} = -\frac{a(N-1)(A-\delta-\rho)}{\rho N^2}\left(1-e^{\rho T}+\right.$$

$$\left. + \frac{b+c(A-\delta-\rho)}{e^{-(3(A-\delta)-2\rho)T}}\left(\log\left(\frac{1-[b+c(A-\delta-\rho)]}{1-[b+c(A-\delta-\rho)]e^{(A-\delta-\rho)T}}\right)\right)\right) + k_0 e^{(A-\delta)T}. \qquad (28)$$

If we call $\pi^* = \pi(q^*(T),\ k^*(T))$ the profit function evaluated at the terminal instant T, we are able to compare it with the profit function $\pi^{CN} = \pi(Q^{CN})$ evaluated at the steady state in the static Cournot problem as shown in Subsection 2.1. Namely, we have that:

$$\pi^* = \left(\frac{a}{Nq_i^*(T)} - c\right)q_i^*(T) - bk_i^*(T) = \frac{a}{N} - cq^*(T) - bk^*(T) =$$

$$= \frac{a}{N^2} - b\left[k_0 e^{(A-\delta)T} - \frac{a(N-1)(A-\delta-\rho)}{N^2}(1-e^{\rho T})+\right.$$

$$\left. + \frac{b+c(A-\delta-\rho)}{\rho e^{-(3(A-\delta)-2\rho)T}}\left(\log\left(\frac{1-[b+c(A-\delta-\rho)]}{1-[b+c(A-\delta-\rho)]e^{(A-\delta-\rho)T}}\right)\right)\right]. \qquad (29)$$

[12]We omit the most tedious calculations, reminding the readers that all of them are available upon request to the authors.

Proposition 12. *Under the hypotheses of Propositions 9 and 10, if $q^*(t)$ and $k^*(t)$ are both feasible at all $t \in [0,\ T]$, then the Cournot-Nash equilibrium profit is larger than the Ramsey-Cournot equilibrium level at the terminal instant.*

Proof. It suffices to consider the difference:

$$\pi^* - \pi^{CN} = \frac{a}{N^2} - bk^*(T) - \frac{a}{N^2} = -bk^*(T)$$

which is strictly negative by the feasibility of $k^*(t)$ at all instants, meaning that the Cournot-Nash equilibrium profit exceeds the Ramsey-Cournot terminal profit. □

Remark 13. *It is worth noting that comparing the two optimal strategies in the static and in the dynamic cases, one immediately sees that the presence of capital accumulation in the dynamic game plays a key role in opening the way towards a solution to the indeterminacy issue affecting the static game as the marginal production cost c of the consumption good drops to zero. Essentially, if $c = 0$, no solution exists for the static game if no strategy space is compact, whereas in the differential game with capacity accumulation $q^*(k,t)$ is well-defined and feasible under suitable parametric conditions even when the marginal cost is zero, both over finite and infinite horizons.*

Propositions 8 and 12 suggest that at the subgame perfect equilibrium of the dynamic game the representative firm may produce either more or less but she earns higher profits when she plays the Cournot-Nash equilibrium of the static game, irrespective of the levels of marginal cost, opportunity cost, intensity of capacity accumulation growth and intertemporal discount rate.

Having characterised the subgame perfect equilibrium of the differential game, we can now proceed to the analysis of its application to horizontal mergers.

3.2 Horizontal mergers

To illustrate the advantages of our approach to the feedback solution of the differential oligopoly game *à la* Ramsey, we illustrate here its applicability to the analysis of the private profitability of a horizontal merger, and its welfare appraisal.

As is well known, a lively debate has taken place on this topic from the 1980's, based upon static oligopoly models. A thorough overview of it is outside the scope of the present paper, and it will suffice to recollect a few essential aspects. Examining a Cournot industry with constant returns to scale, in [29] it is shown that a large proportion of the population of firms has to participate in the merger in order for the latter to be profitable. In particular, a striking result of their analysis is that, in the triopoly case, bilateral mergers are never profitable. Enriching the picture by allowing for the presence of convex variable costs and fixed costs, one may find a way out of this puzzle (in particular, see [23] and [14]).

