ON REVEALED DIVERSITY

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ABSTRACT. We introduce and characterize axiomatically a diversity criterion, capturing individual dissimilarity as 'revealed' by the different best-choices that members of a society select from a set of opportunities. Diversity ordering is induced by a class of frequency-based evaluation functions, the one element of which is the celebrated diversity measure, Shannon entropy.

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"The prospects of peace, tolerance, freedom and democracy in the contemporary world may well lie in the recognition of the plurality of our identities, where personal identity must be understood as an extension of one's own choice of being someone or doing something" (Sen (2006))

1. Introduction

A liberal tradition (see Mill (1859) and Nozick (1974) among others) regards the diversity of a society as a desirable characteristic in itself and considers the freedom of choice of individuals as an adequate tool for guaranteeing such diversity. *How can we measure the diversity of individual choices in a free society*? We answer this question by proposing and characterizing axiomatically a diversity ranking of choice sets that is grounded theoretically and simple to implement. We introduce a new criterion that evaluates the diversity of the best options freely selected by individuals from a suitable set of opportunities. While most of the current economic literature (see e.g. Barberà, Bossert and Pattanaik (2004) or Gravel (2008)) is concerned with evaluation of the diversity of available options from an opportunity set, we measure the extent to which such options allows the revelation of the diversity of individuals. Indeed, the diversity of options in a set does not guarantee the diversity of personal choices. In a democratic society, individuals are always free to select the same option (which is available to all of them), irrespective of the possible diversification of non-valuable opportunities. On the contrary, a non-democratic society could instead force individual diversification irrespective of their actual preferences. Nevertheless, this

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is the case in which none of the opportunity sets offer any freedom of choice to individuals (see Jones and Sugden (1982) and Pattanaik and Xu (1990)) as individual liberties are violated.

In the present work we consider the freedom of choice as a pre-requisite to meaningfully evaluate social diversity. The aim is therefore to capture the diversity within a free society by focusing on the 'diversity of *actual* choices', which *reveals* the freedom of individuals to pursue their own *personal life-plans.*

The concept of *revealed diversity* requires jointly considering a set of opportunities and a preference profile. In our setting, *an opportunity set is a collection of positive-valued options* that may be interpreted as possible individual life-plans, rather than different curricula of a schooling system. In other words, an option can be regarded as a bundle of rights and basic liberties (or functionings *à la Sen*)[1], that everybody may claim to have without preventing others from claiming them as well. Thus, opportunities are seen as both *non-rival* and *excludible.* Moreover, all individuals are assumed to be endowed with a well-defined preference ordering when they choose an option from a set of opportunities. As pointed out by Sen (1991), a sensible analysis of diversity cannot disregard the preferences of the individuals concerned for such diversities. Consider for example an individual having a preference ordering $\Re$ choosing a single option from two menus $A = \{a, b, c\}$ and $B = \{a, d\}$ of alternatives. In the case b and c are considered two undesirable options according to $\Re$, then the choice set from A should be considered better than that from B as long as it provides a minimum diversity among the reasonable[2] alternatives. In other words, we imagine that each individual in a given society selects what she most prefers to be or do (i.e. according to Sen (2006), she claims different meaningful lives), among all the possible opportunities a society offers. As a consequence, each individual is *identified* with the option she claims, and at the same time, her choice '*reveals*' her diversity from others. If someone chooses, for example, to eat certain food and to study for a doctorate in physics she will be identified as *diverse* with respect to another eating special dishes according to religious precepts and leaving school as soon as possible. As much as a society enhances *revealed diversity* among its members, it can be considered better in the sense

[1]A suggested sample of different categories of functionings *à la Sen*, each of which an individual can choose to practise in a (free) society, consists for example in claiming to be a European, an Italian citizen, a Tuscan with Spanish ancestry, a French resident, an economist, a man, a feminist, a strong believer in democracy, a defender of gay and lesbian rights and a nonbeliever in afterlife.

[2]The meaning of "reasonable" could be intended as in Jones and Sugden (1982).

that it allows more "significant choices with respect to various aspects of personal life" (Nozick, (1974)).

In other words, we value a society in which individuals make more heterogeneous claims as a better one. In our case, the importance of diversity comes directly from the value of freedom of choice, i.e. it depends on the liberty people have in their choice processes (see e.g. Sen (2006)). This paper is the first attempt to formalize the idea that a meaningful definition of diversity from a social point of view must be conditional to the individual freedom of choice. Evaluation of diversity depends on what individuals select as an opportunity when they claim their liberty, and not simply on the number of different options they have. We aim to study the diversity of a (free) society in which individual choose (their life plans) revealing indirectly their dissimilarities with the others.

The two primitives of our analysis are: an opportunity set (A) of non-rival and excludible opportunities, and a collection of n individual (linear) preference orderings[3] denoted by $(\Re)$. Each element in the latter selects a single most preferred option from the former. Hence, each pair $(A, \Re)$ originates a set of choices $C(A, \Re)$ of cardinality n. Nothing prevents the set $C(A, \Re)$ to include identical choices. We first analyze a criterion that compares each individual best choice with those of others within the same $C(A, \Re)$ in terms of their revealed diversity. In particular, we consider a single choice to be more dissimilar than another if the number of choices by others which are identical to the latter is lower than those identical to former. Thus, we introduce and characterize the so-called co-cardinality total ordering of dissimilarity, which can be seen as the dual (in the present more general setting) of the celebrated cardinality criterion characterized by Pattanaik and Xu (1990).

Then, we propose a diversity criterion that ranks different pairs of $(A, \Re)$. The ranking is induced by a family of frequency-based measures of revealed diversity. This class of evaluation functions is obtained by monotonic transformations of weighted averages of the evaluations of each single choice. Specifically, for any $(A, \Re)$, we average the dissimilarity of each individual (best) choice in A, normalized by the number of individuals in the reference population. The diversity ranking we obtain is therefore a complete preorder. The characterization of this criterion relies on a new property that rules how the evaluation of $(A, \Re)$ changes when a single element in the preference profile $\Re$ changes. In particular, if an individual, whose preferences

[3] $(\Re)$ could be also be interpreted as the *multiple selves* of an individual rather then a set of *potential* preferences that she actually has in a society as in Jones and Sugden (1982)).

in $\Re$ are changed, now selects a different option allows more revealed diversity (in the sense explained above) than the one selected before, then the aggregate diversity must increase.

