Nonlinear dynamics and chaos in an OLG growth model with endogenous labour supply and multiplicative external habits

Luca Gori • Mauro Sodini

Abstract This paper analyses the mathematical properties of an overlapping generations growth model with endogenous labour supply and multiplicative external habits. The dynamics of the economy is characterised by a two-dimensional map describing the time evolution of capital and labour supply. We show that if the relative importance of external habits in the utility function is sufficiently high, multiple (determinate or indeterminate) fixed points and poverty traps can exist. In addition, cyclical or quasi-cyclical behaviour and/or coexistence of attractors may occur.

Endogenous preferences represent a topic of greater importance in economic theory. In particular, they influence individual choices about consumption and saving paths over time, and can be responsible of endogenous fluctuations and nonlinear dynamics in macroeconomic variables (e.g., per capita output). This study examines an economic growth model with overlapping generations where habit formation is a source of local and global bifurcations.

Keywords External habits; Indeterminacy; Labour supply; OLG model; Nonlinear dynamics

JEL Classification C61; C62; C68; J22; O41

I. INTRODUCTION

The study of economic models with habit formation has received in depth attention in the economic literature. Several works have dealt with this topic in recent years. Among them are the papers by Abel (1990), Boldrin et al. (1997) and Lahiri and Puhakka (1998) that are concerned with the study of habit formation and its relationship with the equity premium puzzle. Boldrin et al. (2001) introduces habit preferences in the standard real business cycle model to better explain the joint behaviour of asset prices and consumption. Chen and Hsu (2007) analyse a continuous time one-sector neoclassical growth model with inelastic labour supply, and show that consumption externalities can be a source of local indeterminacy when the degree of impatience is large enough, while Alonso-Carrera et al. (2008) generalises the model by introducing endogenous labour supply. From an empirical point of view, there exists evidence of the role of habit formation on macroeconomic variables (Ferson and Constantinides, 1991; de la Croix and Urbain, 1998; Carrasco et al., 2005).

It is useful to recall that in the economic literature different concepts of habit formation have been introduced. In particular:

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- (internal) habits refer to the case in which preferences of an individual depend on his/her own consumption as well as on a benchmark level that weights the consumer's own past consumption experience;

- aspirations (or external habits) refer to the case in which preferences of an individual depend on his/her own consumption as well as on a benchmark level that weights the consumption experience of others.

Amongst others, de la Croix (1996) and de la Croix and Michel (1999) study how external habits affect the dynamics of an economy with overlapping generations (OLG), while Carroll et al. (1997, 2000), Alonso-Carrera et al. (2004, 2005, 2007) analyse the role of internal and external habits on individual consumption decisions, savings and economic growth; de la Croix and Michel (2001) analyses a macroeconomic model with external habits and altruistic parents à la Barro (1974).

The present study aims at analysing the role of multiplicative external habits on the dynamics of a two-dimensional OLG economy where an individual works when he/she is young and consumes only when he/she is old (Woodford, 1984, Reichlin, 1986). The study of growth models that generate endogenous deterministic fluctuations dates back to Grandmont (1985), Farmer (1986), Reichlin (1986) and Azariadis (1993). Subsequently, several other authors have dealt with this topic in OLG models either with exogenous labour supply (Yokoo, 2000) or endogenous labour supply (Nourry, 2001; Nourry and Venditti, 2006).[1] The OLG literature related to the present paper is essentially represented by the works of de la Croix (1996) and de la Croix and Michel (1999). Both papers deal, however, with an economy where the labour supply is inelastic and individuals consume in both the first period and second period of life (Diamond, 1965). In particular, de la Croix (1996) shows that when the intensity of aspirations in utility is large, individuals want to increase consumption because the standard of living of their parents is high and then savings becomes low. This can generate a Neimark-Sacker bifurcation when savings experience too high a contraction because of the importance of past consumption levels. De la Croix and Michel (1999), instead, concentrates on the optimality issue in a growth model and show that the negative externality can be corrected through investment subsidies and lump-sum transfers. They also show that the taste externality may cause endogenous fluctuations.

The novelty of this study is the introduction of external habits in an OLG growth model with endogenous labour supply à la Reichlin (1986), i.e. individuals work when they are young and consume when they are old. First, we show that the relative importance of aspirations in utility is responsible for the existence of either one (normalised) fixed point (which can be determinate or indeterminate) or two interior fixed points. Second, some interesting local and global dynamic properties of the two-dimensional decentralised economy emerge: indeed, when the relative importance of aspirations in utility is strong enough cyclical or quasi-cyclical behaviour and/or coexistence of attractors may occur. In particular, this last phenomenon as well as the existence of some global bifurcations may cause global indeterminacy to the model while the stationary equilibria are locally determinate.

It is now useful to clarify the differences between local and global indeterminacy. A fixed point is locally indeterminate if for every arbitrarily small neighbourhood of it and for a given value of the state variable (the stock of capital) close enough to its coordinate at the stationary state, there exists a continuum of values of the control variable (the labour supply) for which equilibrium trajectories converge towards the fixed point. Differently, the system is globally indeterminate when there exist values of the state variable such that different choices on the control variable lead to different invariant sets. In this case, the initial condition of the stock of capital is not sufficient to define the long-term dynamics of the system.

[1] Some applications of nonlinear dynamics and chaos in macroeconomic models can be found in Zhang (1999), Antoci et al. (2004), Chen and Li (2011, 2013) and Fanti et al. (2013).

The rest of the paper is organised as follows. Sec. II outlines the model. Sec. III studies the conditions for the existence of fixed points of the map and analyses local bifurcations and stability. Sec. IV describes the global properties of the two-dimensional map describing the dynamics of capital and labour supply. Conclusions are drawn in Sec. V.

II. THE MODEL

We consider an OLG closed economy populated by a continuum of perfectly rational and identical two-period lived individuals of measure one per generation (Diamond, 1965). Time is discrete and indexed by $t = 0,1,2,...$. A new generation is born in every period. Each generation overlaps for one period with the previous generation and then overlaps for one period with the next generation. In the first period of life (youth), the individual of generation t is endowed with two units of labour and supplies the share $\ell_t \in (0,2)$ to firms, while receiving the wage w_t per unit of labour. The remaining share $2-\ell_t$ is used for leisure activities. Individuals consume only in the second period of life (Woodford, 1984; Reichlin, 1986; Galor and Weil, 1996; Grandmont et al., 1998; Antoci and Sodini, 2009; Gardini et al., 2009; Gori and Sodini, 2011, 2013).