Now take the static Cournot game and examine the incentive for $M > 1$ firms to merge horizontally, out of the initial N. Before proceeding, we deem necessary to clearly define what is meant here as horizontal merger by a subset of the population of firms: the merger creates a single firm whose property is symmetrically distributed across the owners of the previously independent firms, and the resulting merged entity has a single control as well as a single state. This view of the merger can be justified on several grounds. To begin with, our assumptions on product homogeneity imply that productive capital is also homogeneous across firms, and therefore the merging process involves that indeed all previously independent plans are homogenized into a single entity. Additionally, our approach to the modelling of the merger is in line with the antitrust norms currently adopted both in the EU and in North America (US and Canada), establishing that the firms proposing a merger shall reduce their collective capacity endowment lest they attain a dominant position after the merger has taken place (see [8] and ch. 5 in [22], among others). Of course, the capacity and output of the firm resulting from the merger are necessarily larger than those associated to a generic firm if the merger does not occur, but this is simply the natural consequence of the decrease in the population of firms. Therefore, we are going to compare the performance of a firm in two games, alternatively characterized by N or $N - M + 1$ firms, all else equal.

After the merger (if it does take place), there remain $N - M + 1$ firms. If we call $\pi^{CN}(j)$ the profit of a firm in the Cournot static game among j firms, without distinguishing the original ones from those generated by the merger, we can prove the following:

Proposition 14. *In the static Cournot game with hyperbolic inverse demand, the merger is profitable if and only if* $N < M + \sqrt{M}$.

Proof. Profitability holds when

$$\frac{\pi^{CN}(N-M+1)}{M} > \pi^{CN}(N) \iff \frac{a}{M(N-M+1)^2} > \frac{a}{N^2} \iff$$

$$\iff N^2 > M(N-M+1)^2 \iff \ldots \iff N^2 - 2MN - M(1-M) < 0,$$

which entails $N \in \left(M - \sqrt{M},\ M + \sqrt{M}\right)$. Since $N \geq M$, the necessary and sufficient condition boils down to $N < M + \sqrt{M}$. □

It is easily checked that, contrary to [29], if $N = 3$ and $M = 2$, the merger is profitable.

On the other hand, if we consider the terminal outcome of the differential game over finite horizon, in compliance with the above notation, we can assess the profit incentive scheme for an M-firm merger in the dynamic framework too, sticking to the assumption of one state and one control variable per firm.

As for the assumptions of Proposition 9, ensuring the feasibility of the optimal strategy on $[0,\ T]$, we can state the following:

Proposition 15. *If a horizontal merger of M firms is profitable in the Cournot static game, then if* $\delta < A < -\frac{b}{c} + \delta + \rho$, *the same merger is profitable in the Cournot-Ramsey game on the horizon* $[0,\ T]$ *as well.*

Proof. Provided that the net revenue at the equilibrium of the Cournot static game is $\frac{a}{N^2}$ and the net revenue of the same game when a merger of M firms occurs is $\frac{a}{(N-M+1)^2}$, the profitability of a merger in the Cournot-Ramsey game is measured by

$$\frac{1}{M}\left(\frac{a}{(N-M+1)^2} - bk^*_{N-M+1}(T)\right) > \frac{a}{N^2} - bk^*_N(T), \tag{30}$$

where we called $k_l^*(T)$ the capital at time T under circumstances where a merger of $N-l$ firms took place. In order to simplify the notation, call

$$g(T) = \left(1 - e^{\rho T} + \frac{b + c(A-\delta-\rho)}{e^{-(3(A-\delta)-2\rho)T}}\left(\log\left(\frac{1-[b+c(A-\delta-\rho)]}{1-[b+c(A-\delta-\rho)]e^{(A-\delta-\rho)T}}\right)\right)\right)$$

as in (28). The inequality (30) becomes:

$$\frac{a}{M(N-M+1)^2} - \frac{b}{M}\left[-\frac{a(N-M)(A-\delta-\rho)}{(N-M+1)^2}g(T) + k_0 e^{(A-\delta)T}\right] >$$

