House Prices and Monetary Policy*

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Abstract

This paper analyzes global dynamics in an overlapping generations general equilibrium model with housing-wealth effects. It demonstrates that monetary policy cannot burst rational bubbles in the housing market. Under monetary policy rules of the Taylor-type, there exist global self-fulfilling paths of house prices along a heteroclinic orbit connecting multiple equilibria. From bifurcation analysis, the orbit features a boom (bust) in house prices when monetary policy is more (less) active. The paper also proves that booms or busts cannot be ruled out by interest-rate feedback rules responding to both inflation and house prices.

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Keywords: House Prices; Housing-Wealth Effects; Monetary Policy Rules; Equilibrium Dynamics; Global Determinacy; Heteroclinic Orbits.

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1 Introduction

Theoretical developments in monetary economics over the last decades have focused on monetary policy rules capable of ensuring macroeconomic stability. Since the inflationary experience of the 1970s, central banks and academics have tried to design simple rules that promote the credibility and transparency of monetary policy-making. Since the seminal work by Taylor (1993), a large body of research has emphasized the stabilizing properties of active monetary policy rules, whereby the central bank responds to increases in inflation with a more than one-to-one increase in the nominal interest rate.[1] Empirical studies have shown that an active monetary policy stance mimics the Federal Reserve's behavior after 1979, over the Volcker-Greenspan period (e.g., Judd and Rudebush, 1998; Taylor, 1999b; Clarida, Galí and Gertler, 2000).

More recently, Meltzer (2011) and Taylor (2012) set out a distinction between a "rules-based era", from 1985 to 2003, and an "ad hoc era", from 2003 to present. Over the rules-based era, a period characterized by inflation stabilization, the Federal Reserve's policy is well described by a simple Taylor rule whereby the Federal Funds Rate is set as a linear function of inflation and the output gap with coefficients of 1.5 and 0.5, respectively. Over the ad hoc era, a period devastated by the boom and bust in the housing market, the Federal Reserve's policy deviates from the Taylor rule. In particular, from 2002 to 2006 the Federal Funds Rate was 2-3 percentage points below the path prescribed by the Taylor rule for any period since 1980s (Poole, 2007; Taylor, 2007). Significant downward interest-rate gaps from the Taylor rule also occurred over the same period in the OECD countries as a group (Ahrend, Cournède and Price, 2008).

Leamer (2007) and Taylor (2007, 2010, 2011) argue that such a "Great Deviation" from rules-based policy making, resulting in a too accommodating monetary stance, was a major cause of the economic and financial crisis erupted in 2007, since it triggered

[1]See Taylor (1999a), Woodford (2003), Galí (2008), and references therein.

boom-bust dynamics in house prices.

Could the rules-based approach to monetary policy be powerful enough to avoid the possibility of house-price instabilities? This paper studies global dynamics in an overlapping generations general equilibrium model à la Yaari (1965)-Blanchard (1985)-Weil (1989). The overlapping generations setting proves to be a convenient framework for making aggregate demand sensitive to housing wealth and for formalizing the interactions between house-price dynamics and rules-based monetary policies.

It is shown that the rules-based approach to monetary policy is powerless to burst rational bubbles in the housing market. Monetary policy feedback rules of the Taylor-type do not eliminate the possibility of self-fulfilling paths of house prices satisfying agents' transversality conditions. It is demonstrated that global dynamics associated to arbitrary revisions in house-price expectations follow a heteroclinic orbit connecting multiple equilibria. In particular, it is shown that global house-price paths interact with off-target inflation paths and lead the economy into a liquidity trap. From bifurcation analysis, it emerges that the orbit features a boom (bust) in house prices when monetary policy is more (less) active. It is further proved that booms or busts cannot be ruled out by interest-rate feedback rules responding to both inflation and house prices. In particular, it is demonstrated that reacting to house-price inflation more than to consumer-price inflation generates a basin of attraction to the liquidity trap and even leads to local indeterminacy at the steady-state equilibrium away from the trap.

The present paper is linked to both empirical and theoretical literature. Empirical evidence by Muellbauer (2007), Campbell and Cocco (2007), and Carroll, Otsuka and Slacalek (2011) finds highly significant housing-wealth effects on consumption. However, the link between house prices and consumption dynamics is typically overlooked in standard frameworks for monetary policy analysis (e.g., Taylor, 1999; Woodford, 2003; Galí, 2008). In this paper we intend to consider a dynamic general equilibrium framework in which housing-wealth effects do affect the monetary policy transmission mechanism.

Theoretical works by Benhabib, Schmitt-Grohé and Uribe (2001, 2002, 2003) and Schmitt-Grohé and Uribe (2009) show that once global dynamics are taken into account, the usual local stabilizing properties of Taylor (1993, 1999)-type interest-rate feedback rules disappear. In particular, Taylor rules give rise to multiple self-fulfilling decelerating inflation paths converging to a long-run equilibrium around which the monetary authority is no longer capable to ensure aggregate stability. Their model is based on the standard infinite-horizon representative agent setup. Therefore, aggregate demand dynamics only depends on real interest rates. To incorporate housing-wealth effects, we relax this paradigm and develop a monetary framework with overlapping generations à la Yaari (1965)-Blanchard (1985)-Weil (1989). In this way, aggregate demand dynamics will depend not only on real interest rates, but also on housing wealth. As a consequence, monetary policy decisions will affect aggregate demand and inflation through their effects on both the real interest rate and house prices. The model with housing-wealth effects derived in this paper also constitutes a useful theoretical benchmark to investigate the dynamic properties of monetary policy feedback rules whereby the nominal interest rate reacts not only to inflation but also to house prices.

Using a New Keynesian model with households' borrowing constraints, Iacoviello (2005) examines the role of house prices for the business cycle and the design of optimal monetary policy. Consistently with the business cycle literature, the approach relies on local dynamics, thereby abstracting from possible multiplicities of steady-state equilibria. This paper is different in two respects. First, we present an alternative way to analyze the implications of house prices for monetary policy design, since we employ an overlapping generations model.[2] Second, a central focus of this paper is to depart from local analysis. As recently advocated by Cochrane (2011), we use the criterion

[2]Nisticò (2012) studies the interactions between monetary policy and stock-price dynamics in a New Keynesian model with an overlapping generations structure à la Yaari–Blanchard.

of global determinacy to evaluate the connection between monetary policy rules and macroeconomic stability.

The paper is organized as follows. Section 2 presents the model and the monetary regimes. Section 3 analyzes the issue of global equilibrium dynamics under a baseline Taylor-rule framework. Section 4 extends the analysis to the case of monetary policy rules reacting to house prices. Section 5 concludes.

2 The Model

Consider the following monetary version of the Yaari (1965)-Blanchard (1985)-Weil (1989) overlapping generations setup, extended to incorporate housing in the agents' asset menu. Each individual faces a common and constant instantaneous probability of death, $\mu > 0$. Population grows at a constant rate n. At each instant t a new generation is born. The birth rate is $\beta = n + \mu$. Let $N(t)$ denote population at time t, with $N(0) = 1$. So the size of the generation born at time t is $\beta N(t) = \beta e^{nt}$, and the size of the surviving cohort born at time $s \leq t$ is $\beta N(s)\, e^{-\mu(t-s)} = \beta e^{-\mu t} e^{\beta s}$. Total population at time t is given by $N(t) = \beta e^{-\mu t} \int_{-\infty}^{t} e^{\beta s} ds$. As in Blanchard (1985), there is no dynastic altruism. Financial wealth of newly born individuals is therefore zero. Agents supply one unit of labor inelastically, which is transformed one-for-one into output.[3]

The representative agent of the generation born at time $s \leq 0$ chooses the time path of consumption, $\overline{c}(s,t)$, real money balances, $\overline{m}(s,t)$, and housing, $\overline{h}(s,t)$, in order to maximize the expected lifetime utility function given by

$$E_0 \int_0^{\infty} \left[\alpha \log \Lambda \left(\overline{c}(s,t), \overline{m}(s,t) \right) + (1-\alpha) \log \overline{h}(s,t) \right] e^{-\rho t} dt, \tag{1}$$

[3] Brito and Dilão (2010) extend the Yaari–Blanchard continuous time overlapping generations model for an endowment Arrow–Debreu economy with an age–structured population.

where E_0 is the expectation operator conditional on period 0 information, $\rho > 0$ is the pure rate of time preference, and $\Lambda(\cdot)$ is a strictly increasing, strictly concave and linearly homogenous function. Consumption and real money balances are Edgeworth complements, that is, $\Lambda_{cm} > 0$, and the elasticity of substitution between the two is lower than unity. Because the probability at time 0 of surviving at time $t \geq 0$ is $e^{-\mu t}$, the expected lifetime utility function (1) is

$$\int_0^{\infty} \left[\alpha \log \Lambda\left(\overline{c}(s,t), \overline{m}(s,t)\right) + (1-\alpha) \log \overline{h}(s,t)\right] e^{-(\mu+\rho)t} dt. \tag{2}$$

Individuals accumulate their financial assets, $\overline{a}(s,t)$, in the form of real money balances, interest bearing public bonds, $\overline{b}(s,t)$, and housing-wealth, $q(t)\overline{h}(s,t)$, where $q(t)$ is the relative house price. Therefore, $\overline{a}(s,t) = \overline{b}(s,t) + \overline{m}(s,t) + q(t)\overline{h}(s,t)$. The instantaneous budget constraint is given by

$$\begin{aligned} \dot{\overline{a}}(s,t) &= \left(R(t) - \pi(t) + \mu\right) \overline{a}(s,t) + \overline{y}(s,t) - \overline{\tau}(s,t) - \overline{c}(s,t) - \\ &\quad -R(t)\overline{m}(s,t) + \left[\frac{\dot{q}(t)}{q(t)} - \left(R(t) - \pi(t)\right)\right] q(t)\overline{h}(s,t), \end{aligned} \tag{3}$$

where $R(t)$ is the nominal interest rate, $\pi(t)$ is the inflation rate, $\overline{\tau}(s,t)$ are real lump-sum taxes, and $\mu\overline{a}(s,t)$ is an actuarial fair payment that individuals receive from a perfectly competitive life insurance company in exchange for their financial wealth at the time of death, in the spirit of Yaari (1965).[4] Since the asset menu includes housing equity, a reverse-mortgage mechanism à la Eschtruth and Tran (2001) is operative.

Agents are prevented from engaging in Ponzi's games, so that

$$\lim_{t\to\infty} \overline{a}(s,t) e^{-\int_0^t (R(j)-\pi(j)+\mu)dj} \geq 0. \tag{4}$$

[4] Insurance companies collect financial assets from deceased individuals and pay fair premia to current generations. The presence of the life insurance market precludes the possibility for individuals of passing away leaving unintended bequests to their heirs. See Blanchard (1985).