A result of our analysis is that an element of the family of evaluation functions we study is the Shannon entropy measure (see also subsection 3.2 below). For its extreme computational convenience for applications, Shannon's entropy is indeed the most widely used index of diversity (see e.g. Hershey (2009) and Gravel(2008)). However, its axiomatic characterizations usually rely on 'the informativeness of a pair of independent distributions being the sum of their respective levels of informativeness' (see e.g. Theil (1967)), that is a requirement that lacks of compelling justifications when Shannon's entropy is used to measure diversity of actual choices. Indeed, the selection of an option from two independent pairs of $(A, \Re)$ typically does not coincide with the choice from the two sets merged together, unless a particular and arbitrary restriction on the preference domain is adopted. The present paper avoids this major drawback by proposing (indirectly) an alternative characterization of the Shannon entropy that is conducive to a fruitful approach to the issue of diversity in economic environments.

It is worth noticing here that the introduction of frequencies for individual choices is quite novel in economic literature on axiomatic measurement of diversity.[4] It allows us to consider the role of preferences in each single option's contribution to diversity enhancement and prevents Sen's (1991) critique of the so-called objective rankings of opportunity sets. In fact, since the work of Sen (1990), (1991), individual preferences are considered to have a vital role in judgements regarding freedom and/or opportunity. However, the origin of such preferences may seem quite arbitrary if not based on actual choices.[5] In the present work, we endogenously

[4]The more traditional approach focuses on the objective diversity measurement of the options in a given menu. Indeed, according to Gravel (2008), we can distinguish at least three approaches sharing this last view: aggregate cardinal dissimilarity (see Weitzman (1992), (1998); Bossert et al. (2003), Van Hees (2004)), aggregate ordinal dissimilarity (Pattanaik and Xu (2000), Bervoets and Gravel, (2004)), and the valuation of realized attributes (Nehering and Puppe (2002)).

[5]Jones and Sugden (1982) and Sugden (1998) consider *potential preferences*, namely the set of "all possible preference orderings that an individual might reasonably have". For instance, Pattanaik and Xu (1998) proposed a model in which individuals are endowed with a given set of *reasonable preferences*. However, both approaches take the relevant preference orderings as exogenously determined. Dietrich and List (2012) claim that "an agent's preferences are based on certain 'motivationally salient' properties of the alternatives over which preferences are held". In few words, they explain a given set of preferences using other, let us say, deeper preferences, which, in our opinion, need a further recursive explanation.

justify our set of complete and transitive orderings by assuming that they are *revealed* by a decisional process of choice, namely there exists a one-to-one correspondence between the rules of individual choices that satisfy certain plausible properties and the class of preferences we use to rank sets of opportunities in terms of their diversity. The choices made by people from some set X of all possible options (e.g. alternative life-plans) have a rational explanation, or rationalizing ordering (that is a linear order) such that for any $A \subseteq X$, an individual's choice from A is the best element in A according to that ordering. In other words, whatever the choice $a \in A$, it can be explained by an ordering, the maximization of which is consistent with the individuals' behavior (see Kalai, Rubinstein and Spiegler (2002) and in particular Aizerman and Malishevski (1981)). We therefore focus on the foregoing motivation to justify the use of preferences in our setting.

The remainder of our paper is organized as follows. In the next section, we introduce the notation, definitions, and axioms that provide a first result on how to rank options from a set of choices in terms of their (relative) dis-similarity. Section 3 contains our main result and some prominent examples. Section 4 concludes.

2. How to compare individual choices in terms of diversity

2.1. **Notation and definitions.** Let X be the universal set of options, assumed to be finite, and $N = \{1, ..., n\}$ be the set of individuals of a given population. We denote with $\wp(X)$ the set of all non-empty subsets of X. The elements A, B, C, etc. of $\wp(X)$ are the different feasible sets interpreted as opportunity sets. We define with $\mathbf{R}$ the set of all possible preference profiles over X. Then, $\Re = \{\Re_1, .., \Re_i, .., \Re_n\} \in \mathbf{R}$ is a preference profile of n individuals and $\Re_i \in \Re$ is a linear order, namely a transitive, irreflexive, weakly connected binary relation, representing the preferences over X of an individual i belonging to N. For any $x, y \in X$, $x \Re_i y$ means that "x is at least as good as y" according to $\Re_i$. We denote with a_i the choice of the ith individual in A, i.e. a_i is the element in A that i most prefers.

A choice set $C(A, \Re)$ is the set of all maximal elements of an opportunity set A which are selected by the agents with a preference profile $\Re$. In the present framework, we allow an element of any given set A to occur a finite number of times in $C(A, \Re)$. Indeed, $C(A, \Re)$ may include as many copies of the same element as the number of individual preference orderings $\Re_i$ in

$\Re$, for which it is a maximal element.[6] Therefore, a generic choice set $C(A,\Re)$ may have the mathematical structure of a *multiset*,[7] because individuals may select the same option from A, and thus the same option could be allowed to appear more than once in $C(A,\Re)$. Thus, the number of elements in $C(A,\Re)$, i.e. the cardinality of $C(A,\Re)$ denoted as $|C(A,\Re)|$, is equal to the number of individuals in the population considered. The present restriction for a choice set to be a multiset strongly suggests that in our model, opportunities are best regarded as *non-rival*, which means that an option $a \in A$ may be a bundle of rights and basic liberties (or functionings) that everybody could claim in order to live a life worth living. This interpretation is consistent with our idea that an individual simply chooses her own best option (e.g. a life-plan) as opposed (therefore diverse) to that of others without preventing others from doing so and she thus reveals her diversity from the rest of the population.

Now, let $\aleph$ be the set of all possible choice sets built over $\wp(X) \times \mathbf{R}$, then for any $C(A,\Re) \in \aleph$ and any $i \in N$, let $[a_i]$ be the *class of equivalence* of a_i in $C(A,\Re)$, namely the subset of all elements (a_i for $i = 1, ..., n$) of $C(A,\Re)$ that are equal. In particular, if $[a_i] = \{a_i\}$, then we say that a_i is an *isolated* choice (IC), i.e. it identifies the situation in which there is only one individual selecting that particular option as maximal in A. Conversely, if $[a_i] = C(A,\Re)$, every preference ordering in the profile $\Re$ selects the same option. Finally, we note that $[a_i] = \emptyset$ if $i \notin N$.

2.2. **Axioms and characterization.** Recent normative economics has introduced *liberty* as a relevant characteristic for evaluating alternative states of affairs. The basic idea of the foregoing literature is that people are free if they have access to (at least some basic) opportunities to choose from. In particular, an individual must be able to choose what she regards as valuable in terms of her own preferences (Sen, 1991) in the set of *diverse* options open to her. In fact, diversity is a fundamental aspect of freedom of choice in different situations (see e.g. Barberà,

[6]For instance, if we take a set $A = \{a, b, c, d, e, f\}$ and a preference profile $\Re = \{\Re_1, \Re_2, \Re_3, \Re_4\}$ such that individuals $1, 2, 3$ selects as their most preferred option a and individual number 4 chooses c. Then, the resulting choice set $C(A,\Re) = [a, a, a, c]$, where the square brackets denote a set in which the same element can occur several times.