The budget constraint of a young individual of generation t is $s_t = w_t \ell_t$, implying that labour income is entirely saved (s_t) to consume when old (C_{t+1}). Old individuals retire and consumption is constrained by the amount of resources saved when young plus expected interest accrued from time t to time $t+1$, so that $C_{t+1} = R^e_{t+1} s_t$ where R^e_{t+1} is the expected interest factor, which will become the realised interest factor at time $t+1$.

Therefore, the lifetime budget constraint of an individual of generation t can be written as follows:

$$C_{t+1} = R^e_{t+1} w_t \ell_t \,. \tag{1}$$

Individuals have preferences towards leisure when young and consumption when old. In addition, we assume the existence of a reference level against which consumption of the current generation is compared with. This implies that effective consumption of individuals of generation t is negatively affected by the consumption experience of their parents (a_t), which gives rise to a form of external habits in our model with a representative agent (de la Croix, 1996; Carroll et al., 1997, 2000; de la Croix and Michel, 1999; Gori and Sodini, 2013). It is important to specify that we are considering external habit under the flow concept of it.

We assume that the lifetime utility index of generation t is described by a twice continuously differentiable function $U_t(\bullet)$. Since individuals consume only when they are old, consumption of generation $t-1$ (C_t) affects consumption of generation t, so that $a_t = C_t$ for every t. By assuming that external habits take the multiplicative form C_{t+1}/a_t^{ρ} (Abel, 1990; Galí, 1994; Carroll, 2000; Bunzel, 2006; Hiraguchi, 2011), we specify the lifetime utility function of individuals of generation t by using the following Constant Inter-temporal Elasticity of Substitution (CIES) formulation (see Christiano et al., 2010):

$$U_t(\ell_t, a_t, C_{t+1}) = \frac{(2-\ell_t)^{1-\gamma}}{1-\gamma} + B\frac{(C_{t+1}/a_t^{\rho})^{1-\sigma}}{1-\sigma}\,, \tag{2}$$

where B is a scale parameter that allows us to define the normalised fixed point $(1,1)$ when the parameters of the model are continuously changed, $\sigma > 0$ ($\sigma \neq 1$) and $\gamma > 0$ ($\gamma \neq 1$) represent a measure of the constant elasticity of utility with respect to consumption and leisure, respectively, and $\rho \geq 0$ is the aspiration intensity: if $\rho = 0$, external habits are irrelevant; if $\rho = 1$ current and past consumptions are equally weighted in the utility function; if $\rho > 1$ external habits strongly matter.

By taking factor prices and external habits as given, the individual representative of generation t chooses ℓ_t to maximise utility function (2) subject to the lifetime budget constraint (1) and $\ell_t \in (0,2)$. Therefore, the first order conditions for an interior solution are given by:

$$-(2-\ell_t)^{-\gamma} + \frac{B}{\ell_t}(R^e_{t+1} w_t \ell_t / a_t^{\rho})^{1-\sigma} = 0. \quad (3)$$

Different from Grandmont et al. (1998) and Cazzavillan (2001), we do not consider externality in the production sector and assume that at time t identical and competitive firms produce a homogeneous good, Y_t, by combining capital and labour, K_t and L_t, respectively, through the constant returns to scale Cobb-Douglas technology $Y_t = A \cdot F(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha}$, where $A > 0$ and $0 < \alpha < 1$ are a scale parameter and the capital share in production, respectively. The equilibrium supply of labour at time t is given by $L_t = \ell_t$. Then, by assuming that capital fully depreciates at the end of every period and output is sold at the unit price, profit maximisation implies that the marginal productivity of capital (resp. labour) equals the interest factor (resp. the wage rate), that is:

$$R_t = \alpha A K_t^{\alpha-1} \ell_t^{1-\alpha}, \quad (4)$$

$$w_t = (1-\alpha) A K_t^{\alpha} \ell_t^{-\alpha}. \quad (5)$$

The market-clearing condition in the capital market can be expressed as follows:

$$K_{t+1} = s_t = w_t \ell_t. \quad (6)$$

By using (3)-(6) and knowing that: (*i*) individuals have perfect foresight, and (*ii*) the consumption reference for generation t can be expressed as $a_t = C_t = \alpha A K_t^{\alpha} \ell_t^{1-\alpha}$, equilibrium implies:

$$-(2-\ell_t)^{-\gamma} + \frac{B}{\ell_t}\left[\frac{\alpha(1-\alpha)A^2 K_{t+1}^{\alpha-1} \ell_{t+1}^{1-\alpha} K_t^{\alpha} \ell_t^{1-\alpha}}{(\alpha A K_t^{\alpha} \ell_t^{1-\alpha})^{\rho}}\right]^{1-\sigma} = 0, \quad (7)$$

$$K_{t+1} = A(1-\alpha) K_t^{\alpha} \ell_t^{1-\alpha}. \quad (8)$$

The dynamic system described by (7) and (8) defines the variables K_{t+1} and ℓ_{t+1} as functions of K_t and ℓ_t.

III. EXISTENCE OF FIXED POINTS AND LOCAL INDETERMINACY

The aim of this section is to study the existence and stability properties of the fixed point of the system given by (7) and (8). To this purpose, we use the geometrical-graphical method developed by Grandmont et al. (1998).

A. Existence of fixed points

It is well known that in economic models with overlapping generations the existence of fixed points is not generally guaranteed. Thus, we now impose some restrictions on parameters such that the (normalised) fixed point

$$(K, \ell) = (1,1) \quad (9)$$

always exists. This allows us to analyse the effects on stability due to changes in some parameter values by avoiding that the fixed point vanishes (Grandmont et al., 1998; Cazzavillan, 2001). Therefore, by using (7)-(9) we get:

$$A = A^* := \frac{1}{1-\alpha}, \quad (10)$$

$$B = B^* := \left(\frac{\alpha}{1-\alpha}\right)^{-(1-\sigma)(1-\rho)}. \quad (11)$$

By substituting out (10) and (11) into (7) and (8), the two-dimensional system that characterises the dynamics of the economy is the following:

$$M:\begin{cases} K_{t+1}=V(K_t,\ell_t):=K_t^{\alpha}\ell_t^{1-\alpha} \\ \ell_{t+1}=Z(K_t,\ell_t):=K_t^{\frac{\alpha(\rho-\alpha)}{1-\alpha}}\ell_t^{\frac{1+(1-\alpha)(1-\sigma)(\rho-\alpha)}{(1-\alpha)(1-\sigma)}}(2-\ell_t)^{\frac{-\gamma}{(1-\alpha)(1-\sigma)}} \end{cases}. \tag{12}$$

We note that given the couple (K_t,ℓ_t), it is possible to compute its subsequent iterate if and only if we start by a point in the set $D:=\{(K_t,\ell_t)\in R^2\ :\ K_t>0,\ 0<\ell_t<2\}$. However, feasible trajectories lie in a set smaller than D since by starting from an initial condition in D it is possible to have an iterate from which the existence of the subsequent one is not guaranteed. Then, the set of feasible trajectories is $G:=\{(K_0,\ell_0)\in R^2\ :\ K_t>0,\ 0<\ell_t<2,\ \forall t>0\}$. In addition, by the first equation of (12) and $\ell_t<2$, it follows that diverging trajectories cannot exist.