$$> \frac{a}{N^2} - b\left[-\frac{a(N-1)(A-\delta-\rho)}{N^2}g(T) + k_0 e^{(A-\delta)T}\right],$$

which amounts to

$$a\left[\frac{N^2 - M(N-M+1)^2}{MN^2(N-M+1)^2}\right] - bk_0 e^{(A-\delta)T}\left(\frac{1}{M} - 1\right) +$$

$$+ba(A-\delta-\rho)g(T)\left[\frac{N-M}{M(N-M+1)^2} - \frac{N-1}{N^2}\right] > 0. \tag{31}$$

By Proposition 14, the first term of (31) is positive when the M-firm merger is profitable in the static framework, the second term is positive for all $M > 1$, whereas the remaining term is positive for $A < -\frac{b}{c} + \delta + \rho < \delta + \rho$ (the condition ensuring feasibility of the strategy over $[0,\ T]$ by Proposition 9) if and only if the quantity $\frac{N-M}{M(N-M+1)^2} - \frac{N-1}{N^2}$ is negative. In fact, we have that:

$$\frac{N-M}{M(N-M+1)^2} - \frac{N-1}{N^2} = \frac{N-M}{M(N-M+1)^2} - \frac{1}{N} + \frac{1}{N^2} <$$

$$< \frac{N-M}{M(N-M+1)^2} - \frac{1}{N} + \frac{1}{M(N-M+1)^2} = \frac{1}{M(N-M+1)} - \frac{1}{N} =$$

$$= \frac{N - MN + M^2 - M}{NM(N-M+1)} = \frac{(N-M)(1-M)}{NM(N-M+1)} < 0,$$

where we exploited the inequality $N^2 > M(N-M+1)^2$. Hence, this completes the proof. □

Taken together, these facts entail that the interval wherein the M-firm merger is profitable may be the same in the dynamic setup and in the static one, given that the measure of output productivity A is small enough.

The examination of the welfare consequences of a merger is omitted, as it goes without saying that any merger would diminish social welfare, both in the static as well as in the dynamic setting. This is trivially due to the fact that the damage caused to consumer surplus always outweighs the increase in industry profits.[13]

[13]In line of principle, a merger could allow for some reduction in the total opportunity costs for the industry, giving rise to a possible *efficiency defense* argument (see [14]). Although we omit the related calculations for brevity, it is quickly checked that this never outweighs the loss in consumer surplus necessarily generated by any merger. Hence, in this model the efficiency argument cannot be advocated to justify the merger itself.

4 Concluding remarks

Most of the existing literature on oligopoly theory postulates the presence of linear demand functions, this being quite specific in itself and, in general, at odd with empirical evidence. With this in mind, we have constructed a dynamic oligopoly model based on a non-linear demand. In particular, we have characterised the subgame perfect equilibrium of a dynamic Cournot game with inverse hyperbolic demand and costly capacity accumulation, showing that the feedback solution, coincident with the open-loop one, is subgame perfect. We have fully carried out the calculation of the optimal value functions and of the strategies of the differential game subject to a Ramsey-type dynamic constraint. Then, to illustrate the applicability of our framework, we have employed the model to analyse the feasibility of horizontal mergers in both static and dynamic settings, finding out appropriate parametric conditions under which the profitability of a merger in a static game implies the profitability in a dynamic game as well.

Possible future developments of our findings consist in the analysis of the feedback information structure of further differential oligopoly games endowed with a hyperbolic inverse demand function, possibly in presence of more complex dynamic constraints dealing, e.g., with issues related to international trade, environmental and resource economics and R&D activities.

Acknowledgements

We would like to thank Davide Dragone, Mauro Rota, Alessandro Tampieri, Stefan Wrzaczek and the audience at the EURO-INFORMS Joint International Conference in Rome for stimulating discussions and insightful suggestions. We are also grateful to three anonymous referees who greatly helped us improve the paper with their precious comments. The usual disclaimer applies.

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