Letting $\overline{z}(s,t)$ denote total consumption at time t for the agent born at time s, defined as physical consumption plus the interest forgone on real money holdings,

$$\overline{z}(s,t) = \overline{c}(s,t) + R(t)\overline{m}(s,t), \tag{5}$$

the individual optimizing problem can thus be solved using a two-stage procedure.[5]

In the first stage, consumers solve an intratemporal problem of choosing the efficient allocation between consumption, $\overline{c}(s,t)$, and real money balances, $\overline{m}(s,t)$, in order to maximize function $\Lambda(\cdot)$, for a given level of total consumption, $\overline{z}(s,t)$. Optimality implies that the marginal rate of substitution between consumption and real money balances must equal the nominal interest rate, $\Lambda_m(\overline{c}(s,t), \overline{m}(s,t)) / \Lambda_c(\overline{c}(s,t), \overline{m}(s,t)) = R(t)$. Because preferences are linearly homogenous, this optimality condition assumes the following form:

$$\overline{c}(s,t) = \Gamma(R(t))\overline{m}(s,t), \tag{6}$$

where $\Gamma'(R) > 0$.

In the second stage, individuals solve an intertemporal problem of choosing the optimal time paths of total consumption, $\overline{z}(s,t)$, and housing, $\overline{h}(s,t)$, in order to maximize their lifetime utility function (2), given the constraints (3), (4) and the optimal condition (6).[6] Optimality yields

$$\dot{\overline{z}}(s,t) = (R(t) - \pi(t) - \rho)\,\overline{z}(s,t), \tag{7}$$

$$\frac{(1-\alpha)}{\alpha}\frac{\overline{z}(s,t)}{q(t)\overline{h}(s,t)} = (R(t) - \pi(t)) - \frac{\dot{q}(t)}{q(t)}, \tag{8}$$

$$\lim_{t\to\infty} \overline{a}(s,t)e^{-\int_0^t (R(j)-\pi(j)+\mu)dj} = 0. \tag{9}$$

[5] See Deaton and Muellbauer (1980). In the context of the Yaari–Blanchard framework, see Marini and van der Ploeg (1988).

[6] See Appendix A for analytical details.

Substituting the optimality condition (8) into the instantaneous budget constraint (3), integrating forward, applying the transversality condition (9), and using the law of motion of total consumption (7), we can express total consumption as a linear function of total wealth:

$$\overline{z}(s,t) = \alpha(\mu+\rho)\left(\overline{a}(s,t) + \overline{k}(s,t)\right), \tag{10}$$

where $\overline{k}(s,t) \equiv \int_t^\infty \left(\overline{y}(s,t) - \overline{\tau}(s,t)\right) e^{-\int_t^v (R(j)-\pi(j)+\mu)dj} dv$ denotes human wealth, defined as the present discounted value of after-tax labor income. From (5), (6), and (10), it also follows that

$$\overline{c}(s,t) = \frac{\alpha(\mu+\rho)}{L(R(t))}\left(\overline{a}(s,t) + \overline{k}(s,t)\right). \tag{11}$$

Combining next (5), (6) and (7), we obtain the optimal time path of individual consumption:

$$\dot{\overline{c}}(s,t) = \left(R(t) - \pi(t) - \rho - \frac{L'(R(t))}{L(R(t))}\dot{R}(t)\right)\overline{c}(s,t), \tag{12}$$

where $L(R) \equiv 1 + R/\Gamma(R)$ and $L'(R) > 0$. According to (12), the optimal consumption growth rate is identical across all generations. Function $L(R)$ satisfies $L(0) = 1$, $L(\infty) = +\infty$, $L'(0) = \infty$ and $L'(\infty) = 0$. The latter properties are verified, for example, if we assume that $\Lambda(\bar{c}, \bar{m})$ is a CES function.

2.1 Aggregation and Fiscal Policy

We can now derive the evolution of aggregate variables. The population aggregate for a generic variable at individual level, $\overline{x}(s,t)$, is defined as $X(t) \equiv \beta e^{-\mu t}\int_{-\infty}^t \overline{x}(s,t)e^{\beta s} ds$. The corresponding quantity in per capita terms is defined as $x(t) \equiv X(t)e^{-nt} = \beta\int_{-\infty}^t \overline{x}(s,t)e^{\beta(s-t)} ds$.

Suppose that each agent faces identical age-independent income and tax flows, so that $\overline{y}(s,t) = \overline{y}(t)$ and $\overline{\tau}(s,t) = \overline{\tau}(t)$, as in Blanchard (1985). Using $\overline{a}(t,t) = 0$ and $\overline{c}(t,t) = [\alpha(\mu+\rho)/L(R(t))]\,\overline{k}(t,t)$, the budget constraint, the optimal time path of consumption, the optimal time path of house prices, and the transversality condition expressed in per capita terms are, respectively, of the form[7]

$$\begin{aligned}\dot{a}(t) &= (R(t)-\pi(t)-n)\,a(t)+y(t)-\tau(t)-c(t)-\\ &\qquad -R(t)m(t)+\left[\frac{\dot{q}(t)}{q(t)}-(R(t)-\pi(t))\right]q(t)h(t),\end{aligned} \tag{13}$$

$$\dot{c}(t) = \left(R(t)-\pi(t)-\rho-\frac{L'(R(t))}{L(R(t))}\dot{R}(t)\right)c(t)-\frac{\alpha\beta(\rho+\mu)}{L(R(t))}a(t), \tag{14}$$

$$\frac{\dot{q}(t)}{q(t)} = (R(t)-\pi(t))-\frac{(1-\alpha)}{\alpha}\frac{L(R(t))c(t)}{q(t)h(t)}, \tag{15}$$

$$\lim_{t\to\infty} a(t)e^{-\int_0^t (R(j)-\pi(j)+\mu)dj} = 0. \tag{16}$$

From (14), the rate of change of per capita consumption depends on the level of financial wealth $a(t)$, since future cohorts' consumption is not valued by agents currently alive. In particular, older generations are wealthier than younger generations, and so consume more and save less. Only in the limiting case in which the birth rate β is equal to zero, per capita consumption dynamics follows the standard Euler equation prevailing in the infinitely-lived representative agent paradigm.

The flow budget constraint of the government in per capita terms is given by

$$\dot{b}(t)+\dot{m}(t) = (R(t)-\pi(t)-n)\,b(t)-\tau(t)-\pi(t)m(t). \tag{17}$$

To concentrate on the implications of housing-wealth effects, the government is assumed to adopt a tax policy consisting in balancing the budget at all times. It then follows

[7] See Appendix B for analytical details.

that taxes are such that

$$\tau(t) + \pi(t)m(t) = (R(t) - \pi(t) - n)\, b(t). \tag{18}$$

2.2 Monetary Policy Rules

We consider first the case in which the monetary authority follows a conventional Taylor rule, controlling $R(t)$ according to a feedback rule of the form

$$R(t) = T(\pi(t)), \tag{19}$$

where $T(\cdot)$ is a continuous, strictly increasing and strictly positive function. Monetary policy is active when $T'(\pi(t)) > 1$ and passive when $T'(\pi(t)) < 1$. In particular, we may assume, as advocated by Taylor (1993, 1999), a linear rule such as

$$T(\pi(t)) = \tilde{r} + \pi(t) + \gamma(\pi(t) - \tilde{\pi}), \tag{20}$$

where $\tilde{r}$ and $\tilde{\pi}$ are the central bank's targets for the real interest rate and the inflation rate, and $\gamma > 0$ is the policy parameter featuring an active monetary policy.

We shall also consider alternative rules controlling for both general price inflation and the housing price level,

$$T\left(\pi(t), q(t)\right) = \tilde{r} + \pi(t) + \gamma(\pi(t) - \tilde{\pi}) + \epsilon(q(t) - \tilde{q}), \tag{21}$$

where $\tilde{q}, \epsilon > 0$, or controlling for both general price and housing price inflations,

$$T\left(\pi(t), \frac{\dot{q}(t)}{q(t)}\right) = \tilde{r} + \pi(t) + \gamma(\pi(t) - \tilde{\pi}) + \delta\frac{\dot{q}(t)}{q(t)}, \tag{22}$$

where $\delta > 0$.

2.3 Equilibrium

Total output $\overline{y}(t)$ and housing supply $\bar{h}(t)^s$ are assumed to grow at the constant rate n, without loss of generality. It follows that per capita output and housing are constant and can be normalized to one, $y(t) = y = h^s = 1$, for analytical convenience. Equilibrium in the goods market requires that $c(t) = y = 1$. Equilibrium in the housing market requires that $h(t) = h^s = 1$. The balanced budget rule implies $\dot{b}(t) + \dot{m}(t) = 0$, so that total government liabilities are constant over time, $b(t) + m(t) = l$, where l is a constant. For analytical convenience, we can study the dynamic properties of the model normalizing the constant l to zero.

Using the law of motion of per capita consumption (14), the equilibrium real interest rate is given by

$$R(t) - \pi(t) = \rho + \frac{L'(R(t))}{L(R(t))}\dot{R}(t) + \frac{\alpha\beta(\rho+\mu)}{L(R(t))}q(t). \tag{23}$$

Then, the nominal interest rate dynamics are given by

$$\dot{R}(t) = \frac{1}{L'(R(t))}\left[\left(R(t) - \pi(t) - \rho\right)L(R(t)) - \alpha\beta(\rho+\mu)q(t)\right]. \tag{24}$$

From (15), house price dynamics are given by

$$\dot{q}(t) = \left[R(\pi(t)) - \pi(t)\right]q(t) - \frac{(1-\alpha)}{\alpha}L\left[R(\pi(t))\right]. \tag{25}$$

The equilibrium dynamic system is completed by introducing a monetary policy rule. Consider a generic rule as

$$R(t) = T\left(\pi(t), q(t), \dot{q}(t)/q(t)\right), \tag{26}$$

where $T(\pi, q, \dot{q}/q)$ is increasing and additively separable in all its components. We can solve equation (26) for π to get

$$\pi(t) = P(R(t), q(t), \dot{q}(t)/q(t)). \tag{27}$$

An active policy is such that $\partial T/\partial\pi > 1$, which implies that $0 < \partial P/\partial R < 1$ for any $(R, q, \dot{q}/q)$. To be an equilibrium, the paths $(R(t), q(t), \pi(t))_{t\in[0,\infty)}$, solving equations (24), (25) and (27), should verify the no-Ponzi game and the transversality conditions.

3 Conventional Taylor Rule

We analyze initially local and global equilibrium dynamics under the baseline Taylor-rule framework.

3.1 Steady-State and Local Dynamics

A generic conventional Taylor rule implies $\pi = P(R) = T^{-1}(R)$, where $0 < P^{'}(R) < 1$. We can rewrite the system (24)-(25) as

$$\begin{aligned} \dot{q}(t) &= (R(t) - P(R(t)))\,(q(t) - \Psi(R(t)))\,, && (28)\\ \dot{R}(t) &= \frac{\alpha\beta(\rho+\mu)}{L'(R(t))}\,(\Phi(R(t)) - q(t))\,, && (29) \end{aligned}$$

where

$$\begin{aligned} \Psi(R) &\equiv \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{L(R)}{R-P(R)}\right), && (30)\\ \Phi(R) &\equiv \left(\frac{R-P(R)-\rho}{\alpha\beta(\rho+\mu)}\right)L(R). && (31) \end{aligned}$$

Equilibrium steady states are bounded values for R and q, such that the transversality condition (16) holds. Hence $\lim_{t\to\infty}(R(t) - P(R(t)) + \mu) > 0$.