Since $K=\ell$ always holds as a coordinate of a stationary state of map M, then stationary-state coordinate values of ℓ are determined as solutions of $\ell=Z(\ell,\ell)$ or they are equivalently obtained by solving the following equation:

$$g(\ell):=\ell^{\rho+\sigma(1-\rho)}(2-\ell)^{-\gamma}=1. \tag{13}$$

Of course, $\ell=1$ is a solution of (13) for every constellation of parameters. Then, the following proposition holds.

Proposition 1. *[Existence of fixed points]. If either $\sigma<1$ or $\sigma>1$ and $\rho<\frac{\sigma}{\sigma-1}$, then $(1,1)$ is the unique fixed point of map M. If $\sigma>1$ and $\frac{\sigma}{\sigma-1}<\rho<\frac{\sigma+\gamma}{\sigma-1}$ (resp. $\sigma>1$ and $\rho>\frac{\sigma+\gamma}{\sigma-1}$), then another fixed point exists with $\ell<1$ (resp. $\ell>1$).*

Proof. From (13) we have that $\lim_{\ell\to 2} g(\ell)=+\infty$, while $\lim_{\ell\to 0} g(\ell)=0$ (resp. $\lim_{\ell\to 0} g(\ell)=+\infty$) if and only if (*a*) $\sigma<1$ or (*b*) $\sigma>1$ and $\rho<\frac{\sigma}{\sigma-1}$ (resp. $\sigma>1$ and $\rho>\frac{\sigma}{\sigma-1}$). In addition, with direct computation we have that

$$\text{sgn}\{g'(\ell)\}=\text{sgn}\{2[\sigma-\rho(\sigma-1)]-\ell[\sigma-\rho(\sigma-1)-\gamma]\},$$

and

$$\text{sgn}\{g'(1)\}=\text{sgn}\{\sigma+\gamma-\rho(\sigma-1)\}.$$

It follows that $g(\ell)$ is a monotone function or unimodal function. In the former case, no solution other than $\ell=1$ of (13) does exist. In the latter case, there exists another solution of (13) on the left (resp. on the right) of $\ell=1$ if $\rho<\frac{\sigma+\gamma}{\sigma-1}$ (resp. $\rho>\frac{\sigma+\gamma}{\sigma-1}$). **Q.E.D.**

Proposition 1 shows the crucial role played by both the reciprocal of the elasticity of marginal utility of effective consumption and intensity of aspirations in utility (i.e., the relative importance of the taste externality), in determining the existence of either one or two interior fixed points. In particular, when σ is low (i.e., the elasticity of substitution of effective consumption) and/or the importance of the taste externality ρ is low, a unique (normalised) fixed point does exist. When σ raises together with the relative degree of aspirations, a second fixed point appears with leisure being either lower or higher than the level corresponding to the normalised equilibrium. In particular, when the relative degree of aspirations is sufficiently high, the supply of labour becomes higher than 1 because individuals want to increase the amount of time spent at work when they are young in order to increase consumption possibilities when they are old, since the relative importance of past consumption is high in that case.

B. Local bifurcations and stability

This section starts by analysing the local dynamics around the normalised fixed point. In the present model, the stock of capital K_t is a state variable, so its initial value K_0 is given, while the supply of labour ℓ_t is a control variable. It follows that individuals of the first generation ($t=0$) choose the initial value ℓ_0 . If the normalised fixed point is a saddle and the initial condition of the stock variable K is close enough to 1, then, given the expectations on the interest rate, there exists a unique initial value of ℓ_t (ℓ_0) such that the orbit that passes through (K_0, ℓ_0) approaches the fixed point. In contrast, when the fixed point is a sink, given the initial value K_0 and expectations on the interest factor, there exists a continuum of initial values ℓ_0 such that the orbit that passes through (K_0, ℓ_0) approaches the fixed point. As a consequence, the orbit that the economy will follow is "locally indeterminate" because it depends on the choice of ℓ_0.

The Jacobian matrix of map M evaluated at $(1,1)$ is:

$$J = \begin{pmatrix} \alpha & 1-\alpha \\ \dfrac{\alpha(\rho-\alpha)}{1-\alpha} & \dfrac{(\rho-\alpha)(1-\sigma)(1-\alpha)+\gamma+1}{(1-\sigma)(1-\alpha)} \end{pmatrix}. \tag{14}$$

The trace and determinant of (14) are the following:

$$Tr(J) = \frac{1+\gamma}{(1-\alpha)(1-\sigma)} + \rho , \tag{15}$$

$$Det(J) = \frac{\alpha(1+\gamma)}{(1-\alpha)(1-\sigma)} . \tag{16}$$

Ceteris paribus, when ρ varies the point

$$(P_1, P_2) := \left(\frac{1+\gamma}{(1-\alpha)(1-\sigma)} + \rho, \frac{\alpha(1+\gamma)}{(1-\alpha)(1-\sigma)} \right), \tag{17}$$

drawn in the $(Tr(J), Det(J))$ plane, describes a horizontal half-line T_1 that starts from

$$(\overline{P}_1, \overline{P}_2) := \left(\frac{1+\gamma}{(1-\alpha)(1-\sigma)}, \frac{\alpha(1+\gamma)}{(1-\alpha)(1-\sigma)} \right), \tag{18}$$

when $\rho = 0$. When α varies, the point $(\overline{P}_1, \overline{P}_2)$ in Eq. (18), drawn in $(Tr(J), Det(J))$ plane, describes a half-line T_2 (with slope equal to α) that starts from $\left(\frac{1+\gamma}{1-\sigma}, 0\right)$ when $\alpha = 0$. If $\sigma \in (0,1)$ (resp. $\sigma > 1$), then $(\overline{P}_1, \overline{P}_2) \to (+\infty, +\infty)$ (resp. $(\overline{P}_1, \overline{P}_2) \to (-\infty, -\infty)$) for $\alpha \to 1$. Moreover, regardless of the value of σ, $(P_1, P_2) \to \left(+\infty, \frac{\alpha(1+\gamma)}{(1-\alpha)(1-\sigma)} \right)$ for $\rho \to +\infty$.