[7]Note that a finite multiset on X is defined as a function $m : X \to \mathbb{Z}_+$ such that $\sum_{x \in X} m(x) < \infty$, i.e. each member of a multiset has a multiplicity, which is a natural number indicating (loosely speaking) how many memberships it has in the multiset. Like sets, multisets support operations to insert and withdraw items and the basic set operations of union, intersection, and difference.

Bossert and Pattanaik (2004)). As long as a plurality of *different* choices exists, there is also a plurality of *diverse* individual possibilities to be or to do. In other words, if we want to evaluate the liberty of members of a society, we have to look at the diversity of choices that the society allows its citizens as well as to their preferability.

We now proceed with our analysis of revelaed diversity by introducing a notion of *dissimilarity* of items in a choice set. In other words, given any choice set $C(A, \Re)$, we define a binary relation $\succeq$, over the classes of equivalence of personal choices, which establishes that for any $[a_i]$,$[a_j] \subset C(A, \Re)$, all elements in $[a_i]$ provide at least as much *revealed diversity* (or just diversity from here onwards) as all elements in $[a_j]$, i.e. $[a_i] \succeq [a_j]$.[8] In particular, we characterize the following measure of diversity of choices $\succeq_d$:

Definition 1. *Given $C(A, \Re) \in \aleph$, for any $[a_i]$,$[a_j] \subset C(A, \Re)$, we say that*

$$[a_i] \succeq_d [a_j] \quad \text{if and only if} \quad d([a_i], C(A, \Re)) \geq d([a_j], C(A, \Re)), \tag{2.1}$$

where for each $[a_s] \in C(A, \Re)$, $d([a_s], C(A, \Re)) \equiv |[C(A, \Re) \setminus [a_s]]|$.

In words, we say that, given a choice set $C(A, \Re)$, the choice a_i provides at least as much diversity as a_j if and only if the number of individuals (represented by the orderings $\Re_i$ in $\Re$) choosing an option different from a_i is not smaller than the number of individuals choosing an option different from a_j.[9] The ordering $\succeq_d$ is the so-called co-cardinality total preordering induced by our notion of dis-similarity and relies on the information provided by a cardinally meaningful numerical distance between the objects of the choice set in question.

We now characterize $\succeq_d$ using the following list of suitable properties:

Axiom 1 (Total Preorder - **TP**)**.** *The binary relation $\succeq$ is a complete preorder.*

Axiom 2 (Indifference - **I**)**.** *Given $C(A, \Re) \in \aleph$ and for any $[a_i]$,$[a_j] \subseteq C(A, \Re)$ that are IC, $[a_i] \sim [a_j]$.*

Axiom 3 (Dominance - **D**)**.** *Given $C(A, \Re) \in \aleph$, for any $[a_i] \subseteq C(A, \Re)$ that is IC and any $[a_j] \subseteq C(A, \Re)$ that is not, $[a_i] \succ [a_j]$.*

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[8]Note that $\succ$ and $\sim$ represent the asymmetric and symmetric parts of $\succeq$, respectively.

[9]In terms of our previous example with $C(A, \Re) = [a, a, a, c]$, c, that among other things is a IC, is a more diverse choice than a or differently individual number 4 has undertaken a choice different from the other members of the society.

Let $\{\Re^{(k)}\}_{k=1}^{\ell}$ be a finite ℓ-partition of $\Re \in \mathbf{R}$, counting ℓ elements, so that $\left(\cup_{k=1}^{\ell}\Re^{(k)}\right) = \Re$ and $\Re^{(k)} \cap \Re^{(q)} = \emptyset$ for any $k, q \in \{1, ..., \ell\}$. Given $\Re^{(k)} \in \{\Re^{(k)}\}_{k=1}^{\ell}$, let $[a_{i_k}] \subseteq C\left(A, \Re^{(k)}\right)$ denote the class of equivalence of the choice $a_i \in C(A, \Re)$ restricted to $\Re^{(k)}$. Then,

Axiom 4 (Independence - **N**)**.** *Given $C(A,\Re) \in \aleph$, if there exist at least a 2-partition of $\Re$, namely $\left\{\Re^{(1)}, \Re^{(2)}\right\}$ such that:*

$$\begin{aligned} \textit{for } [a_{i_1}], [a_{j_1}] &\in C\left(A, \Re^{(1)}\right), \quad [a_{i_1}] \succeq [a_{j_1}] \textit{ and} \\ \textit{for } [a_{i_2}], [a_{j_2}] &\in C\left(A, \Re^{(2)}\right), \quad [a_{i_2}] \sim [a_{j_2}], \end{aligned}$$

then, for $[a_i], [a_j] \in C(A, \Re)$, $[a_i] \succeq [a_j]$.

To our knowledge, all diversity rankings discussed in economic literature and used in practice are assumed to be reflexive (any set is at least as diverse as itself), transitive and complete, because, for example, a social decision-maker or government agency that intends to measure the degree of biodiversity of different ecological environments must be able to establish that one environment is more or less diverse than another or that both have the same level of diversity. Hence, the first axiom has its own rational.

The other three axioms have a natural interpretation. Indifference establishes that any class of equivalence that is a singleton set provides the same diversity. This property is satisfied by most indices used in current economic literature. However, some scholars claim that conceptions of diversity that focus on the attributes of the objects in a set rather than on the objects themselves have no reason to observe this property: in principle, there is no reason to consider two ecological environments with only mosquitoes or human beings as indifferent in terms of diversity they provide. This criticism does not apply to our framework. We compare options that are bundles of *positive*-valuable items, such as individual rights and personal liberties, on which there are no *a priori* preferences. In other words, in the present setting, saying that the choice of being, for example, a painter by a person who would like to be different from others is better than an analogous choice of another of being a lawyer is totally arbitrary or requires (a class of meta-) preferences that are, difficult to justify and on the whole unnecessary for the aim of the present analysis of diversity. We can therefore rely on this axiom.

Since isolated choices represent sets of maximal diversity, Dominance requires that sharing a choice with others leads to a set that is worse in terms of diversity according to $\succ$ than any choice taken in isolation. As dominance-type axioms tend to rule out rankings of sets that are based on 'total-goodness' criteria with respect to $\succ$ (see Fishburn (1988)), the dominance axiom appears to be a plausible requirement in the present work.

Independence makes it possible to consider unions of preference orderings so that implications can be derived for potential choices ruled by I or D under larger preference sets. More specifically, Independence concerns the order of two classes of equivalence of a choice set obtained on a given set of options A after merging two different preference orderings. In such a case, indifference between classes of equivalence in one of the two choice sets on A, yielded by the two starting orderings, is neutral for the determination of the order of the corresponding classes of equivalence in the final choice set on A.