Defining

$$r^* \equiv \frac{\rho + \sqrt{\rho^2 + 4\beta(1-\alpha)(\rho+\mu)}}{2} \tag{32}$$

and observing that $r^* > \rho > 0$, the following proposition holds.

Proposition 1 *Assume a conventional Taylor rule in which monetary policy is globally active. Then: (a) if* $P(0) \geq 0$*, there is a unique steady state equilibrium* $(R^*, q^*) = (R_1^*, q_1^*)$*, where*

$$q_1^* = \left(\frac{1-\alpha}{\alpha}\right)\frac{L(R_1^*)}{r^*} > 0, \tag{33}$$

and $R_1^* = \{R : \ R - P(R) = r^*\}$ *is the unique element; (b) if* $0 > P(0) > -r^*$*, there are two equilibrium steady states* $(0, q_0^*)$ *and* (R_1^*, q_1^*)*, where*

$$q_0^* = -\left(\frac{1-\alpha}{\alpha}\right)\frac{1}{P(0)} > 0; \tag{34}$$

and (c) if $P(0) \leq -r^*$*, there is a unique steady state equilibrium* $(R^*, q^*) = (0, q_0^*)$*.*

Proof. See Appendix C.

As a result, interest-rate feedback rules of the Taylor-type may give rise to multiple steady-state values for house prices. The long-run inflation rates associated to the two steady states q_0^* and q_1^* are $\pi_0^* = P(0) < 0$ and $\pi_1^* = P(R_1^*) = R_1^* - r^* > \pi_0^*$, respectively.[8]

Compare the steady-state level of house prices associated with a zero nominal interest rate, q_0^*, with the steady-state level of house prices associated with a positive nominal interest rate, q_1^*. From Proposition 1, it follows that $q_0^* > q_1^*$ if and only if $r^* + P(0)L(R_1^*) > 0$.[9] The latter condition is sufficient for the existence of two steady

[8] For the case of a linear rule à la Taylor (1993, 1999), we can determine R_1^* explicitly. Substituting equation (20) yields $R_1^* = (1+\gamma)(P(0)+r^*)/\gamma = r^* + \tilde{\pi} + (r^* - \tilde{r})/\gamma$, because $P(0) = (\gamma\tilde{\pi} - \tilde{r})/(1+\gamma)$. In this case, a necessary condition for the existence of two steady states is $(1+\gamma)r^* - \gamma\tilde{\pi} > \tilde{r} > \gamma\tilde{\pi}$.

[9] Therefore, the relationship between the house prices for the two possible long-run nominal interest rates depends upon the long-run real interest rate, the substitutability between real money balances and consumption, and the characteristics of the Taylor rule.

states, since it satisfies $r^* + P(0) > 0$. However, if $r^* + P(0) > 0 \geq r^* + P(0)L(R_1^*)$, we have two steady states and $q_0^* \leq q_1^*$. If we assume a linear conventional Taylor rule, the case in which $q_0^* > q_1^*$ occurs when

$$\Gamma\left(\frac{(1+\gamma)r^* + \gamma\tilde{\pi} - \tilde{r}}{\gamma}\right) > \frac{\tilde{r} - \gamma\tilde{\pi}}{\gamma}, \tag{35}$$

which tends to be verified when the long-run real interest rate r^*, the inflation target $\tilde{\pi}$, the elasticity of substitution between real money balances and consumption, and/or the monetary policy feedback parameter γ are sufficiently large.

Explore now local equilibrium dynamics. Linearizing equations (28) and (29) in the neighborhood of any point (R, q), we obtain the Jacobian

$$J = \begin{pmatrix} R - P(R) & (1 - P'(R))(q - \Psi(R)) - (R - P(R))\Psi'(R) \\ -\frac{\alpha\beta(\rho+\mu)}{L'(R)} & \frac{\alpha\beta(\rho+\mu)}{(L'(R))^2}\left(\Phi'(R)L'(R) - (\Phi(R) - q)L''(R)\right) \end{pmatrix}. \tag{36}$$

The trace and the determinant of the Jacobian matrix are

$$\mathrm{tr}J = R - P(R) + \frac{\alpha\beta(\rho+\mu)}{L'(R)}\left(\Phi'(R) - (\Phi(R) - q)\frac{L''(R)}{L'(R)}\right)$$

and

$$\det J = \frac{\alpha\beta(\rho+\mu)}{L'(R)}\left[\begin{array}{c} (R - P(R))\left(\Phi'(R) - \Psi'(R) + (q - \Phi(R))\frac{L''(R)}{L'(R)}\right) + \\ +(1 - P'(R))(q - \Psi(R)) \end{array}\right].$$

We readily observe that

$$\Psi'(R) = \Psi(R)\left(\frac{L'(R)}{L(R)} - \frac{1 - P'(R)}{R - P(R)}\right)$$

and

$$\Phi'(R) = \Phi(R)\left(\frac{1-P'(R)}{R-P(R)-\rho} - \frac{L'(R)}{L(R)}\right).$$

Hence, the following proposition holds.

Proposition 2 *Let the assumptions in Proposition 1 hold, such that there are two steady state equilibria. Then, the steady state* $(0, q_0^*)$ *is a singular saddle point and the steady state* (R_1^*, q_1^*) *is a source.*

Proof. See Appendix D.

From Proposition 2, since both $R(t)$ and $q(t)$ are jump variables, the steady state in which $R^* = R_1^* > 0$ is locally determinate and the steady state in which $R^* = 0$ is locally indeterminate.

As a result, even in the presence of housing-wealth effects, an active monetary policy stance in the spirit of Taylor (1993, 1999b) exhibits the usual property of local determinacy. In particular, in the neighborhood of (R_1^*, q_1^*), and in the absence of exogenous fundamental shocks, the only equilibrium path is $R(t) = R_1^*$, and $q(t) = q_1^*$.

Nevertheless, in a small neighborhood around the steady state in which $R^* = 0$, local indeterminacy applies, i.e., there exist infinite equilibrium paths of $R(t)$ and $q(t)$ converging asymptotically to the steady state: for any initial $R(0)$ there exists a $q(0)$ such that the time paths of $R(t)$ and $q(t)$ satisfying the system (28)-(29) will converge asymptotically to that steady state. As this is a singular steady state, in the sense that the eigenvalues for system (R, q) are infinite, the speed of approach is locally very high. Singular steady states appear in economic theory from the existence of static constraints in some macroeconomic models, as in Leeper and Sims (1994), and Barnett and He (2004, 2006, 2010). However, in our case, the singularity has a different nature: it is related to the properties of function $\Gamma(R)$, since when R tends to zero the relationship between consumption and money demand becomes locally insensitive to the nominal interest rate. This type of singularity seems not to have been examined previously and can only be analyzed by investigating global dynamics.

3.2 Global Dynamics

We now conduct a global dynamics analysis of the effects of the baseline Taylor-rule framework. Thereafter, we shall investigate whether changing the Taylor rule by incorporating the housing prices significantly modifies the dynamics.

Proposition 3 *Let the assumptions in Proposition 1 hold, such that there are two steady state equilibria. Then, there is a heteroclinic orbit joining steady states* (R_1^*, q_1^*) *and* $(0, q_0^*)$. *The orbit has a positive slope near the steady state* (R_1^*, q_1^*) *and has a zero slope near the steady state* $(0, q_0^*)$.

Proof. See Appendix E.

Figures 1 and 2 present the phase diagrams for the cases in which $q_0^* > q_1^*$ and $q_0^* < q_1^*$. In both cases, there is a heteroclinic trajectory joining the two steady states, and this trajectory is positively sloped at the neighborhood of (R_1^*, q_1^*). This implies that the nominal interest rate follows a non-monotonous trajectory in the case depicted in Figure 1. The boom in house prices stimulates aggregate demand and thus inflation via the positive wealth effect on aggregate consumption. However, the aggressive increase in the real interest rate implied by the Taylor rule makes the economy spiral down into an off-target decelerating-inflation path, leading the economy to the liquidity trap equilibrium.

As a consequence of Proposition 3, there is not only local indeterminacy at the steady state $(q_0^*, 0)$, but also global indeterminacy: any point along the heteroclinic orbit is an equilibrium and, if it is not located at any of the two steady states, there is a transition dynamics which converges asymptotically to the liquidity trap. Next, we set out the rationale for our results.

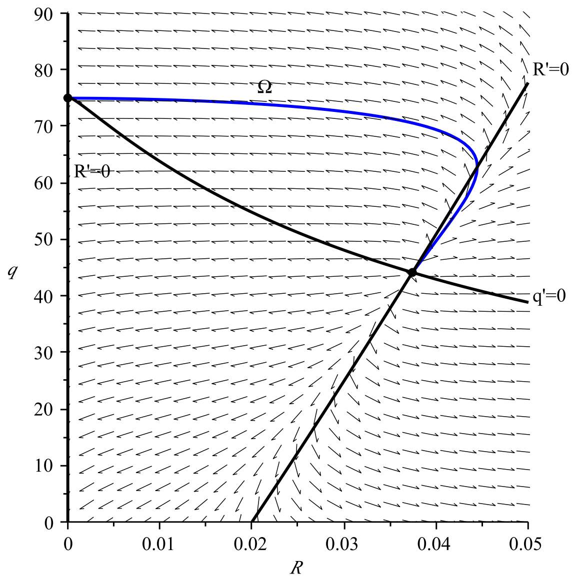

Figure 1: Phase diagram for a conventional Taylor rule: $r^* + P(0)L(R_1^*) > 0$. The figure is built by assuming $L(R) = 1 + \xi R^{1-\zeta}$, a linear rule $R = \tilde{r} + \pi + \gamma(\pi - \tilde{\pi})$, and the values of the parameters $\alpha = 0.5$, $\beta = 0.01$, $\xi = 1$, $\zeta = 0.4$, $\mu = 0.01$, $\rho = 0.02$, $\gamma = 0.5$, $\tilde{r} = 0.03$ and $\tilde{\pi} = 0$.