From the above geometrical findings and Proposition 1, we can state the following proposition with regard to local bifurcations.

Proposition 2. *[Local bifurcation]. Let* $\rho_{fl} := \frac{2+\sigma(\alpha-1)+\gamma(1+\alpha)}{(1-\alpha)(\sigma-1)}$ *and* $\rho_{tc} := \frac{\sigma+\gamma}{\sigma-1}$ *hold. Then, (1) if* $\sigma \in (0,1)$*, the normalised (unique) fixed point is determinate; (2) if* $\sigma > 2+\gamma$ *and* $\alpha < \frac{\sigma-2-\gamma}{\sigma+\gamma}$ *the normalised fixed point is indeterminate for* $\rho \in (0, \rho_{tc})$*, it undergoes a transcritical bifurcation for* $\rho = \rho_{tc}$*, and it is a saddle for* $\rho > \rho_{tc}$*; (3) if* $\sigma > 2+\gamma$ *and*

$\alpha > \frac{\sigma - 2 - \gamma}{\sigma + \gamma}$ or $1 < \sigma < 2 + \gamma$, the normalised fixed point is determinate for $\rho < \rho_{fl}$, it undergoes a supercritical flip bifurcation for $\rho = \rho_{fl}$, it is indeterminate for $\rho \in (\rho_{fl}, \rho_{tc})$, it undergoes a transcritical bifurcation for $\rho = \rho_{tc}$, and it is a saddle for $\rho > \rho_{tc}$.

Proof. In order to find the bifurcation values of ρ, we impose the condition that (P_1, P_2) belongs to: (*i*) the straight line $1 - Tr(J) + Det(J) = 0$, to obtain the transcritical bifurcation value ρ_{tc}, and (*ii*) the straight line $1 + Tr(J) + Det(J) = 0$, to obtain the flip bifurcation value ρ_{fl}. Then, we identify Cases 1-3 by considering the position of the starting points $(\overline{P}_1, \overline{P}_2)$ and (P_1, P_2) with respect to the stability triangle delimited by $1 \pm Tr(J) + Det(J) = 0$ and $Det(J) = 1$ (see Grandmont et al., 1998 for details). **Q.E.D.**

Proposition 2 is represented in Fig. 1. In particular, the first quadrant of it is referred to Point (1) of the proposition, the second and third quadrants to Points (2) and (3).

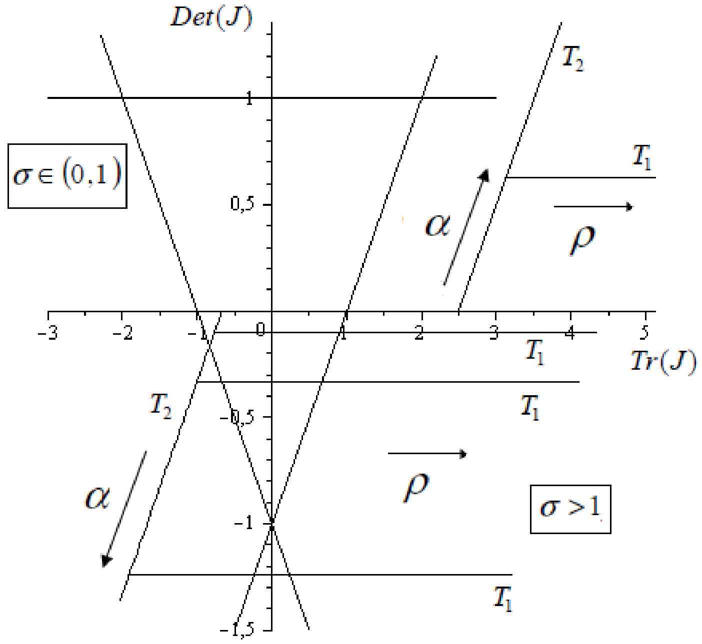


FIG. 1. Stability triangle and local indeterminacy. If $\sigma \in (0,1)$ a unique fixed point (saddle) exists. If $\sigma > 1$ multiple fixed points and local indeterminacy may occur.

It is important to note that map M can be not defined at the point $(0,0)$. In particular, this happens if and only if either $\rho < \alpha$ or $\sigma > 1$ and $\rho < \frac{1 + (\sigma - 1)(1 - \alpha)\alpha}{(\sigma - 1)(1 - \alpha)}$. In the remaining cases, however, the map results to be not differentiable in such a point. Then, it is not possible to apply the linearization method with the purpose of studying the stability of the map at $(0,0)$. However, we can classify the local properties of the map at the point $(0,0)$ by considering the sign of $\Delta K \equiv K_{t+1} - K_t$ and $\Delta \ell \equiv \ell_{t+1} - \ell_t$ in the phase plane. By this study, we can deduce that $(0,0)$ is an attracting fixed point when two interior stationary equilibria exist. In this case the fixed point $(0,0)$ can be interpreted as a poverty trap (e.g., Azariadis, 1996; Chakraborty, 2004).

IV. GLOBAL ANALYSIS

The importance of the global analysis for economic models is recognised by the fact that studying just the local behaviour of a map does not give information with regard to the structure of the basins of attraction and their qualitative changes when parameters vary. Since in economics it is also important to understand the long-term behaviour of variables given initial conditions, a characterisation of the basins of attraction is indeed necessary if one wants to explain phenomena that occur by starting from initial conditions far away from the fixed point or an attracting set.

In this section we show how the study of (*i*) the dynamics around the non-normalised fixed point, and (*ii*) the global structure of map M permit us to explain some interesting events that cannot be investigated with the local analysis (Pintus et al., 2000). We start the global analysis by showing that map M is invertible. The invertibility of a map is an important result when the global properties of a dynamic system are studied. For instance, it implies that the basins of attraction of any attracting set of a map are connected sets. In addition, by making use of the inverse map, we can obtain the boundary of the attracting sets and, more generally, the stable manifolds of saddle points.

With regard to map M, the following lemma holds.