The above axioms fully characterize the total preordering $\succeq_d$, as the following result shows:

Theorem 1. *Let $\succeq$ be a complete preorder on $C(A, \Re) \in \aleph$, Then $\succeq$ satisfies I, D, N if and only if $\succeq = \succeq_d$.*

Rule 2.1 differs from the cardinality total preorder rule characterized by Pattanaik and Xu (1990). Indeed, it compares items of the same choice set, rather than sets, and avoids the fundamental criticisms of Sen (1991) according to which "*the idea of effective freedom cannot be dissociated from our preferences*". More important, our criterion is the first step to precisely formalize the notion that having a number of similar alternatives available does not provide the same degree of freedom as having the same number of distinct options. Definition 1 considers individuals' freedom to choose as an effective means to analyze their diversity and consequently the diversity of the society represented by the choice set under consideration as a desirable feature in itself.

3. On the comparison of choice sets in terms of diversity

3.1. **New axioms and a characterization.** In what follows, we compare pairs of sets of opportunity and individual preference orderings in terms of *aggregate revealed diversity.* In order to do so, we introduce a binary relation $\succeq$ defined over $\aleph$ such that, for any $C(A, \Re'), C(B, \Re'') \in \aleph$, $C(A, \Re') \succeq C(B, \Re'')$ if and only if the choice set $C(A, \Re')$ provides at least as much aggregate diversity as the choice set $C(B, \Re'')$,

In particular, we study the following prominent notion of aggregate diversity:

Definition 2. *For any $A, B \in \wp(X)$ and any two profiles of preferences $\Re^{'} = \{\Re_1, .., \Re_i, .., \Re_m\}$ and $\Re^{''} = \{\Re_1, .., \Re_i, .., \Re_n\}$, $\succeq_D$ is an* aggregate-diversity *total preorder, defined by the following* *rule:*

$$C(A,\Re') \succeq_D C(B,\Re'') \quad \textit{if and only if} \quad E(C(A,\Re')) \geq E(C(B,\Re''))$$

where

$$E(C(Z,\Re)) = \frac{1}{|C(Z,\Re)|}\sum_{i=1}^{n}\frac{d([z_i],C(Z,\Re))}{|C(Z,\Re)|} \tag{3.1}$$

for any $Z \in \wp(X)$ and any $\Re \in \mathbf{R}$, where $d([z_i],C(Z,\Re))$ is the measure of diversity reflecting $\succeq_d$.

In words, for a given choice set $C(Z,\Re)$, $E(C(Z,\Re))$ is the average of the (normalized) measure of diversity $\succeq_d$ of *any* choice in $C(Z,\Re)$.[10] The criterion underlying 3.1 takes into account the degree of dissimilarity between alternatives. It establishes that the diversity of a (choice) set is obtained by aggregating the dissimilarities between the elements of that set. We now examine under what circumstances this is true by axiomatically characterizing $\succeq_D$ as follows:

Axiom 5 (Replication Principle - **R**). *For any $C(A,\Re) \in \aleph$,*

$$C\left(A,(\Re)^t\right) \sim C(A,\Re)$$

where $(\Re)^t \equiv \{\underbrace{\Re}_{1},\underbrace{\Re}_{2},...,\underbrace{\Re}_{t-1},\underbrace{\Re}_{t}\}$ denotes the t-replication of $\Re$.

Axiom 6 (Option Anonymity - **A**). *Given a preference profile $\Re \in \mathbf{R}$ and any $A,B \in \wp(X)$, such that $B \equiv \{(A\backslash\{a\})\cup\{b\}\}$, with $a \neq b \in X$, if $b = b_i \in C(B,\Re)$ for all and only i for which $a = a_i \in C(A,\Re)$, then:*

$$C(A,\Re) \sim C(B,\Re).$$

Axiom 7 (Preference Substitution - **P**). *For any $A \in \wp(X)$ and any $\Re \in \mathbf{R}$, consider a single preference substitution $\Re' := \Re\backslash\Re_j \cup \Re_h$ with $\Re_j \neq \Re_h$ and $\Re_h \in \mathbf{R}$, then:*

1. *(*weak dominance*) if $a_h = a_i$ and $[a_i] \succ_d [a_j]$, then $C(A,\Re') \succeq C(A,\Re)$,*
2. *(*strict dominance*) if $a_h = a_i$ and $[a_i] \sim_d [a_j]$, then $C(A,\Re') \prec C(A,\Re)$,*
3. *(*preference anonymity*) if $a_h = a_j$, then $C(A,\Re') \sim C(A,\Re)$.*

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[10]For example, if $C(A,\Re) = \{a_1,a_2,a_3\}$ with $a_1 = a_2 \neq a_3$ then $E(C(Z,\Re)) = 1/3(1/3+1/3+2/3)$.

The Replication Principle just states that the aggregate diversity has to be neutral with respect to the number of individuals, i.e. the diversity of a given choice set does not change if we consider a (t-fold) repetition of its elements. Option anonymity implies that the substitution of a single option, which does not affect the distribution of the choices in a given choice set, does not modify the value of the aggregate diversity of the new choice set. The Preference substitution axiom rules instead the changes in aggregate diversity after a single individual preference substitution. Notice that the A-axiom is generally more demanding than the P-axiom. In fact, a change of a single preference never affects other personal choices, whereas the substitution of a single option can unpredictably change the distribution of the choices since a new option is now available. This is why A is conditional on a certain property of the preference ordering profile ensuring that an option can be changed without affecting the distribution of choices. One can make sequential use of P to satisfy such condition (on preferences). Example 1 below shows how A and P can be used jointly to compare two generic choice sets.

We are now ready to state that:

Theorem 2. *Let $\succeq$ be a complete preorder on $\aleph$, then $\succeq$ satisfies R, A and P if and only if* $\succeq = \succeq_D$.

This result on ranking sets of opportunities in terms of the diversity revealed by individual choice captures the freedom of a social structure at an abstract level: a society that allows more pluralistic choices can be considered better than another in terms of the freedom/diversity it provides to its members. For intuition on how the system of axioms works, consider the following:

Example 1. Suppose two different opportunity sets A and B with $|A| \geq |B|$, and two preference orderings, namely $\Re = \{\Re_1, \Re_2\}$ and $\Re''' = \{\Re_3, \Re_4, \Re_5\}$. Suppose the two corresponding choice sets $C(A, \Re) = \{a_1, a_2\}$ and $C(B, \Re^{'''}) = \{b_3, b_4, b_5\}$ where $[b_3] = \{b_3, b_4\}$, while all other choices in both choice sets are isolated.