The isocline $\dot{q} = 0$ has just one branch, $q = \Psi(R)$, while the isocline $\dot{R} = 0$ has two branches, $R = 0$ and $q = \Phi(R)$. The slopes of the last two equations are, respectively,

$$\left.\frac{dq}{dR}\right|_{\dot{q}=0} = \Psi'(R) = \Psi(R)\left(\frac{L'(R)}{L(R)} - \frac{1 - P'(R)}{R - P(R)}\right) \tag{37}$$

and

$$\left.\frac{dq}{dR}\right|_{\dot{R}=0} = \Phi'(R) = \Phi(R)\left(\frac{1 - P'(R)}{R - P(R) - \rho} + \frac{L'(R)}{L(R)}\right). \tag{38}$$

With the assumptions made at Proposition 1, the isocline $\dot{q} = 0$ has a positive value at $R = 0$ and an asymptote at this point, because $q = \Psi(0) > 0$ if $P(0) < 0$ and $\Psi'(0) = \Psi(0)(L'(0) + (1 - P'(0))/P(0)) = +\infty$. On the other side, $\Psi'(\infty) = 0$. For positive nominal interest rates, the slope depends upon the relationship between the wealth effect on consumption and the monetary rule. The schedule has a negative slope everywhere if $\Psi'(R) < 0$, that is, if $L'(R)/L(R) < (1 - P'(R)/(R - P(R))$. This is

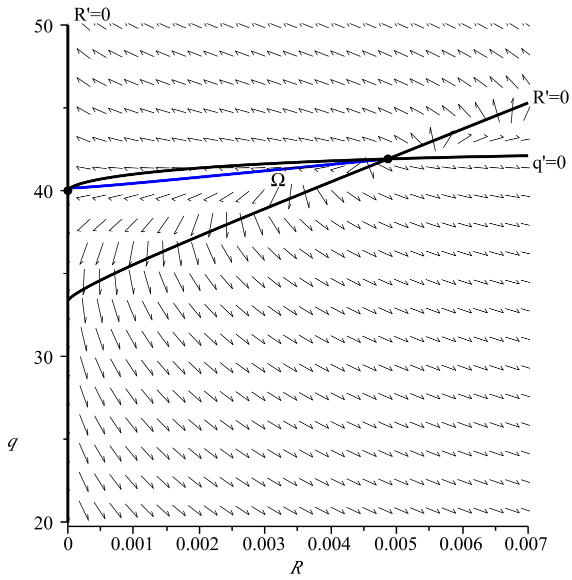

Figure 2: Phase diagram for a conventional Taylor rule: $r^* + P(0)L(R_1^*) < 0$. We use the same functional forms and parameters as in Figure 1 with $\gamma = 0.2$ instead.

the case depicted in Figure 1. However, if this condition does not hold, that is locally $L'(R)/L(R) > (1 - P'(R)/(R - P(R))$, the two isoclines are locally increasing. This is the case depicted in Figure 2.

Let us consider the following example: if the monetary rule is linear and $\Lambda(c, m)$ is a CES function,

$$\Lambda(\bar{c}, \bar{m}) = \left[\eta \bar{c}^{\frac{\zeta}{\zeta-1}} + (1-\eta)\, \bar{m}^{\frac{\zeta}{\zeta-1}}\right]^{\frac{\zeta-1}{\zeta}}, \tag{39}$$

with $0 < \eta, \zeta < 1$, the first case occurs if

$$-\left(\frac{1+\gamma}{\gamma}\right) P(0) = \frac{\tilde{r} - \gamma\tilde{\pi}}{\gamma} < \left[\frac{1}{\xi}\left(\frac{\zeta}{1-\zeta-\gamma}\right)^{\zeta}\right]^{1/(1-\zeta)},$$

where $\xi \equiv [\eta/(1-\eta)]^{1-\zeta}$. There are two cases. First, if $\gamma > 1 - \zeta$, the isocline $\dot{q} = 0$ is locally increasing. Second, if $\gamma < 1 - \zeta$, the isocline $\dot{q} = 0$ may (not) be locally increasing, for an active Taylor rule, if the target for the real interest rate is relatively high (low), the target for the inflation rate is relatively low (high), and/or the degree of reactiveness of the nominal interest rate to inflation is relatively low (high).

If the first case occurs, then we always have $q_1^* < q_0^*$. In the second case, we may have $q_1^* > q_0^*$, depending upon the deep parameters for the consumer demand and the monetary rule. A necessary condition for $q_1^* > q_0^*$ is that locally $\Psi'(R) > 0$.

The equilibrium point $R^* = 0$ always exists and is, geometrically, in the intersection of isocline $\dot{q} = 0$ with the first branch of isocline $\dot{R} = 0$. The second equilibrium point, which exists under the conditions of Proposition 1, is in the intersection of the isocline $\dot{q} = 0$ with the second branch of the isocline $\dot{R} = 0$, whose slope is given by equation (38). For the range in which $q > 0$, we have $\Phi(R) > 0$, which means that the branch of the isocline, in which $q = \Phi(R)$, is globally increasing. Since

$$\Psi(0) - \Phi(0) = \frac{(P(0) + r_+)(P(0) + r_-)}{\alpha\beta(\rho + \mu)P(0)},$$

where

$$r_+ \equiv \frac{\rho + \sqrt{\rho^2 + 4\beta\,(1-\alpha)\,(\rho+\mu)}}{2},\ r_- \equiv \frac{\rho - \sqrt{\rho^2 + 4\beta\,(1-\alpha)\,(\rho+\mu)}}{2},$$

and $P(0) < 0$, a necessary condition for the existence of the steady state (R_1^*, q_1^*), which was introduced in Proposition 1, is $P(0) + r_+ > 0$, implying that $\Psi(0) - \Phi(0) > 0$, which means that the isocline $\dot{q} = 0$ cuts the $q-$axis above the isocline $\dot{R} = 0$. In Proposition 2, we proved that this steady state is locally a source.

Furthermore, we may conjecture that the unstable manifold associated to the equilibrium point (R_1^*, q_1^*) and the stable manifold associated to the equilibrium point $(0, q_0^*)$ intersect. The proposition proves that this conjecture is right, which means that a heteroclinic orbit defined as

$$\Omega \equiv \left\{(R, q) \in \mathcal{W} :\ \lim_{t\to\infty}(R(t), q(t)) = (0, q_0^*),\ \lim_{t\to-\infty}(R(t), q(t)) = (R_1^*, q_1^*)\right\}$$

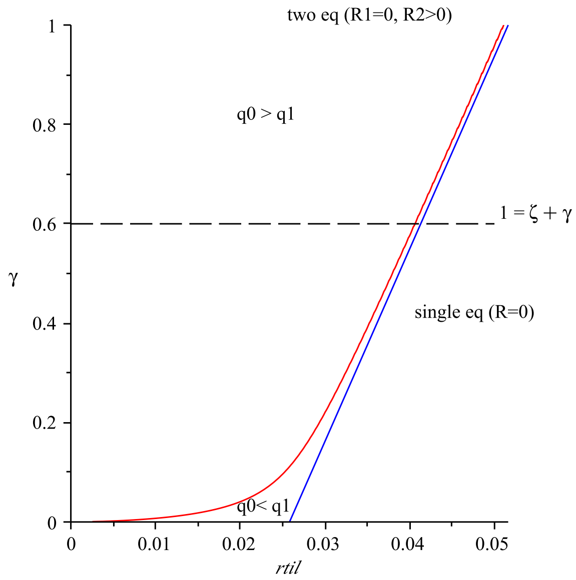

Figure 3: Bifurcation diagram for the conventional Taylor rule in the space $(\tilde{r}, \gamma)$. The values for the other parameters are as in Figure 1.

exists in $\mathcal{W} \subseteq \mathbb{R}^2_+$. In our case, Ω corresponds to the set of all the equilibrium values for the nominal interest rate and house prices. All other points, in $\mathcal{W}/\Omega$, lead to a violation of the transversality and/or the no-Ponzi game condition.

This means that there is global indeterminacy: any initial point $(R(0), q(0)) \in \Omega$ is an equilibrium point and converges asymptotically to $(0, q_0^*)$. Condition (35) implies that if the monetary policy is more (less) active, then the reduction of the nominal interest rate is correlated with an increase (decrease) in the prices of houses. Along a liquidity-trap path, in fact, the central bank adopting the Taylor-rule framework tries to stimulate aggregate demand and thus inflation by decreasing the nominal interest rate more than proportionally with the decline in inflation. This policy triggers a fall in the real interest rate. If the interest rate cuts are sufficiently aggressive, house prices will increase even along the decelerating-inflation trajectory, as it emerges from the bifurcation diagram in Figure 3.

4 Alternative Taylor Rules

We now examine local and global equilibrium dynamics for the case of Taylor rules that incorporate house prices. We start with rule (21) and next we study rule (22).

4.1 Taylor Rule Depending on the Level of Housing Prices

For the case of rule (21), we have $\pi = P(R, q)$, where

$$P(R(t), q(t)) = \frac{1}{1+\gamma} R(t) - \frac{\epsilon}{1+\gamma} \tilde{q} + P(0,0), \tag{40}$$

with $P(0,0) \equiv (\gamma\tilde{\pi} - \tilde{r} - \epsilon\tilde{q})/(1+\gamma)$.

Hence, the dynamic system assumes the form

$$\begin{aligned} \dot{q}(t) &= (R(t) - P(R(t), q(t)))\,(q(t) - \Psi(R(t), q(t)))\,, \qquad (41) \\ \dot{R}(t) &= \frac{\alpha\beta(\rho+\mu)}{L'(R(t))}\,(\Phi(R(t), q(t)) - q(t))\,, \qquad (42) \end{aligned}$$

where $\Psi(R, q)$ and $\Phi(R, q)$ are as in equations (30) and (31), in which $P(R)$ is substituted by $P(R, q)$ as in equation (40).

This case does not present substantial changes as regards the conventional Taylor rule, as we shall see in the following Proposition.

Proposition 4 *Assume a modified Taylor rule depending on the level of housing prices as in equation* (21) *in which the monetary policy is active, meaning that* $0 < \partial P/\partial R < 1$*, and define*

$$BP(0,0) \equiv \frac{P(0,0)}{2} + \left[\left(\frac{P(0,0)}{2}\right)^2 + \frac{\epsilon(1-\alpha)}{\alpha(1+\gamma)}\right]^{1/2} > 0.$$

Hence:

(a) if $r_+ + P(0,0) \leq B(P(0,0)$*, then there is an unique equilibrium steady state* $(R^*, q^*) = (0, q_0^*)$ *where*

$$q_0^* = \frac{1+\gamma}{\epsilon} BP(0,0); \tag{43}$$

(b) if $r_+ + P(0,0) > B(P(0,0)$*, then there are two equilibrium steady states* $(R^*, q^*) = (0, q_0^*)$ *and* $(R^*, q^*) = (R_1^*, q_1^*)$ *where*

$$q_1^* = \frac{(1+\gamma)(P(0,0)+r_+) - \gamma R_1^*}{\epsilon} \tag{44}$$

and

$$R_1^* = \left\{ R : \ L(R) = \left(\frac{\alpha(1+\gamma)r_+}{(1-\alpha)\epsilon} \right) \left(P(0,0) + r_+ - \frac{\gamma}{1+\gamma} R \right) \right\},$$

and there is a heteroclinic trajectory connecting those two equilibrium steady states, starting from equilibrium (R_1^*, q_1^*)*, which is locally a source, and converging asymptotically to equilibrium* $(0, q_0^*)$*.*

Proof. See Appendix F.