Lemma 1. *Map M is invertible on the set D.*

Proof. Notice that it is impossible to have a closed-form expression of the inverse map of M, i.e. M^{-1}. However, after some algebraic manipulations it is possible to find that M^{-1} is solution of the following system:

$$M^{-1}: \begin{cases} \dfrac{(2-\ell_t)^{\gamma}}{\ell_t} = \left(\dfrac{K_{t+1}^{\rho-\alpha}}{\ell_{t+1}^{1-\alpha}}\right)^{1-\sigma} \\ K_t = K_{t+1}^{\frac{1}{\alpha}} \ell_{t+1}^{\frac{\alpha-1}{\alpha}} \end{cases} . \tag{19}$$

The right-hand side of the first equation in (19) defines a bijective function on R_+. It follows that the first equation of system (19) admits a unique solution of ℓ_t. The second equation of the system implies the result. **Q.E.D.**

Before performing the global analysis of map M, we recall the definitions of both the stable manifold

$$W^s(p) = \{x : M^{zn}(x) \to p \; as \; n \to +\infty\}, \tag{20}$$

and unstable manifold

$$W^u(p) = \{x : M^{zn}(x) \to p \; as \; n \to -\infty\}, \tag{21}$$

of a periodic point p of period z. If the periodic point $p \in R^2$ is a saddle, then the stable (resp. unstable) manifold is a smooth curve through p, tangent at p to the eigenvector of the Jacobian matrix evaluated at p corresponding to the eigenvalue λ with $|\lambda| < 1$ (resp. $|\lambda| > 1$), see, e.g., Guckenheimer and Holmes (1983). Outside the neighbourhood of p, the stable and unstable manifolds may even intersect each other with dramatic consequences on the global dynamics of the model (see Guckenheimer and Holmes, 1983, p. 22).

Non-trivial intersection points of stable and unstable manifolds of a unique saddle cycle are known as homoclinic points. However, when multiple saddle cycles exist, heteroclinic bifurcations may also occur. We remember that given two saddle cycles h_1 and h_2, a heteroclinic bifurcation is defined as the birth of a non trivial point E of intersection between the stable manifold of one cycle and the unstable manifold of the other cycle. Starting from this new configuration, it is possible to find a path on the two manifolds that connects the cycles. This phenomenon is interesting from an economic point of view because it is related to global indeterminacy. We recall that global indeterminacy occurs when, starting from the same initial condition K_0 of the state variable K_t, different fixed points or other ω-limit sets can be reached according to the initial value ℓ_0 of the jumping variable ℓ_t chosen by individuals of the first generation (Agliari and Vachadze, 2011; Gori and Sodini, 2011).

In addition, when saddle cycles exist heteroclinic connections can be another source (alongside Neimark-Sacker bifurcations) of the existence of repelling or attracting closed invariant curves (Agliari et al., 2005). Let us remind that the dynamics of the restriction of a map to a closed invariant curve is either quasiperiodic or periodic often of very high period (so that) numerically indistinguishable from a quasiperiodic one.

In the analysis we are going to perform we fix these parameter values: $\alpha = 0.35$ (Gollin, 2002), $\gamma = 1.2$, $\sigma = 3$ and let ρ vary. We start the analysis with $\rho = 2.2$, corresponding to which the normalised fixed point $(1,1)$ is a saddle (in economics this is referred to be saddle stable), i.e. it is locally determinate (see Sec. 3.2) with $(\hat{K}, \hat{\ell})$ and $(0,0)$ being indeterminate, and their basins of attraction are separated by the stable manifold of the normalised fixed point. Even if $(1,1)$ is locally determinate, a heteroclinic connection exists between $(1,1)$ and $(\hat{K}, \hat{\ell})$, and $(1,1)$ and $(0,0)$, each of which is given by one of the branches of the unstable manifold of the saddle (see Fig. 2.a). This implies that by considering a small neighbourhood of $(1,1)$, there exist feasible trajectories that converge either to the interior equilibrium or to $(0,0)$. While this result is not surprising and generically occurs when the saddle point belongs to the border of the basin of attraction of an attracting fixed point, the economic literature on local (in)determinacy has not stressed the importance of the existence of a continuum of equilibria around the determined fixed point (an exception is Agliari and Vachadze, 2011), even though this can be of interest especially from a policy perspective. This implies that given an initial condition K_0 close enough to 1, there exist several feasible trajectories converging to other attractors of the system.

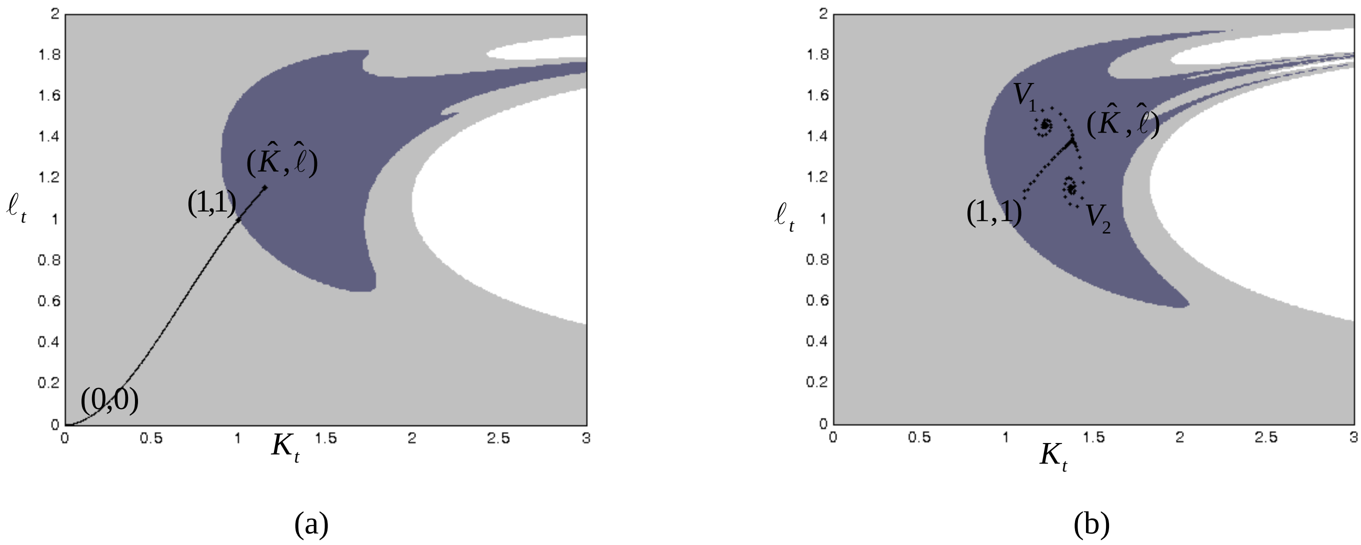
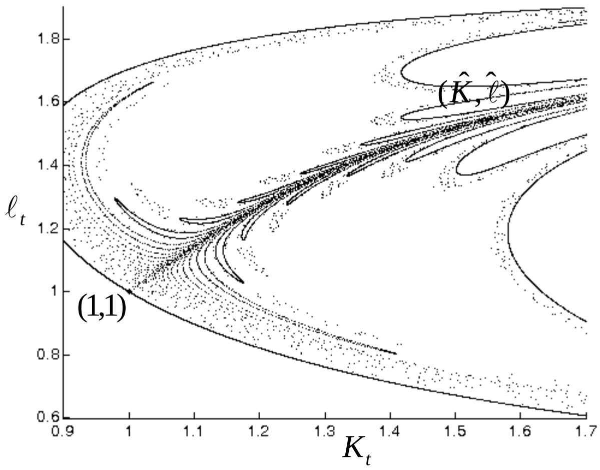