By direct application of I and D we know $[b_3] \prec [b_5]$ and $[a_1] \sim [a_2]$. In order to compare $C(A, \Re)$ and $C(B, \Re''')$ according to $E(\cdot)$, we first establish that $C(A, \Re) \sim C(A, (\Re)^3)$ by R. Also consider the single preference substitution of a copy of $\Re_2$ in $(\Re)^3$ with a copy of $\Re_1$ so that we can define $\Re' = \{\Re_1, \Re_2, \Re_1, \Re_2, \Re_1, \Re_1\}$ after a preference substitution in $(\Re)^3 = \{\Re_1, \Re_2, \Re_1, \Re_2, \Re_1, \Re_2\}$. Therefore, $C(A, \Re') = \{[a_1], [a_2]\}$ with $|[a_1]| = 4$ and $|[a_2]| = 2$.

Applying $P.1$ and R, we get $C(A,\Re') \prec C(A,\Re)$. We now build up $\Re'' = \{\Re_8, \Re_9, \Re_{10}, \Re_8, \Re_9, \Re_{10}\}$ such that: i) b_3 is $\Re_8''$-maximal and $\Re_9''$-maximal in $\{A \cup B\}$ and a_1 is $\Re_8''$-maximal and $\Re_9''$-maximal in $\{(A \cup B) \setminus \{b_3\}\}$, ii) b_5 is $\Re_{10}''$-maximal in $\{A \cup B\}$ and a_2 is $\Re_{10}''$-maximal in $\{(A \cup B) \setminus \{b_5\}\}$. Then $C(A,\Re'') = \{[a_8], [a_{10}]\}$ and $C(B,\Re'') = \{[b_8], [b_{10}]\}$ where $|[a_8]| = |[b_8]| = 4$ and $|[a_{10}]| = |[b_{10}]| = 2$ with $a_8 = a_9 = a_1$, $a_{10} = a_2$ and $b_8 = b_9 = b_3$, $b_{10} = b_5$. By iterated application of A-axiom, we can state that $C(A,\Re'') \sim C(B,\Re'')$ and by iterated single preference substitutions $P.3$, we also have $C(A,\Re') \sim C(A,\Re'')$ and $C(B,\Re'') \sim C(B,(\Re''')^2)$ and finally $C(B,\Re'') \sim C(B,\Re''')$ by R. Thus, by transitivity, we obtain that $C(A,\Re) \succ C(B,\Re''')$. Accordingly, our measure of aggregate diversity yields: $E(C(A,\Re)) = 1/2 > 4/9 = E(C(B,\Re'''))$.

Pattanaik and Xu (1998) characterize a criterion for ranking sets of opportunity in terms of freedom of choice. It simply counts the number of distinct options selected by at least one individual in the reference set in order to establish if one set (of opportunity) is better than another. As already observed, our insight is rather that the options people choose have to be evaluated proportionally with respect to the diversity they allow, i.e. the distribution of all individual choices matters as long as it actually reveals the *differentiation of people* through choices. The diversity criterion (3.1) we propose is therefore not a refinement of the one characterized by Theorem 1 in Pattanaik and Xu (1998), as the following example shows:

Example 2. Suppose $A, B \in \wp(X)$ and $\Re = \{\Re_1, .., \Re_i, .., \Re_{10}\}$ are such that:

$$\begin{aligned} C(A,\Re) &= \{[a_1], [a_4], [a_7]\} \\ C(B,\Re) &= \{[b_1], [b_8], [b_9], [b_{10}]\} \end{aligned}$$

where $|[a_1]| = |[a_4]| = 3$, $|[a_7]| = 4$, $|[b_1]| = 7$ and $|[b_8]| = |[b_9]| = |[b_{10}]| = 1$. Then, according to Pattainak and Xu (1998) $C(A,\Re) \prec_M C(B,\Re)$ because $M(C(A,\Re)) = 3 < M(C(B,\Re)) = 4$, where $M(C(Z,\Re))$ is the number of classes of equivalence in $C(Z,\Re)$ which induces the $\prec_M$-ranking. On the contrary, applying (3.1), we get that $C(A,\Re) \succ_D C(B,\Re)$ because $E(C(A,\Re)) = 0.66 > E(C(B,\Re)) = 0.48$.

In particular, our criterion (3.1) prevents some paradoxical situations resulting from the use of $\prec_M$ in Pattanaik and Xu (1998), as shown in the following:

Example 3. We want to compare three different schooling systems to offer to a set of individuals with a given preference profile $\Re$. The first system A provides a scientific (s) and a humanistic (h) curriculum; system B offers a generalist (g) and an artistic (a) curriculum; system C has a humanistic and domestic science (d) curriculum. Let us imagine that $\Re$ is such that when faced with *i*) A, half the individuals prefer the scientific curriculum and the other half the humanistic one; *ii*) B, only one individual chooses (a) and the others choose (g); *iii*) C, everybody prefers the generalist curriculum (g).

According to Pattanaik and Xu's criterion $\prec_M$, A and B are evaluated equally and higher than C, since the cardinality of the $\Re$-maximal sets is equal to two for A and B and equal to one for C. Instead, according to (3.1) A is better than B in terms of diversity for preference profile $\Re$ because it allows higher *average* differentiation in the society. Moreover, it still maintains that B is undoubtedly a better opportunity set than C according to $\Re$, because it allows at least to one individual to reveal his diversity. However, the improvement in diversity is only marginal and it is still rated much lower than that of A.

3.2. **Diversity, preferences and entropy.** As a final remark we point out the connection between the class of evaluation functions that induces our diversity criterion (3.1) and the classical entropy measure advocated by Suppes (1996) and Erlander (2005) as a suitable tool for ranking opportunity sets in terms of freedom of choice. In fact, for any choice set $C(Z,\Re)$, the Shannon entropy measure, denoted as $Ent(\cdot)$, belongs to the class of frequency-based functions in (3.1). To show that it is enough to write $Ent(\cdot)$ as a frequency-weighted average of the order-preserving images of $d([z_i], C(Z,\Re))$ according to $-\log(1-x)$, namely:

$$Ent(C(Z,\Re)) = -\frac{1}{|C(Z,\Re)|}\sum_{i=1}^{n}\log\left(1-\frac{d([z_i],C(Z,\Re))}{|C(Z,\Re)|}\right). \tag{3.2}$$