The equilibrium $(0, q_0^*)$ is again singular and behaves globally as a saddle point: for $R^* = 0$, we have $\dot{q} \lesseqqgtr 0$ if and only if $q \lesseqqgtr q_0^*$, and the vector field is horizontal on the space $\mathcal{W}$. The equilibrium steady state (R_1^*, q_1^*) also displays local determinacy, because the local Jacobian has positive eigenvalues for the admissible values of the parameters.

The global dynamics is as in the version of the model with a conventional Taylor rule (see Figure 4). There is a heteroclinic orbit connecting equilibria (R_1^*, q_1^*) to $(0, q_0^*)$. The only combinations of equilibrium nominal interest rates and housing prices are those along the orbit, which means that there is global indeterminacy.

Therefore, this Taylor rule does not change qualitatively the dynamics, but only

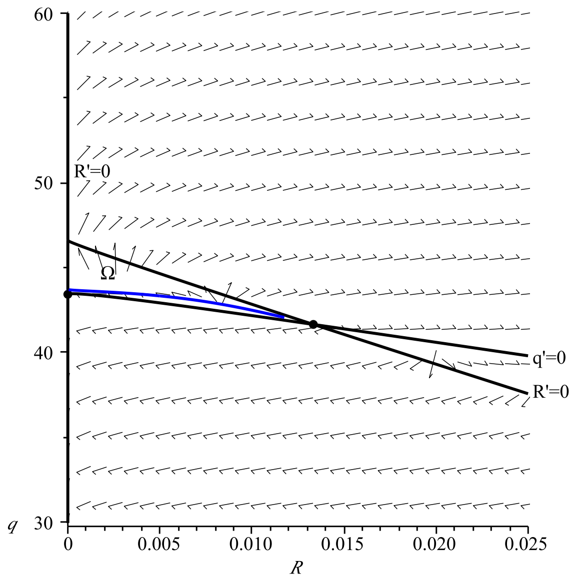

Figure 4: Phase diagram for Taylor rule (21). The figure is built by assuming the same functional forms and the same parameters as Figure 1, and $\epsilon = 0.002$ and $\tilde{q} = 50$.

quantitatively. However, since

$$P(0,0) = \frac{\gamma\tilde{\pi} + \epsilon\tilde{q} - \tilde{r}}{1+\gamma}$$

can have any sign, although it has all parameters positive, the definition of the target values is less stringent than in the case of the conventional Taylor rule. Furthermore, we can exclude the case in which there is a unique steady state equilibrium (R_1^*, q_1^*). That is, the equilibrium with a zero nominal interest rate always exists.

4.2 Taylor Rule Depending on the Rate of Growth of Housing Prices

For the case of rule (22), the inflation rate becomes a function of the rate of change of housing prices,

$$\pi(t) = \frac{R(t) + \gamma\tilde{\pi} - \tilde{r} - \delta\dot{q}(t)/q(t)}{1+\gamma},$$

and the general equilibrium dynamic system is

$$\dot{q}(t) = \frac{\gamma R(t) + \tilde{r} - \gamma\tilde{\pi}}{1+\gamma-\delta}\left(q(t) - \Psi(R(t))\right), \tag{45}$$

$$\dot{R}(t) = \frac{\alpha\beta(\rho+\mu)}{L'(R(t))}\left(\Phi(R(t)) - q(t)\right) + \frac{\delta}{1+\gamma}\frac{L(R(t))}{L'(R(t))}\frac{\dot{q}(t)}{q(t)}, \tag{46}$$

where $\Psi(R)$ and $\Phi(R)$ are similar to equations (30) and (31),

$$\Psi \equiv \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{(1+\gamma)}{\gamma R + \tilde{r} - \gamma\tilde{\pi}}\right) L(R), \tag{47}$$

$$\Phi \equiv \left(\frac{\gamma R + \tilde{r} - \gamma\tilde{\pi} - (1+\gamma)\rho}{\alpha\beta(\rho+\mu)(1+\gamma)}\right) L(R). \tag{48}$$

Therefore, we can state the following proposition.

Proposition 5 *Let the same assumptions as in Proposition 1 hold. Then the existence, multiplicity and magnitudes of the steady-state equilibria are as in Proposition 1. In addition:*

(a) if $1+\gamma > \delta$, then the steady state $(0, q_0^*)$ (if it exists) is singular and is a generalized saddle point and the steady state (R_1^*, q_1^*) (if it exists) is a source, and there is a heteroclinic orbit joining them.

(b) if $1+\gamma < \delta$, then the steady state $(0, q_0^*)$ (if it exists) is singular and is a generalized sink and the steady state (R_1^*, q_1^*) (if it exists) is a saddle point. If there are two steady state equilibria, the saddle manifold associated to (R_1^*, q_1^*) is the boundary of the basin of attraction of equilibrium $(0, q_0^*)$.

Proof. See Appendix G.

As in the version of the model with the conventional Taylor rule (see Proposition 1), the steady-state equilibria of housing prices are given in equations (34) and (33). From now on, consider the the values for the parameters such that there are two steady

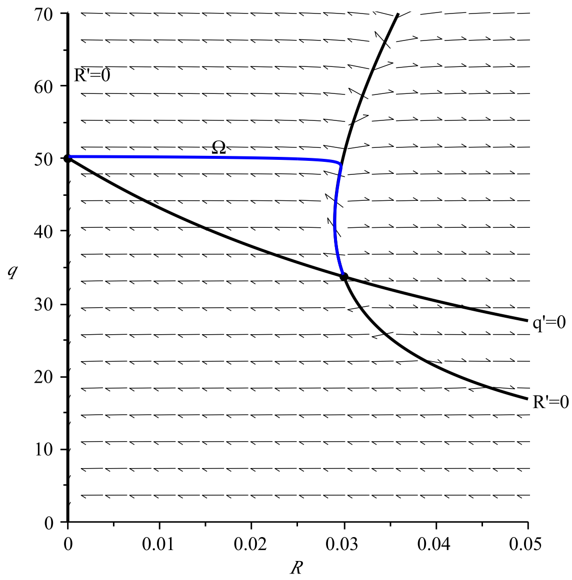

Figure 5: Phase diagram for Taylor rule (22). The figure is built by assuming the same functional forms and the same parameters as in Figure 1, and $\delta = 0.5 < 1 + \gamma$.

state equilibria. The associated phase diagrams are given in Figure 5, for the case in which $1 + \gamma > \delta$, and in Figure 6, for the case in which $1 + \gamma < \delta$.

If $\delta < 1+\gamma$, whereby the central bank reacts more to consumer-price inflation than to house-price inflation, the global dynamics is similar to the version of the model with a conventional Taylor rule (see Figure 5): there is global indeterminacy in the sense that there is an interval for initial values of R, $R(0) \in (0, R_1^*)$ which are equilibrium values and the nominal interest rate tends (possibly in a non-monotonous way) to the steady state $R^* = 0$. For a given initial value of the nominal interest rate in that interval, there is a single initial value for the equilibrium house prices. Next, the pair $(R(t), q(t))$ will follow along the heteroclinic orbit Ω converging to $(0, q_0^*)$.

If $\delta > 1+\gamma$, whereby the central bank reacts more to house-price inflation than to consumer-price inflation, the global dynamics changes substantially (see Figure 6): the two steady-state equilibria are both locally and globally indeterminate. In particular, the steady state (R_1^*, q_1^*) is locally indeterminate of order one and the steady state $(0, q_0^*)$ is locally indeterminate of order two. The orders of indeterminacy are given by the

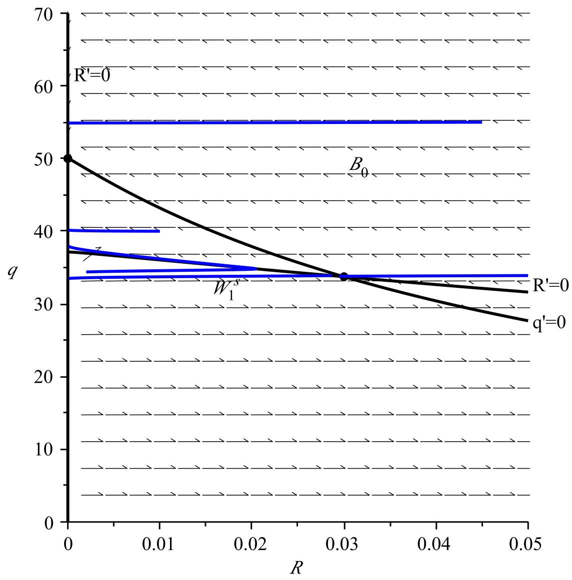

Figure 6: Phase diagram for Taylor rule (22). $\mathcal{B}_0$ is the basin of attraction of equilibrium $(0, q_0^*)$ and $\mathcal{W}_1^s$ is the stable sub-manifold associated to (R_1^*, q_1^*). The figure is built by assuming the same functional forms and the same parameters as in Figure 1, and $\delta = 5 > 1 + \gamma$.

dimension of the local stable manifolds. Globally, the stable sub-manifold associated to equilibrium (R_1^*, q_1^*), W_1^s, bounds the basin of attraction, B_0, of the steady state $(0, q_0^*)$. In our case, this means that the set $B_0 \cup W_1^s$ is the space of equilibrium states of the economy, in the sense that all the trajectories starting in it are equilibrium trajectories. However, if $(R(0), q(0)) \in B_0$, then the equilibrium trajectory will converge to $(0, q_0^*)$, while, if $(R(0), q(0)) \in W_1^s$, then the equilibrium trajectory will converge to (R_1^*, q_1^*). This means that if we have an initial nominal interest rate $R(0) \neq 0$, there will be one single initial value $q(0)$, say $q_1(0)$, such that the pair $(R(t), q(t))$ will converge to the steady state (R_1^*, q_1^*), but an infinite number of initial values $q(0) > q_1(0)$ such that $(R(t), q(t))$ will converge to the steady state $(0, q_0^*)$. Indeterminacy of order two indeed implies that there is an infinite number of initial equilibrium values for $q\,(0)$, given $R(0)$, and not just one as in the version with a conventional Taylor rule.

5 Conclusions

The interaction between monetary policy and house prices arguably has been one of the most debated topics in recent years. The Federal Reserve's accommodating monetary policy of 2002-2006 - the so-called Great Deviation from the Taylor rule - is argued to be responsible for the global crisis started in 2007, since it generated boom-bust patterns in house prices. In this paper we have presented an overlapping generations economy with housing-wealth effects and demonstrated that house-price instabilities can well occur even if the central bank follows monetary policy feedback rules of the Taylor-style. In particular, global equilibrium dynamics satisfying agents' tranversality conditions follow a heteroclinic orbit connecting multiple steady-state equilibria. Rules-based monetary policies, even when they aim to fight aggressively booms or busts in house prices, can well induce an off-target self-fulfilling trajectory in which house prices and inflation converge to a liquidity trap. Reacting to house-price inflation more than to consumer-price inflation gives rise to a basin of attraction to the liquidity trap, and introduces local indeterminacy at the off-trap steady state. Bifurcation analysis reveals that, along a liquidity-trap path, a central bank adopting a sufficiently aggressive monetary policy conforming to the Taylor principle may bring about an off-target self-fulfilling boom in house prices, even if the economy is spiraling down into a decelerating inflation dynamics. Hence, active monetary policy cannot be used to prevent the occurrence of bubbles.