FIG. 2. (a) ($\rho = 2.2$). The fixed point $(\hat{K}, \hat{\ell})$ ($\hat{K} = \hat{\ell} \cong 1.1532$) is the unique attractor of the system (indeterminate equilibrium). The normalised fixed point $(1,1)$ belongs to the boundary of the attractor. The basin of the attraction of $(\hat{K}, \hat{\ell})$ is dark-grey-coloured. The region of unfeasible trajectories is light-grey-coloured. The curve that connects $(\hat{K}, \hat{\ell})$ and $(1,1)$ is the simulated unstable manifold of $(1,1)$ (i.e., the heteroclinic connection), obtained by iterating a small segment in the direction of the unstable eigenvector. If we consider an economy that starts on the stable manifold that converges to $(1,1)$, a small change (shock) in expectations about the interest factor R^e_{t+1} may cause the convergence to $(\hat{K}, \hat{\ell})$, where the capital stock is the same and the

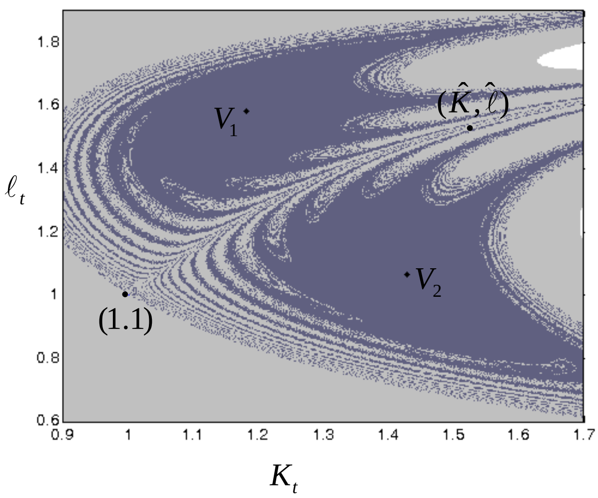
labour supply higher than the normalised fixed point. (b) ($\rho = 2.4$). Map M has an attracting two-period cycle of points V_1 and V_2. The stable manifold of $(1,1)$ defines the boundary of the basin of attraction of the two-period cycle (dark-grey region). Trajectories that start in the light-grey region converge to $(0,0)$. The white region describes the set of initial conditions that generate unfeasible trajectories.

If we let ρ increase, a flip bifurcation occurs at $\rho \cong 2.3078$ for $(\hat{K},\hat{\ell})$, and a two-period cycle captures almost all feasible trajectories beyond the stable manifold of $(1,1)$. At this stage, the point $(1,1)$ is globally indeterminate in a more general form: in fact, even if we restrict the model to a small neighbourhood there exist (*i*) an infinite number of trajectories that converge to the attractors of the system, and (*ii*) a unique trajectory, i.e. one of the branches of the unstable manifold of $(1,1)$, that converges to the interior saddle $(\hat{K},\hat{\ell})$.

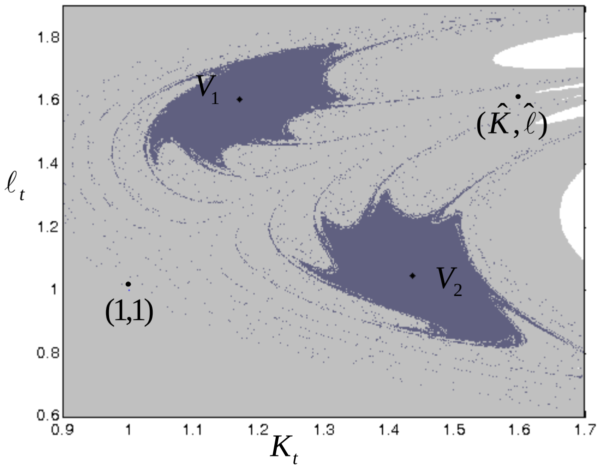
Trajectories, as those drawn in Fig. 2.b, starting in the dark-grey region just out the left or the right of the stable manifold of $(\hat{K},\hat{\ell})$, follow the curve almost until the saddle $(\hat{K},\hat{\ell})$ but converge to a two-period cycle of points V_1 and V_2. This implies that, after the transient, the economy oscillates between the two points. With regard to economic implications, by considering a historically given value of the capital stock K_0 close to 1, individuals may either coordinate themselves on one of the (stable or unstable) manifolds of the system or choose another feasible value of labour supply ℓ_t with very different long-term behaviours.



(a)



(b)



(c)

FIG. 3. (a) The stable and unstable manifolds of $(1,1)$ when $\rho = 2.58$. (b) Basin of attraction of M^2 when $\rho = 2.58$. (c) Evolution of the basin of attraction of M^2 when $\rho = 2.628$.