In particular notice that order-preserving transformation of $d([z_i], C(Z,\Re))$ equally satisfy the relation $\succeq_d$defined over $(Z,\Re)$. Therefore Theorem 2 equally applies to 3.2. Shannon entropy has been widely used in biology to measure the diversity of ecosystems, since entropy is a measure of the "disorder" of a system. Translated into our setting, a set of opportunities that is maximally "disordered", namely has the greatest variety of dissimilar options, is considered maximally diverse. Note that Suppes (1996) and Erlander (2005) proposed a entropy-based measure of freedom of choice, but did not characterize it axiomatically. Our work could also be seen as the first axiomatic foundation for using entropy as a measure of diversity of choices. Suppes

(1996) and Erlander (2005) motivated application of this measure by stochastic utility theory of logit models (see e.g. MacFadden (1974)).[11] However, the usual entropy interpretation and its well-known characterizations in physics and biology cannot directly be applied in an economic environment. Indeed, additivity[12], the key-property of almost all entropy characterizations, does not find a proper meaning in the economic context of revealed diversity unless we severely restrict the domain of individual preference orderings. Two distinct populations, choosing their best options from two different opportunity sets such that the resulting choice sets have a null intersection, will not typically select the same options when both populations and opportunity sets are merged together. In other words, it is not generally true that the entropy of choices satisfies additivity once the whole set of individuals has to select from the union of the two opportunity sets. This only happens in some very special cases after appropriate restriction of individual preference orderings. Our axiomatic method avoids this difficulty, making entropy a measure applicable to the context. In fact, joint application of the *preference substitution* axiom with *option anonymity* and the *replication principle* shows the direction in which the aggregate diversity evaluation of a generic $C(Z, \Re)$ changes after a single change in the preference profile (see Example 1).

4. Concluding note

In the present paper, we have explored the problem of ranking opportunity sets (the elements of which could be interpreted as bundle of rights and basic liberties), in terms of their diversity after the individuals (with well-defined preference profiles) of a population have selected their best choice. Since the choice concerns various aspects of personal life, it *reveals* the diversity of people in a society. A society that enhances (more) *revealed diversity* among its members can be considered better than a society where individuals make homogeneous claims, because diversity draws its value from greater freedom of choice (see e.g. Sen (2006)). If the set of opportunities a society provides to its members contains only one suitable option, 'human identities are formed by membership of a single social group' (see Sen, (2006)) and 'everyone is locked up in tight little boxes from which she emerges only to attack one another' (see Sen (2006)). "The prospects

[11]Indeed, in that perspective, the utility function is the propensity to choose and no longer a deterministic device as in standard utility theory. The analysis relies on the concept of statistical equilibrium as defined in e.g. Foley (1994).

[12]The entropy of a joint distribution of two variables is bounded (or equal to in the case of independent variables) from above by the sum of the entropies of the two distributions.

of peace, tolerance, freedom and democracy in the contemporary world may well lie in the recognition of the plurality (hence *diversity*) of our identities, where personal identity must be understood as an extension of one's own choice of being someone or doing something" (Sen (2006)). This study was devoted to providing a rationale for this insight, in an attempt to open new research perspectives in the analysis of freedom of choice and (individual) diversity.

5. Appendix: Proofs

Proof of Theorem 1. ($\Rightarrow$) That $\succeq_d$ be a total preorder and satisfy I, D, N is trivial;

($\Leftarrow$) Conversely, for any $C(A,\Re) \in \aleph$, take $[a_i], [a_j] \subset C(A,\Re)$ such that $d([a_i], C(A,\Re)) > d([a_j], C(A,\Re))$ and suppose $[a_i] \prec [a_j]$. The fact $d([a_i], C(A,\Re)) > d([a_j], C(A,\Re))$ implies that $[a_i]$ has less elements than $[a_j]$. Specifically, suppose without loss of generality that $|[a_i]| = \ell < n = |[a_j]|$. Construct an ℓ-partition $\{\Re^{(k)}\}_{k=1}^{\ell}$ of $\Re$, such that for $k < \ell$, $[a_{i_k}]$,$[a_{j_k}] \in C\left(A, \Re^{(k)}\right)$ are both Is. Hence, according to I $[a_{i_k}] \sim [a_{j_k}]$. By construction it follows that $[a_{i_\ell}]$,$[a_{j_\ell}] \in C\left(A, \Re^{(l)}\right)$ such that $[a_{i_\ell}]$ is IS and $[a_{j_\ell}]$ is not IS. Thus, by D, $[a_{i_\ell}] \succ [a_{j_\ell}]$. Therefore, since $\succ$ is a complete preorder, by $\ell - 1$ iterated applications of N, we get that $[a_i] \succeq [a_j]$, hence a contradiction.

Now, let $d([a_i], C(A,\Re)) = d([a_j], C(A,\Re))$ but suppose $[a_i] \nsim [a_j]$. That is, assume that $[a_i]$ is not indifferent to $[a_j]$, or without loss of generality that $[a_i] \prec [a_j]$. The fact $d([a_i], C(A,\Re)) = d([a_j], C(A,\Re))$ means that $|[a_i]| = |[a_j]| = \ell$. For $\ell = 1$ a contradiction arises from direct application of I. For $\ell > 1$, exactly as before, construct an ℓ-partition $\{\Re^{(k)}\}_{k=1}^{\ell}$ of $\Re$ counting ℓ elements, such that for $k < \ell$, $[a_{i_k}]$,$[a_{j_k}] \in C\left(A, \Re^{(k)}\right)$ are both IS. Hence again according to I, $[a_{i_k}] \sim [a_{j_k}]$ for each $k < \ell$. But now, by construction it must be that $[a_{i_\ell}]$,$[a_{j_\ell}] \in C\left(A, \Re^{(l)}\right)$ are also both IS. Therefore, since $\succ$ is a complete preorder, by $\ell - 1$ iterated applications of N, we obtain that $[a_i] \sim [a_j]$, hence a contradiction.[13] □

Proof of Theorem 2. ($\Rightarrow$) To check that $\succeq_D$ is a total preorder and satisfies R, A and $P.3$ is straightforward. To show that $\succeq_D$ also satisfies $P.1$ and $P.2$, take any $C(A,\Re) \in \aleph$ and for any

[13] In the present setting, since $\succeq_d$ is the dual of the cardinality total preordering of opportunity sets characterized by Pattanaik and Xu (1990), we do not provide examples of the independence of the axioms used, but can supply them on request.