A Solution of the Representative Consumer's Problem

In the intertemporal optimization problem, the representative consumer born at time s chooses the optimal time path of total consumption, $\overline{z}(s,t)$, to maximize the lifetime utility function (2), given (6) and the constraints (3) and (4). Using the definition of total consumption, $\overline{z}(s,t) \equiv \overline{c}(s,t) + R(t)\overline{m}(s,t)$, and the optimal intratemporal condition (6), we can write

$$\log \Lambda\left(\overline{c}(s,t), \overline{m}(s,t)\right) = \log \upsilon(t) + \log \overline{z}(s,t), \tag{49}$$

where $\upsilon(t) \equiv \Lambda\left(\frac{\Gamma(R(t))}{\Gamma(R(t))+R(t)}, \frac{1}{\Gamma(R(t))+R(t)}\right)$ is the same for all generations. Therefore, the intertemporal optimization problem can be formalized in the following terms:

$$\max_{\{\overline{z}(s,t),\, \overline{h}(s,t)\}} \int_0^\infty \left[\alpha\left(\log \upsilon(t) + \log \overline{z}(s,t)\right) + (1-\alpha)\log \overline{h}(s,t)\right] e^{-(\mu+\rho)t} dt, \tag{50}$$

subject to

$$\begin{aligned} \dot{\overline{a}}(s,t) &= (R(t) - \pi(t) + \mu)\,\overline{a}(s,t) + \overline{y}(s,t) - \overline{\tau}(s,t) - \overline{z}(s,t) + \\ &\quad + \left[\frac{\dot{q}(t)}{q(t)} - (R(t) - \pi(t))\right] q(t)\overline{h}(s,t), \end{aligned} \tag{51}$$

and given $\overline{a}(s,0)$. The optimality conditions are

$$\dot{\overline{z}}(s,t) = (R(t) - \pi(t) - \rho)\overline{z}(s,t), \tag{52}$$

$$\frac{1-\alpha}{\alpha}\overline{z}(s,t) = \left[(R(t) - \pi(t)) - \frac{\dot{q}(t)}{q(t)}\right] q(t)\overline{h}(s,t), \tag{53}$$

$$\lim_{t\to\infty} \overline{a}(s,t)e^{-\int_0^t (R(j)-\pi(j)+\mu)dj} = 0. \tag{54}$$

Therefore, the individual budget constraint (51) can be expressed as

$$\begin{aligned}
\dot{\overline{a}}(s,t) &= (R(t)-\pi(t)+\mu)\,\overline{a}(s,t)+\overline{y}(s,t)-\overline{\tau}(s,t)- \\
&\quad -\overline{z}(s,t)+\left[\frac{\dot{q}(t)}{q(t)}-(R(t)-\pi(t))\right]q(t)\overline{h}(s,t) \\
&= (R(t)-\pi(t)+\mu)\,\overline{a}(s,t)+\overline{y}(s,t)-\overline{z}(s,t)-\frac{1-\alpha}{\alpha}\overline{z}(s,t) \\
&= (R(t)-\pi(t)+\mu)\,\overline{a}(s,t)+\overline{y}(s,t)-\frac{1}{\alpha}\overline{z}(s,t). \qquad (55)
\end{aligned}$$

Integrating forward (55), using the transversality condition (54) and (52), total consumption turns out to be a linear function of total wealth:

$$\overline{z}(s,t)=\alpha(\mu+\rho)\left(\overline{a}(s,t)+\overline{k}(s,t)\right), \qquad (56)$$

where $\overline{k}(s,t)$ is human wealth, defined as the present discounted value of after-tax labor income, $\overline{k}(s,t)\equiv\int_t^\infty\left(\overline{y}(s,t)-\overline{\tau}(s,t)\right)e^{-\int_t^v(R(j)-\pi(j)+\mu)dj}dv$. From (6),

$$\overline{z}(s,t)=L(R(t))\overline{c}(s,t), \qquad (57)$$

where $L(R(t))\equiv 1+R(t)/\Gamma\left(R(t)\right)$. Time-differentiating (57) yields

$$\dot{\overline{z}}(s,t)=L'(R(t))\overline{c}(s,t)\dot{R}(t)+L(R(t))\dot{\overline{c}}(s,t). \qquad (58)$$

Therefore, the dynamic equation for individual consumption is

$$\dot{\overline{c}}(s,t)=(R(t)-\pi(t)-\rho)\,\overline{c}(s,t)-\frac{L'(R(t))\dot{R}(t)}{L(R(t))}\overline{c}(s,t). \qquad (59)$$

B Aggregation

The per capita aggregate financial wealth is given by

$$a(t) = \beta \int_{-\infty}^{t} \overline{a}(s,t) e^{\beta(s-t)} ds. \tag{60}$$

Differentiating with respect to time yields

$$\dot{a(t)} = \beta \overline{a}(t,t) - \beta a(t) + \beta \int_{-\infty}^{t} \dot{\overline{a}}(s,t) e^{\beta(s-t)} ds, \tag{61}$$

where $\overline{a}(t,t)$ is equal to zero by assumption. Using (3) yields

$$\begin{aligned}
\dot{a(t)} &= -\beta a(t) + \mu a(t) + \left(R(t) - \pi(t)\right) a(t) + y(t) - \tau(t) - c(t) - \\
&\quad -R(t)m(t) + \left[\frac{\dot{q}(t)}{q(t)} - \left(R(t) - \pi(t)\right)\right] q(t)h(t) \\
&= \left(R(t) - \pi(t) - n\right) a(t) + y(t) - \tau(t) - c(t) - \\
&\quad -R(t)m(t) + \left[\frac{\dot{q}(t)}{q(t)} - \left(R(t) - \pi(t)\right)\right] q(t)h(t).
\end{aligned} \tag{62}$$

Using (11), the per capita aggregate consumption is given by

$$c(t) = \frac{\alpha(\mu + \rho)}{L(R(t))} \left(a(t) + k(t)\right), \tag{63}$$

where $k(t) = \int_t^{\infty} \left(y(t) - \tau(t)\right) e^{-\int_t^{v}(R(j)-\pi(j)+\mu)dj} dv$ is the per capita aggregate human wealth. Next differentiate with respect to time the definition of per capita aggregate consumption, to obtain

$$\dot{c(t)} = \beta \overline{c}(t,t) - \beta c(t) + \beta \int_{-\infty}^{t} \dot{\overline{c}}(s,t) e^{\beta(s-t)} ds. \tag{64}$$

Note that $\overline{c}(t,t)$ denotes consumption of the newborn generation. Since $\overline{a}(t,t) = 0$, from (11) we have

$$\overline{c}(t,t) = \frac{\alpha(\mu+\rho)}{L(R(t))}\left(\overline{k}(t,t)\right). \tag{65}$$

Using (12), (63) and (65) into (64) yields the time path of per capita aggregate consumption:

$$\dot{c}(t) = (R(t) - \pi(t) - \rho)\, c(t) - \frac{L'(R(t))\dot{R}(t)}{L(R(t))} c(t) - \frac{\alpha\beta(\rho+\mu)}{L(R(t))} a(t). \tag{66}$$

C Proof of Proposition 1

Any equilibrium steady states, (R^*, q^*), are bounded points $(R, q) \in (\mathbb{R}_+, \mathbb{R}_{++})$ such that $\dot{q} = \dot{R} = 0$ and the transversality condition holds if $R^* - P(R^*) > -\mu$. From equation (28), $\dot{q} = 0$ if and only if $q = \Psi(R)$. From equation (29), $\dot{R} = 0$ if and only if $R = 0$, because $L'(0) = \infty$, or $q = \Phi(R)$. Therefore, there are two candidates for equilibrium steady states. First, we have $R = 0$ and $q = q_0^* \equiv \Psi(0) = -(1-\alpha)/(\alpha P(0))$ which constitute indeed an equilibrium point only if $P(0) < 0$. In this case, the transversality condition is verified, because the condition $-P(0) > -\mu$ always holds. Second, from the condition $q = \Psi(R) = \Phi(R)$, there is a second candidate if the set $\{R \geq 0 : \ \Psi(R) = \Phi(R) > 0\}$ is non-empty. We have

$$\Psi(R) = \Phi(R) \Leftrightarrow (R - P(R) - \rho)(R - P(R)) - \beta\,(1-\alpha)\,(\rho+\mu) = (r - r_+)(r - r_-) = 0,$$

where

$$r_+ \equiv \frac{\rho + \sqrt{\rho^2 + 4\beta\,(1-\alpha)\,(\rho+\mu)}}{2}, \; r_- \equiv \frac{\rho - \sqrt{\rho^2 + 4\beta\,(1-\alpha)\,(\rho+\mu)}}{2}$$

and $r = R - P(R) > -\mu$ from the transversality condition. Because $\beta\,(1-\alpha)\,(\rho+\mu) > 0$, then $r_+ > \rho > 0 > r_-$. As $L(R) > 1$, then the condition $r > \rho$ must hold, which implies that the transversality condition is always met. Therefore, an equivalent condition for the existence of a second steady state is $R_1^* = \{R \geq 0 : \ R - P(R) = r_+\}$. From the assumption that the monetary policy is globally active, $R - P(R)$ is monotonically increasing in R, which means that $R - P(R) \in [-P(0), +\infty)$. Therefore, the steady state (R_1^*, q_1^*) exists if $P(0) + r_+ \geq 0$, where $q_1^* = \Psi(R_1^*) = \Phi(R_1^*)$. As a result, there is multiplicity if $0 > P(0) \geq -r_+$, there is only one steady state (R_1^*, q_1^*) if $P(0) > 0$, and there is only steady state $(0, q_0^*)$ if $P(0) + r_+ < 0$. We set $r^* = r_+$.

D Proof of Proposition 2

For the steady-state equilibrium $(R^*, q^*) = (R_1^*, q_1^*)$, we have the trace and determinant of the Jacobian given by

$$\mathrm{tr}J(R_1^*) = r^* + \alpha\beta(\rho+\mu)\frac{\Phi'(R_1^*)}{L'(R_1^*)} = 2r^* - \rho + \frac{(1-P'(R_1^*)L(R_1^*)}{L'(R_1^*)}$$

and

$$\det J(R_1^*) = r^*\alpha\beta(\rho+\mu)\left(\frac{\Phi'(R_1^*) - \Psi'(R_1^*)}{L'(R_1^*)}\right) = \frac{(2r^*-\rho)(1-P'(R_1^*))L(R_1^*)}{(r^*-\rho)L'(R_1^*)} > 0,$$

because

$$\Phi'(R_1^*) = q_1^*\left(\frac{1-P'(R_1^*)}{r^*-\rho} + \frac{L'(R_1^*)}{L(R_1^*)}\right) > 0,$$

yielding the eigenvalues $\lambda_1 = 2r^* - \rho > 0$ and $\lambda_2 = (1 - P'(R_1^*))L(R_1^*)/L'(R_1^*) > 0$ if the monetary policy is locally active, $1 - P'(R_1^*) > 0$. Then the steady state (R_1^*, q_1^*) is always a source.