If we let ρ increase, it can be seen through numerical simulations that more and more convolutions of the boundary of the unstable manifold of $(\hat{K},\hat{\ell})$ evolve. This is a sign that a heteroclinic bifurcation is close to occur. In particular, for $\rho \cong 2.5$ the unstable manifold of $(\hat{K},\hat{\ell})$ has a tangential contact with the stable manifold of $(1,1)$. Starting from this value of ρ new heteroclinic orbits exist from a neighbourhood of $(\hat{K},\hat{\ell})$ towards $(1,1)$. This implies that even if the economy lies on the path that converges to $(\hat{K},\hat{\ell})$, a small change in the expectations on the interest factor may lead the economy lie on a path that converges to $(1,1)$. From the numerical study, it seems that heteroclinic orbits tend to survive at least until the attractors exist. If we let ρ increase further (see Figs. 3.a and 3.b), a new phenomenon can be observed: the stable manifold of $(1,1)$ begins to show more and more convolutions, and the stable and unstable manifolds of $(1,1)$ seem to get arbitrarily close each other (Brock and Hommes, 1997). This causes important changes in the boundary of the basin of attraction of the cycle of period two. It can also have interesting economic consequences because even if the 2-period cycle is stable for map M starting from values of the capital stock around $\hat{K}$, only a few values of ℓ_0 (Fig. 3.c) can generate trajectories converging to the attractor or interior fixed points, while the remaining values of ℓ_0 generate trajectories approaching to $(0,0)$. From an economic point of view this result suggests the importance of coordination of individuals on the choice of the labour supply for the long-term economic consequences.

The last part of this section is devoted to the study of periodic and/or quasi-periodic orbits generated by the system by considering higher values of ρ. In order to better illustrate the dynamic properties of the map, we study through numerical simulations the second (forward) iterate of map M, namely M^2. We now recall that fixed points of M hold as fixed points of M^2, while the two-period cycles of M become fixed points of M^2. In other words, the flip bifurcation found in the study of M is a pitchfork bifurcation of M^2 and two attracting fixed points V_1 and V_2 exist. In what follows we concentrate on the evolution of the dynamics on the upper part of the domain D around V_1 (the dynamics on the lower part around V_2 being symmetric).

The bifurcation diagram depicted in Fig. 4, which is obtained by using $(K_0,\ell_0) = (1.12,1.6)$ as the initial condition, shows some apparent discontinuities starting from $\rho \cong 2.6295$. They are caused by a 5-period cycle born through a saddle node bifurcation that captures the given initial condition for some ranges of the parameter ρ. At this stage, the dynamics may converge to the interior fixed point, $(0,0)$ or to the 5-period cycle. The basin of attraction of the 5-period cycle is defined by the stable manifold of the saddles (see Fig. 5.a).

Following now the evolution of the fixed point V_1 of M^2, we can see that it undergoes a supercritical Neimark-Sacker bifurcation at $\rho \cong 2.629732$ (see Fig. 4) and an attracting invariant curve (Γ) may be observed around the interior equilibrium of M^2 (see Fig. 5.b, where the coexistence of the attractors is illustrated when $\rho = 2.6302$ and two trajectories are drawn), corresponding to which the dynamics may be cyclical or quasi-cyclical.

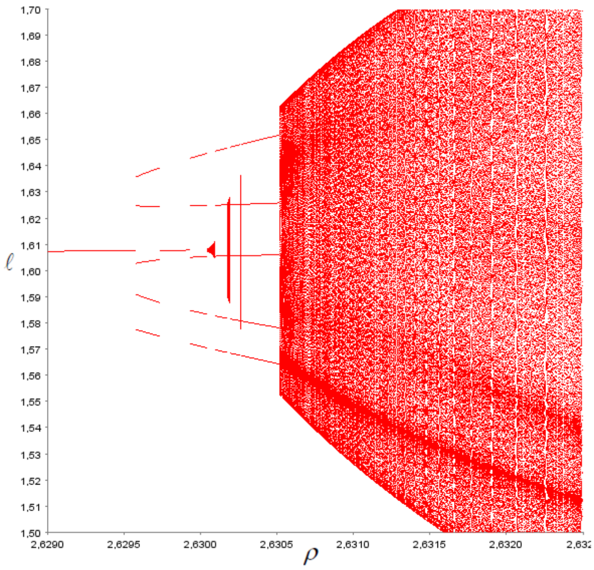
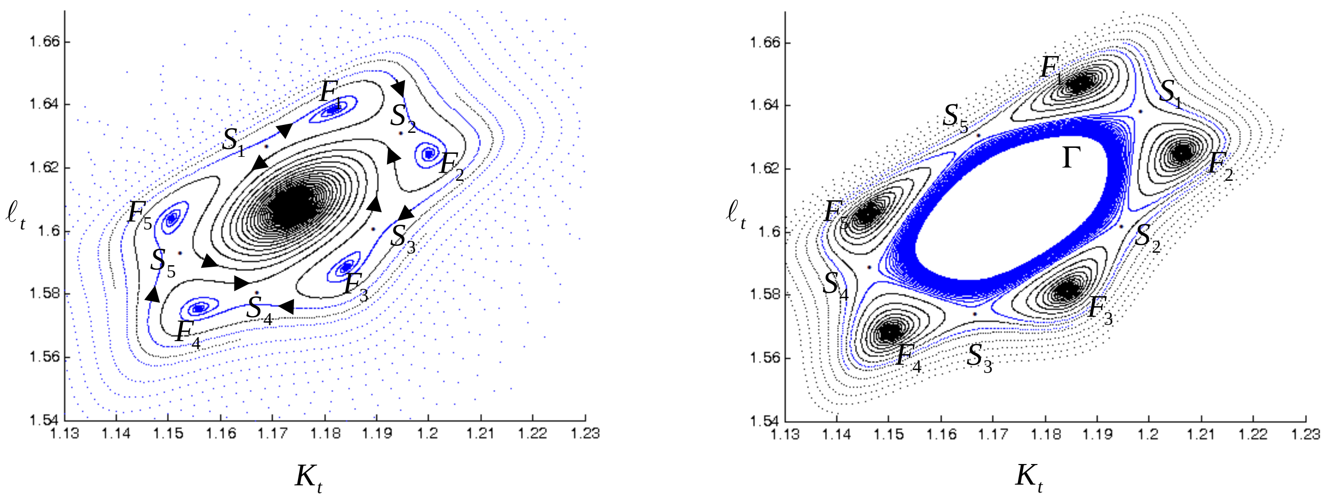
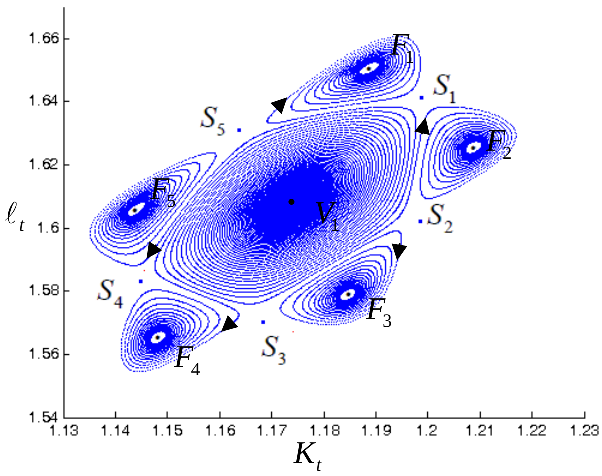


FIG. 4. Bifurcation diagram for ρ. We follow the long-term evolution of the starting point $(K_0, \ell_0) = (1.12, 1.6)$ when ρ increases. The discontinuities in the picture are due to the birth of another attractor. The apparent superposition of the curves just beyond $\rho = 2.63$ is due to the projection of the dynamics on the ℓ axis.