$[a_i]$,$[a_j] \in C(A, \Re)$ suppose that $k_i = |[a_i]|$, $k_j = |[a_j]|$ and $|\Re| = m$ without loss of generality. By definition:

$$E(C(A, \Re)) = \left(1/m^2\right)[k_i(k_j + k_{-i,j}) + k_j(k_i + k_{-i,j}) + \mathbf{k}],$$

where $k_{-i,j} = m - (k_i + k_j)$ and $\mathbf{k}$ is a real number depending on the distribution of $(k_i + k_{-j})$ and the potential choices outside the set $\{[a_i] \cup [a_j]\}$. Now take $\Re' \in \wp(\mathbf{R})$ such that $\Re' = \Re \backslash \Re_j \cup \Re_h$ with $\Re_h \in \mathbf{R}$ and compute $E(C\left(A, \Re'\right))$. In the case $a_h = a_i$, we have:

$$E(C\left(A, \Re'\right)) = \left(1/m^2\right)((k_i + 1)(k_j - 1 + k_{-i,j}) + (k_j - 1)(k_i + 1 + k_{-i,j}) + \mathbf{k}),$$

in which both the distribution of potential choices outside the set $\{[a_i] \cup [a_j]\}$ and the sum $(k_i + k_j)$ are unaffected by the preference substitution. Thus, the difference $E(C\left(A, \Re'\right)) - E(C(A, \Re)) = \left(2/m^2\right)(-k_i + k_j - 1)$ does not depend on $\mathbf{k}$, so

$$E(C\left(A, \Re'\right)) \geq E((A, \Re)) \text{ if and only if } k_j \geq k_i + 1. \tag{5.1}$$

If $[a_i] \succ_d [a_j]$ then $k_j > k_i$ and therefore $C\left(A, \Re'\right) \succeq C(A, \Re)$, as required by $P.1$. If $[a_i] \sim_d [a_j]$, then $k_j = k_i$ and $E(C\left(A, \Re'\right)) < E(C(A, \Re))$ and therefore $C\left(A, \Re'\right) \prec C(A, \Re)$ as required by $P.2$. We therefore conclude that $\succeq_D$ also satisfies weak dominance and strict dominance in P.

($\Leftarrow$) To show that if the total preorder $\succeq$ satisfies R, A and P then $\succeq = \succeq_D$, take $A, B \in \wp(X)$ and first suppose without loss of generality $|A| \geq |B|$. Then, take $\Re$, $\Re''' \in \wp(\mathbf{R})$ such that $|\Re| = m$ and $|\Re'''| = n$ and suppose $E(C(A, \Re)) > E(C\left(B, \Re'''\right))$, but $C(A, \Re) \prec C\left(B, \Re'''\right)$.

First step. Given that $C(A, (\Re)^n) \sim C(A, \Re)$ and $C\left(B, \left(\Re'''\right)^m\right) \sim C\left(B, \Re'''\right)$ by R, then $E(C(A, (\Re)^n)) = E(C(A, \Re)) > E\left(C\left(B, \Re'''\right)\right) = E\left(C\left(B, \left(\Re'''\right)^m\right)\right)$. Since $|A| \geq |B|$ and $|C(A, (\Re)^n)| = \left|C\left(B, \left(\Re'''\right)^m\right)\right|$, it is possible to obtain an $\Re' \in \wp(\mathbf{R})$, where $|\Re'| = n \times m$, by iterated preference substitutions such that $E(C\left(A, \Re'\right)) = E\left(C\left(B, \left(\Re'''\right)^m\right)\right)$. To show that $C(A, \Re) \succeq C\left(A, \Re'\right)$, consider a finite sequence $\Re^{(0)}, .., \Re^{(q)}, .., \Re^{(s)}$ obtained by iterated single preference substitutions such that $\Re^{(0)} = (\Re)^n$ and $\Re^{(s)} = \Re'$ and $E(C\left(A, \Re^{(q)}\right)) \geq E(C\left(A, \Re^{(q+1)}\right))$ for any $q = 0, ..., s-1$. In order to construct such a sequence use the double implication in (5.1). That is, at any step q, substitute an $\Re_j$ with an $\Re_h$ such that $a_h = a_i$ with $k_j \leq k_i + 1$. The latter implies that $[a_j] \succ_d [a_i]$ if $k_j < k_i$ and $[a_j] \sim_d [a_i]$ if $k_j = k_i$. Therefore, by $P.1$ and $P.2$, we have $C\left(A, \Re^{(q)}\right) \succeq C\left(A, \Re^{(q+1)}\right)$ for any $q = 0, .., s-1$, hence by transitivity we have $C(A, (\Re)^n) \succeq C\left(A, \Re'\right)$.

Second step. Since $E(C\left(A,\Re'\right))=E\left(C\left(B,\left(\Re'''\right)^m\right)\right)$ then $C\left(A,\Re'\right)$ and $C\left(B,\left(\Re'''\right)^m\right)$ have the same distribution of choices.[14] Therefore there exists a finite number of pairs $([a_i],[b_j])$ with $[a_i]\in C\left(A,\Re'\right)$, $[b_j]\in C\left(B,\left(\Re'''\right)^m\right)$ with the same cardinality (i.e. $|[b_j]|=|[a_i]|$) that form a partition of the set $\{C\left(A,\Re'\right)\cup C\left(B,\left(\Re'''\right)^m\right)\}$. Now, build up an $\Re''\in\mathbf{R}$ with $|\Re''|=n\times m$, such that, for each pair $([a_i],[b_j])$ with the same cardinality ℓ, there exist ℓ elements $\Re''_h\in\Re''$ such that b_j is $\Re''_h$-maximal in $\{A\cup B\}$ and a_i is $\Re''_h$-maximal in $\{(A\cup B)/\{b_j\}\}$.

By repeated applications of $P.3$, we obtain $C\left(A,\Re'\right)\sim C\left(A,\Re''\right)$. Notice that $C\left(A,\Re''\right)$ allows iterated applications of A-axiom to obtain $C\left(B,\Re''\right)\sim C\left(A,\Re''\right)$. Indeed, $\Re''$ avoids changes in distribution of choices in $C\left(A,\Re''\right)$ with sequential substitution of options in A, so that A applies. Again, using $P.3$, we get $C\left(B,\left(\Re'''\right)^m\right)\sim C\left(B,\Re''\right)$.

Thus $C\left(A,\Re\right)\sim C\left(A,(\Re)^n\right)\succeq C\left(A,\Re'\right)\sim C\left(A,\Re''\right)\sim C\left(B,\Re''\right)\sim C\left(B,\left(\Re'''\right)^m\right)\sim C\left(B,\Re'''\right)$ and by transitivity $C\left(A,\Re\right)\succeq C\left(B,\Re'''\right)$ which is a contradiction.

Third step. Suppose $E(C\left(A,\Re\right))=E(C\left(B,\Re'''\right))$, but $C\left(A,\Re\right)\nsim C\left(B,\Re'''\right)$ and in particular, without loss of generality, that $C\left(A,\Re\right)\prec C\left(B,\Re'''\right)$. Repeat the second step with due correspondences to show $C\left(A,\Re\right)\sim C\left(A,\Re''\right)\sim C\left(B,\Re''\right)\sim C\left(B,\Re'''\right)$ so that by transitivity $C\left(A,\Re\right)\sim C\left(B,\Re'''\right)$, which is a contradiction.