Evaluating the trace and the determinant for equilibrium $(R^*, q^*) = (0, q_0^*)$, with $q_0^* = \Psi(0)$, we get

$$\mathrm{tr}J(0) = -P(0) + \alpha\beta(\rho+\mu)\left(\frac{\Phi'(0)}{L'(0)} + (\Psi(0)-\Phi(0))\frac{L''(0)}{(L'(0))^2}\right)$$

and

$$\det J(0) = -P(0)\frac{\alpha\beta(\rho+\mu)}{L'(0)}\left(\Phi'(0) - \Psi'(0) + (\Psi(0) - \Phi(0))\frac{L''(0)}{L'(0)}\right).$$

As

$$\frac{\Psi'(0) - \Phi'(0)}{L'(0)} = \Psi(0) - \Phi(0) = \frac{(P(0)+r_+)(P(0)+r_-)}{\alpha\beta(\rho+\mu)P(0)} > 0,$$

we have

$$\begin{aligned}
\mathrm{tr}J(0) &= -(2P(0)+\rho) + \frac{(P(0)+r_+)(P(0)+r_-)}{P(0)}\frac{L''(0)}{(L'(0))^2}, \\
\det J(0) &= (P(0)+r_+)(P(0)+r_-)\left(1 - \frac{L''(0)}{(L'(0))^2}\right).
\end{aligned}$$

Recall that this steady state exists only if $P(0) < 0$, and observe that $L'(0) = \infty$ and $L''(0)/(L'(0))^2 = -\infty$. Then

$$\mathrm{tr}J(0) = \det J(0) = \begin{cases} -\infty, & \text{if } P(0) + r_+ > 0 \\ +\infty, & \text{if } P(0) + r_+ < 0 \end{cases}$$

which means that the eigenvalues of the Jacobian are infinite and the steady state is singular. If $P(0) + r_+ > 0$, the steady state is multiple (case (b) in Proposition 2) and is a kind of non-regular saddle point, and, if $P(0) + r_+ < 0$, the steady state is unique (case (c) in Proposition 2) and is a kind of non-regular source. In any case, the flow approaches or diverges from $(0, q_0^*)$ at an infinite speed and is non-differentiable locally.

In order to study local dynamics, we can use several methods, such as, first, finding a de-singularized projection and studying its local dynamics, and, second, studying global dynamics.

If we use the first method, the natural way to remove the singularity introduced by

$L'(R)$ at $R = 0$, would be to recast the system in variables (L, q),

$$\begin{aligned} \dot{q} &= q\,(R - P(R)) - \left(\frac{1-\alpha}{\alpha}\right) L, \\ \dot{L} &= L'(R)\dot{R} = L\,(R - P(R) - \rho) - \alpha\beta(\rho + \mu)q, \end{aligned}$$

where $L \geq 1$, $R = R(L)$ is increasing and $R(1) = 0$, $R'(1) = 0$. However, in this case there is an unique steady state $(L(R_1^*), q_1^*)$. Therefore, this method does not solve our de-singularization problem. We use the second method in the proof of Proposition 3. There, we show that the singular steady state $(0, q_0^*)$, for the case $P(0) + r_+ > 0$ (i.e., case (b) in Proposition 2) behaves as a generalized saddle point.

E Proof of Proposition 3

As the system (28)-(29) does not have an explicit solution, we must employ qualitative methods in order to study global dynamics. One possible method is to find a first integral of system (28)-(29), that is, a Lyapunov function $V(.)$ such that $V(R, q) =$ constant. We could not find this function. Another method is to determine a trapping area for the heteroclinic orbit. The rationale is the following: as the steady state (R_1^*, q_1^*) is a source, the unstable manifold is the set $\mathbb{R}_+/(R_1^*, q_1^*)$; as the steady state $(0, q_0^*)$ is a saddle point, the stable manifold is, locally, composed by a single trajectory belonging to $\mathbb{R}_+$; therefore there is an intersection of the unstable manifold of the first point and of the stable manifold of the second which is non-empty. In order to prove that it exists, and to characterize it, we consider a trapping area for the heteroclinic orbit. In order to prove this, we start by determining the slopes of the heteroclinic orbit in the neighborhoods of the two equilibria, we build a trapping area enclosing the heteroclinic, and demonstrate that all the trajectories starting inside the trapping area escape from it, with the exception of those starting at any point along the heteroclinic orbit.

E.1 Slopes of the Eigenspaces Associated to the Two Equilibria

The unstable eigenspace $\mathcal{E}_1^u$ is the tangent space to the unstable manifold associated to equilibrium (R_1^*, q_1^*), in the space where $(R, q) \in \mathcal{W} \subseteq \mathbb{R}_+^2$ lie,

$$\mathcal{W}_1^u \equiv \left\{ (R, q) \in \mathcal{W} : \lim_{t \to -\infty} (R(t), q(t)) = (R_1^*, q_1^*) \right\}.$$

The unstable eigenspace $\mathcal{E}_1^u$ is the linear space which is tangent to $\mathcal{W}_1^u$ and is spanned by the eigenvectors $(V_1, 1)^\top$ and $(V_2, 1)^\top$ which are associated to eigenvalues λ_1 and λ_2, respectively, where

$$V_1 = -\frac{r_+ \Psi'(R_1^*)}{r_+ - \rho}, \; V_2 = \frac{(r_+ - \rho) L'(R_1^*)}{L(R_1^*)}.$$

In general, $V_2 > 0$ and the sign of V_1 is ambiguous. Observe that the the slope of $\mathcal{E}_1^u$, $\frac{dq}{dR}\big|_{\mathcal{E}_1^u}$, is the opposite to the slope of isocline $\dot{q} = 0$, locally at the steady state (R_1^*, q_1^*). Let us call $\mathcal{E}_{1,+}^u$ ($\mathcal{E}_{1,-}^u$) the eigenspace related to the dominant (non-dominant) eigenvalue. The following conditions can be proved: (1) if $r_+ > (1 - P'(R_1^*))L(R_1^*)/L'(R_1^*)$, then $2r_+ - \rho > (1 - P'(R_1^*))L(R_1^*)/L'(R_1^*)$, which is equivalent to $\lambda_1 > \lambda_2$, and therefore $\mathcal{E}_{1,+}^u = \{(R, q) \in \mathcal{W} : \; (q - q_1^*) = V_1(R - R_1^*)\}$ and $\mathcal{E}_{1,-}^u = \{(R, q) \in \mathcal{W} : \; (q - q_1^*) = V_2(R - R_1^*)\}$. In this case, $V_1 < 0$ and $V_2 > 0$ and the slope associated to the dominant eigenvalue is negative and the slope associated to the non-dominant eigenvalue is positive; (2) if $\lambda_1 < \lambda_2$, which is equivalent to $2r_+ - \rho < (1 - P'(R_1^*))L(R_1^*)/L'(R_1^*)$, then $r_+ < (1 - P'(R_1^*))L(R_1^*)/L'(R_1^*)$, and $\mathcal{E}_{1,+}^u = \{(R, q) \in \mathcal{W} : \; (q - q_1^*) = V_2(R - R_1^*)\}$ and $\mathcal{E}_{1,-}^u = \{(R, q) \in \mathcal{W} : \; (q - q_1^*) = V_1(R - R_1^*)\}$. In this case, $V_1 > V_2 > 0$ and the slope associated to the both eigenvalues are both positive, but the one associated with $\mathcal{E}_{1,-}^u$ is steeper.

The stable manifold associated to steady state $(0, q_0^*)$ is defined as

$$\mathcal{W}_0^s \equiv \left\{ (R, q) \in \mathcal{W} : \lim_{t \to \infty} (R(t), q(t)) = (0, q_0^*) \right\}.$$

However, we saw that the projection of the steady state $(0, q_0^*)$ in the space $\mathcal{W}$ is singular. This means that the solution approaches the singular steady state asymptotically with an infinite speed. In order to characterize the dynamics in the space $\mathcal{W}$ in the neighborhood of $(0, q_0^*)$, we have to take a different approach: observe that, as $R'(1) = 0$, then a naïve calculation for the slope of the stable manifold in the neighborhood of the singular equilibria could be $dq/dR = (r_+ - \rho)/(\alpha\beta(\rho + \mu)R'(1)) = \infty$.

Instead, observe that along the singular surface $R = 0$ we have $\dot{R} = 0$ and $\dot{q} = P(0)(\Psi(0) - q)$. Then, from any point along this surface where $q \neq q_0^* = \Psi(0)$, an unstable trajectory unfolds. This means that any trajectory coming from $R > 0$ will be deflected away from the equilibrium point on hitting the surface $R = 0$, with the exception of the one which converges to the equilibrium point $(0, q_0^*)$. The (global) direction of the vector field generated by equations (28)-(29) is given by

$$\left.\frac{dq}{dR}\right|_{(\dot{q}, \dot{R})} = \frac{L'(R)(R - P(R))(q - \Psi(R))}{\alpha\beta(\rho + \mu)(\Phi(R) - q)}. \tag{67}$$

In order to determine the slope of the trajectory which converges to the singular steady state, we determine the slope of the vector field hitting the surface $R = 0$ using equation (67). We have

$$\left.\frac{dq}{dR}\right|_{R=0} = \frac{-L'(0)P(0)(q - \Psi(0))}{\alpha\beta(\rho + \mu)(\Phi(0) - q)} = \begin{cases} \infty, & \text{if } q \neq q_0^* \\ 0, & \text{if } q = q_0^* \end{cases}$$

because $L'(0) = \infty$. Therefore, the stable manifold associated to the singular steady state $(0, q_0^*)$ is horizontal in the space $\mathcal{W}$.

The heteroclinic orbit, $\Omega = \mathcal{W}_0^s \cap \mathcal{W}_1^u$, is tangent to a horizontal line in the neighborhood of the equilibrium point $(0, q_0^*)$ and is positively slopped in the neighborhood of (R_1^*, q_1^*), because it is tangent to $\mathcal{E}_{1,-}^u$.