(a) (b)



(c)

FIG. 5. (a) ($\rho = 2.6297$). A 5-period cycle coexists with an attracting fixed point. Two converging trajectories are drawn (black-coloured and blue-coloured). F_i (S_i) indicates the i th point of the attracting (saddle) 5-period cycle. (b) ($\rho = 2.6302$). Coexistence of both an attracting 5-period cycle and closed invariant curve (Γ). (c) ($\rho = 2.63044$). A basin boundary bifurcation has destroyed the closed invariant curve and a unique attracting 5-period cycle survives.

When the dynamics is characterised by three coexisting interior attractors (see Figs. 5.a and 5.b), some exogenous changes in the parameters as well as in the expectations of individuals about the future interest rate, may cause the switch to another attractor of the system: for instance, if a trajectory is converging to the interior equilibrium (as in Fig. 5.a), the coordination on a large value of ℓ causes the switch to an attractor with larger oscillations. The enlargement of the invariant curve causes a collision between the stable manifold of the 5-period cycles and the curve itself at $\rho \cong 2.63044$ (see Fig. 5.c). This causes the death of the invariant curve and a unique attracting 5-period cycle of the system does exist.

Now, to understand the sudden explosion in the bifurcation diagram at $\rho \cong 2.63052$ (see Fig. 4), we refer to the theoretical results on invariant curves proposed by Agliari et al. (2005). Essentially, a global bifurcation involving both the stable and unstable manifolds of the 5-period cycle (see Fig. 6.a) has occurred at $\rho = 2.63056$ (saddle node connection): a larger invariant curve (Δ) surrounds an attracting 5-period cycle. For higher values of ρ, another basin boundary bifurcation causes the death of the 5-period cycle of M^2 and the invariant curve remains the unique attractor of the system (see Fig. 6.b plotted for $\rho = 2.6312$).

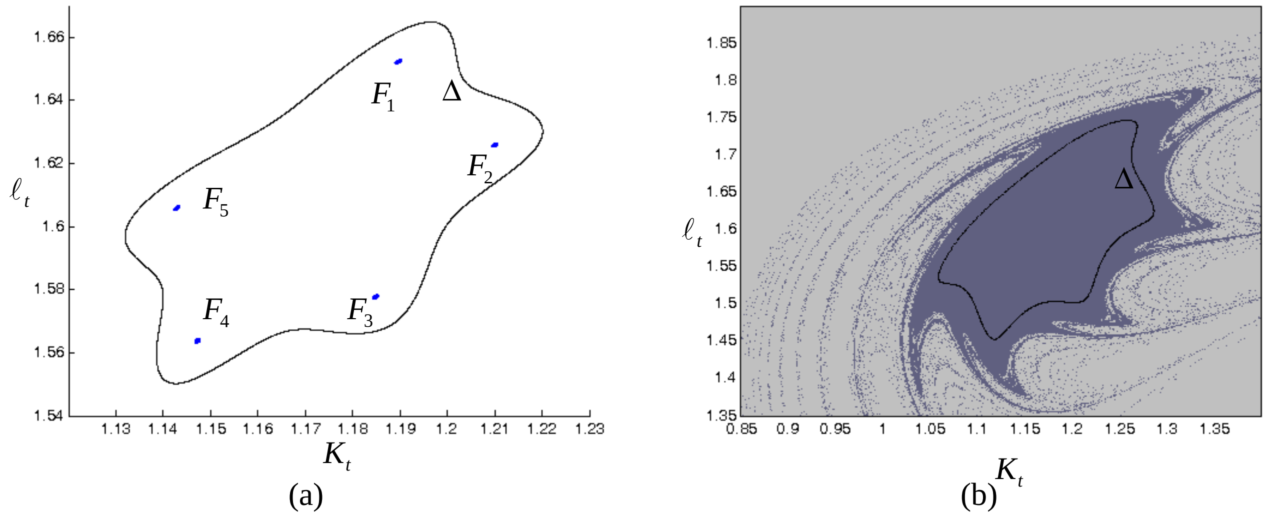


FIG. 6. (a) A large invariant curve (Δ) surrounds a 5-period cycle ($\rho = 2.63056$). (b) When ρ increases ($\rho = 2.633$), the invariant closed curve (Δ) becomes larger and remains the unique interior attractor of the system. Note that: (*i*) in Fig. 6.a the basin of attraction of the attractors is not reported because of the long transient, and (*ii*) in Fig. 6.b both the attractor and its basin of attraction are depicted.

V. CONCLUDING REMARKS

This paper has concerned with the study of the dynamic properties of a two-dimensional overlapping generations growth model with endogenous labour supply (Reichlin, 1986) and multiplicative external habits (aspirations). By following de la Croix (1996) and de la Croix and Michel (1999), we have assumed that preferences of an individual that belong to the current generation are affected by the consumption experience of an individual that belong to the past generation. In these works, the authors assume that an individual supplies labour inelastically and aspirations are considered as a stock variable. The dynamics of the economy is characterised by a two-dimensional system because of the accumulation of capital and the stock of aspirations. Different from them, we assume that the allocation of labour over time is chosen by an individual through the maximisation of his/her utility function and aspirations are considered to be a flow variable. Our model generates a two-dimensional system because of the accumulation of capital and the evolution of the supply of labour of individuals. We have shown that the intensity of aspirations in utility matters for the existence either of one (normalised) fixed point (when the intensity of aspirations is sufficiently low) or three attractors (when the intensity of aspirations is sufficiently high). In addition, some interesting local and global stability properties arise when the taste externality gradually increases. In particular, coexistence of attractors may cause global indeterminacy even if the stationary equilibria are locally determinate.

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