The above characterization is tight. To check the validity of this claim, consider the following examples.

i - Completeness: Independence of the completeness requirement is immediately demonstrated by considering the binary relational system $(\wp(X)\times\wp(\mathbf{R}),\geqslant_c)$, defined as follows: for any $A,B\in\wp(X)$ and any $\Re,\Re'\in\wp(\mathbf{R})$

$$C\left(A,\Re\right)\geqslant_c C\left(B,\Re'\right)\quad\text{if and only if}\quad\begin{cases}\left|C\left(A,\Re\right)\right|\geq\left|C\left(B,\Re'\right)\right| & \text{and}\\ E\left(C\left(A,\Re\right)\right)\geq E\left(C\left(B,\Re'\right)\right)\end{cases}.$$

ii - Transitivity: Independence of the transitivity requirement can be shown by considering the binary relational system $(\wp(X)\times\wp(\mathbf{R}),\geqslant_t)$, defined as follows: for any $A,B\in\wp(X)$ and any $\Re,\Re'\in\wp(\mathbf{R})$:

$$\begin{aligned}C\left(A,\Re\right)&\geqslant{}_tC\left(B,\Re'\right)\text{ if and only if there exists }k\in\mathbb{Z}_+,\ k\geq1\\ \text{such that }\ E\left(C\left(A,\Re\right)\right)&=E\left(C\left(B,\Re'\right)\right)+k\left(\left|C\left(A,\Re\right)\right|-\left|C\left(B,\Re'\right)\right|\right).\end{aligned}$$

[14] This is a property of the Shannon entropy measure (see Theil (1967) chap.5) that also holds because of 3.2.

iii - Replication Principle: To prove independence of the R property from the other conditions, let us consider the binary relational system $(\wp(X) \times \wp(\mathbf{R}), \geqslant_r)$, defined as follows: for any $A, B \in \wp(X)$ and any $\Re, \Re' \in \wp(\mathbf{R})$:

$$C(A,\Re) \geqslant {}_r C\left(B,\Re'\right) \text{ if and only if } E'(C(A,\Re)) = E'\left(C\left(B,\Re'\right)\right), \text{ where}$$

$$E'(C(Z,\Re)) = \sum_{i=1}^{I} d([z_i], C(Z,\Re)) \quad \text{for any } Z \in \wp(X) \text{ and any } \Re \in \wp(\mathbf{R}).$$

iv - Opportunity Anonymity: To establish independence of the A property from the others, let us introduce the binary relational system $(\wp(X) \times \wp(\mathbf{R}), \geqslant_a)$, defined as follows: for any $A, B \in \wp(X)$ and any $\Re, \Re' \in \wp(\mathbf{R})$:

$$C(A,\Re) \geqslant {}_a C\left(B,\Re'\right) \text{ if and only if } E^*(C(A,\Re)) \geq E^*\left(C\left(B,\Re'\right)\right), \text{ where}$$

$$E^*(C(Z,\Re)) = \frac{1}{|C(Z,\Re)|} \sum_{i=1}^{I} \frac{\alpha_i d([z_i], C(Z,\Re))}{|C(Z,\Re)|} \quad \text{for any } Z \in \wp(X) \text{ and any } \Re \in \wp(\mathbf{R}),$$

$$\text{with } \begin{cases} \alpha_i = 2 \text{ if } \alpha_i = \overline{x} \in X, \\ \alpha_i = 0 \text{ otherwise} \end{cases}.$$

v - Preference substitution: (a) strict dominance: To check independence of the Strict Dominance property in the Preference Substitution axiom from the others, let us introduce the binary relational system $(\wp(X) \times \wp(\mathbf{R}), \geqslant_{st})$, defined as follows: for any $A, B \in \wp(X)$ and any $\Re, \Re' \in \wp(\mathbf{R})$:

$$C(A,\Re) \geqslant {}_{st} C\left(B,\Re'\right) \text{ if and only if } E^+(C(A,\Re)) \geq E^+\left(C\left(B,\Re'\right)\right),$$

$$\text{where } E^*(C(Z,\Re)) = \frac{1}{|C(Z,\Re)|} \sum_{i=1}^{I} \frac{\max\{\beta, d([z_i], C(Z,\Re))\}}{|C(Z,\Re)|} \quad \text{for any } Z \in \wp(X), \text{ any } \Re \in \wp(\mathbf{R}) \text{ and } \beta > 0.$$

vi - Preference substitution: (b) weak dominance: In order to prove the independence of the Weak Dominance property in the Preference Substitution axiom from the others, let us introduce the binary relational system $(\wp(X) \times \wp(\mathbf{R}), \geqslant_{wt})$, defined as follows: for any $A, B \in \wp(X)$ and any $\Re, \Re' \in \wp(\mathbf{R})$:

$$C(A,\Re) \geqslant {}_{wt} C\left(B,\Re'\right) \text{ if and only if } E^{\circ}(C(A,\Re)) \geq E^{\circ}\left(C\left(B,\Re'\right)\right), \text{ where}$$

$$\begin{cases} E^{\circ}(C(Z,\Re)) = 1 \text{ if and only if } [z_i] \sim_d [z_j] \text{ for any } i, j \in I \\ E^{\circ}(C(Z,\Re)) = 0 \quad \text{otherwise} \end{cases}$$

for any $Z \in \wp(X)$ and any $\Re \in \wp(\mathbf{R})$.

vii - Preference substitution: (c) preference anonymity: To check independence of the Preference Anonymity property in the Preference Substitution axiom from the others, let us introduce the binary relational system $(\wp(X) \times \wp(\mathbf{R}), \geqslant_{pa})$, defined as follows: for any $A, B \in \wp(X)$ and any $\Re, \Re' \in \wp(\mathbf{R})$:

$$C(A,\Re) \geqslant {}_{pa}C\left(B,\Re'\right) \text{ if and only if } E^{\#}(C(A,\Re)) \geq E^{\#}\left(C\left(B,\Re'\right)\right), \text{ where}$$

$$E^{\#}(C(Z,\Re)) = \frac{1}{|C(Z,\Re)|}\sum_{i=1}^{I}\frac{\alpha_i d([z_i], C(Z,\Re))}{|C(Z,\Re)|} \quad \text{for any } Z \in \wp(X) \text{ and any } \Re \in \wp(\mathbf{R}),$$

$$\text{with } \begin{cases} \alpha_i = 2 \ \text{ if } \alpha_i = \bar{i} \in I, \\ \alpha_i = 1 \ \text{ otherwise} \end{cases}.$$

□

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