E.2 Trapping Area

Next we consider the case in which $\lambda_1 < \lambda_2$, which is depicted in Figure 1. Recall that Ω is tangent to a line $dq/dR = 0$ in the neighborhood of point $(0, q_0^*)$. Observe that, in the neighborhood of point (R_1^*, q_1^*), the slope of the eigenspace associated to the non-dominant eigenvalue (λ_1) and of the isocline $\dot{R} = 0$ are both positive, but the former is less steep that the later because (see Figure 7)

$$\left.\frac{dq}{dR}\right|_{\dot{R}} = \left.\frac{dq}{dR}\right|_{\mathcal{E}_{1,-}^u} = \Phi'(R_1^*) + \frac{r_+}{r_+ - \rho}\ \Psi'(R_1^*) = -\frac{2r_+ - \rho}{r_+ - \rho} q_1^* \left(\frac{L'(R_1^*)}{L(R_1^*)}\right) < 0.$$

As the heteroclinic Ω is tangent to that eigenspace in the heighborhood of that equilibrium point, it will lie between the isocline $\dot{R} = 0$ and $\mathcal{E}_{1,-}^u$ and will never cross this line.

This allows to consider the trapping area whose sides are given by the segment of the isocline $\dot{q} = 0$ between the two equilibria (recall that the two equilibria lay along this isocline), by a line passing through the steady state (R_1^*, q_1^*), whose slope is given by the eigenvector which is associated to the non-dominant eigenvalue and by the horizontal segment such that $q = q_0^*$, between the q axis and the previous eigenvector-line.

Formally, the trapping area $[A, B, C]$ is defined by the vertices $A \equiv (0, q_0^*)$, $B \equiv (R_1^*, q_1^*)$, and $C \equiv (R_C, q_0^*) = (R_1^* - (q_1^* - q_0^*)/V_1, q_0^*)$ and the sides $(A, B) = \{(R, q) \in (0, R_1^*) \times (q_1^*, q_0^*) :\ \dot{q} = 0\}$, $(A, C) = \{(R, q) : R \in (0, R_1^*),\ q = q_0^*\}$, and $(B, C) = \{(R, q) \in (R_1^*, R_C) \times (q_1^*, q_0^*) : q = q_1^* + V_1(R - R_1^*)\}$.

Next we have to demonstrate that the (global) direction of the vector field, given by equation (67), which is generated by equations (28)-(29), allows us to prove that all the trajectories hitting the three boundaries of the trapping exit the trapping area.

At side (A, B) we have $\dot{q} = 0$ and $\dot{R} < 0$, because

$$\dot{R}\Big|_{(A,B)} = \frac{\alpha\beta(\rho+\mu)\,(\Phi(R) - \Psi(R))}{L'(R)} = \frac{L(R)(R - P(R) - r_+)(R - P(R) - r_-)}{L'(R)(R - P(R))} < 0,$$

given the fact that $-r_- < R - P(R) < r_+$ if $R \in [0, R_1^*)$. Then, as the slope of the vector field in the interval is

$$\frac{dq}{dR}\Big|_{(A,B)} = 0,$$

then the vector field points globally out of the trapping area within (A, B) with a horizontal slope.

At side (A, C), the vector field corresponds to a horizontal line between point A and point C, which is in the intersection of a horizontal line passing through the q-axis and the direction defined by the eigenvector associated to the dominant eigenvalue at point B. These two lines meet at point C. Along (A, C), we have $\dot{R} < 0$ for $R \in (0, \Phi^{-1}(q_0^*))$, $\dot{R} = 0$ at point $R = \Phi^{-1}(q_0^*)$ and $\dot{R} > 0$ for $R \in \Phi^{-1}(q_0^*), R_C$, and, we have $\dot{q} > 0$ everywhere. As the slope of the vector field is given by

$$\frac{dq}{dR}\Big|_{(A,C)} = \frac{L'(R)(R - P(R))(q_0^* - \Psi(R))}{\alpha\beta(\rho+\mu)(\Phi(R) - q_0^*)} \begin{cases} < 0, & \text{if } R \in (0, \Phi^{-1}(q_0^*) \\ \infty & \text{if } R = \Phi^{-1}(q_0^*) \\ > 0, & \text{if } R \in (\Phi^{-1}(q_0^*), R_C) \end{cases}$$

then the vector field points globally out of the trapping area, at all points located at (A, C).

At side (B, C), which is a segment of the eigenspace $\mathcal{E}^u_{1,-}$ between points (R_1^*, q_1^*) and (R_C, q_0^*), the vector field has local time-variations given by $\dot{R} > 0$ and $\dot{q} > 0$. As this side has a slope given by V_1, then the trajectories exit the trapping area if the slope of the vector field is less steep than V_1, that is, if and only if

$$\left.\frac{dq}{dR}\right|_{(B,C)} - \left.\frac{dq}{dR}\right|_{\mathcal{E}^u_{1,-}} = \frac{L'(R)(R-P(R))(q-\Psi(R))}{\alpha\beta(\rho+\mu)(\Phi(R)-q)} + \frac{r_+\Psi'(R_1^*)}{r_+-\rho} < 0.$$

The numerator is equivalent to

$$L'(R)(R-P(R))(q-\Psi(R)) + (\Phi(R)-q)\left(r_+L'(R_1^*) - L(R_1^*)(1-P'(R_1^*))\right) <$$
$$L'(R)(R-P(R))(q-\Psi(R)) + (\Phi(R)-q)\left((R-P(R))L'(R_1^*) - L(R_1^*)(1-P'(R_1^*))\right) <$$
$$L'(R_1^*)(R-P(R))(\Phi(R)-\Psi(R)) - (\Phi(R)-q)L(R_1^*)(1-P'(R_1^*)) <$$
$$(\Phi(R)-q)\left(L'(R_1^*)(R-P(R)) - L(R_1^*)(1-P'(R_1^*))\right) <$$
$$(\Phi(R)-q)\left(L'(R_1^*)r_+ - L(R_1^*)(1-P'(R_1^*))\right) < 0.$$

It is negative because $R > R_1^*$ implies $L'(R) < L'(R_1^*)$, $R - P(R) > r_+$, $q > \Psi(R)$, and because we assume $L'(R_1^*)r_+ < L(R_1^*)(1-P'(R_1^*)$ from $\lambda_1 < \lambda_2$. Then, all the trajectories reaching segment (B, C) will exit the trapping area.

Therefore, there is an unique trajectory starting from point B, (R_1^*, q_1^*), that does not hits the boundaries of the trapping area, and therefore converges to point A, $(0, q_0^*)$. This is the heteroclinic trajectory Ω, and it happens to cross the isocline $\dot{R} = 0$.

F Proof of Proposition 4

We use the same methods as for the proof of Propositions 1, 2 and 3.

First, the steady-state conditions are $q = \Psi(R, q)$ and $R = 0$ or $q = \Phi(R, q)$, and a steady state is an equilibrium if $R^* - P(R^*, q^*) + \mu > 0$. A steady state exists and is an equilibrium if there is a $q^* > 0$ such that $q^* = \Psi(0, q^*)$ and $-P(0, q^*) > -\mu$. Using the Taylor rule (21), the equilibrium condition is equivalent to $(q - q_+)(q - q_-) = 0$,

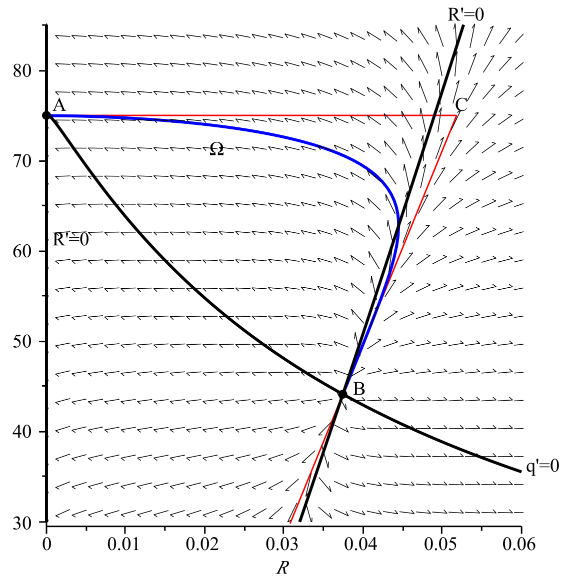

Figure 7: Proof of Proposition 3.

where

$$q_\pm = \left(\frac{1+\gamma}{\epsilon}\right)\left\{\frac{P(0,0)}{2} \pm \left[\left(\frac{P(0,0)}{2}\right)^2 + \frac{1-\alpha}{\alpha}\frac{\epsilon}{1+\gamma}\right]^{1/2}\right\}.$$

As $q_- < 0 < q_+$, then

$$q_0^* = q_+ = \left(\frac{1+\gamma}{\epsilon}\right) BP(0,0) > 0,$$

which holds for any $P(0,0)$. As

$$\begin{aligned} -P(0,q_0^*) &= \frac{\epsilon}{1+\gamma}q_0^* - P(0,0) = BP(0,0) - P(0,0) = \\ &= -\frac{P(0,0)}{2} + \left[\left(\frac{P(0,0)}{2}\right)^2 + \frac{1-\alpha}{\alpha}\frac{\epsilon}{1+\gamma}\right]^{1/2} > 0, \end{aligned}$$

then the transversality condition holds without further conditions.

An interior steady state is determined from the non-negative values of (R,q) such that $q = \Psi(R,q) = \Phi(R,q) > 0$. If we define $r(R,q) \equiv R - P(R,q)$, this condition is equivalent to $(r-r_+)(r-r_-) = 0$. As a necessary condition for a positive q is $r(R,q) > \rho$, then $r(R,q) = r_+ > -\mu$, which means that the transversality condition is automatically verified. This is equivalent to $\gamma R + \epsilon q = (1+\gamma)(P(0,0)+r_+)$. Substituting in $q = \Psi(R,q)$, we get the equation

$$L(R) = \frac{\alpha}{1-\alpha}\left[\left(\frac{1+\gamma}{\epsilon}\right)(P(0,0)+r_+) - \frac{\gamma}{\epsilon}R\right] r_+.$$

After some algebra, we can prove that this equation has a non-negative solution for R if and only if $r_+ + P(0,0) \geq BP(0,0) > 0$.

The local dynamics are determined in the same way as in the proofs of Propositions 2 and 3. But, in this case, the eigenvalues for the steady state (R_1^*, q_1^*) are

$$\begin{aligned}
\lambda_1 &= 2r_+ - \rho > 0, \\
\lambda_2 &= \frac{\epsilon}{1+\gamma} q_1^* + \frac{\gamma}{1+\gamma}\frac{L(R_1^*)}{L'(R_1^*)} > 0
\end{aligned}$$

for the values of the parameters such that there are two steady states.

G Proof of Proposition 5

It is easy to see that the steady state conditions are exactly the same as in the case of the conventional Taylor rule. The only thing that may change is related to the local and global dynamics of the model.

Applying the same methods as for the proof of Proposition 2, we find the eigenvalues for the steady state (R_1^*, q_1^*):

$$\begin{aligned}
\lambda_1 &= 2r_+ - \rho > 0, \\
\lambda_2 &= \frac{\gamma}{1+\gamma-\delta}\frac{L(R_1^*)}{L'(R_1^*)},
\end{aligned}$$

where $\mathrm{sign}(\lambda_2) = \mathrm{sign}(1+\gamma-\delta)$.

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