A flow network analysis of direct balance-sheet contagion in financial networks

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Abstract

This paper put forward a novel approach to the analysis of direct contagion in financial networks. Financial systems are here represented as flow networks – i.e., directed and weighted graphs endowed with source nodes and sink nodes – and the propagation of losses and defaults, originated by an exogenous shock, is here represented as a flow that crosses such a network. In establishing existence and uniqueness of such a flow function, we address a know problem of indeterminacy that arise, in financial networks, from the intercyclicity of payments. Sufficient and necessary conditions for uniqueness are pinned down. We embed this result in an algorithm that, while computing the propagation caused by a shock, controls for the emergence of possible indeterminacies. We then apply some properties of network flows to investigate the relation between the structure of a financial network – i.e., the size and the pattern of obligations – and its exposure to default contagion. We characterise

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first and final contagion thresholds (i.e., the value of the smallest shock capable of inducing default contagion and the value of the smallest shock capable of inducing the default of all agents in the network, respectively) for some classes of networks, namely the complete, star-shaped, incomplete regular, and cycle-shaped networks. Finally, we show that the exposition to default contagion of a generic network – both in terms of contagion thresholds and of number of defaults induced by a shock – monotonically grows with the ratio between internal and external debts, where the former are the intra-network obligations and the latter are the debts that the agents in the network owe to final claimants who do not belong to the network.

JEL classification: C63, G01, G33.

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1 Introduction

In this paper we put forward a novel approach, based on the theory of *flow networks*,[1] for the analysis of direct contagion in networks of agents connected among themselves by financial obligations. Financial contagion is broadly defined as the transmission of financial distress across agents, sectors or regions of the economy. The literature has distinguished among three different forms of financial contagion, also known as *systemic risk*, corresponding to different possible channels of propagation:[2] 1) *Informational contagion,* that can occur in imperfectly informed financial markets and in banking systems,where depositors' expectations about the possibility of a crisis can lead to bank runs; 2) *Direct contagion* transmitted via networks of financial obligations. In banking and financial systems, such networks arises from three sources: *i)* loans and deposits in the interbank money market, *ii)* 'over-the-counter' trading in assets and derivatives, and *iii)* payment systems; while, in the manufacturing sector, networks of financial obligations arise from trade credit.[3] 3) *Common exposure* to drops in the value of assets, drops that can be exogenous or endogenous, the latter being the case of fire sales of illiquid assets induced by liquidity shortages. In this paper we forego informational contagion, as well as any analysis of agents' behaviour, and focus on the mechanics of direct balance-sheet contagion using a framework that takes common exposures into account.

[1] See Ahuja et al. (1993) for a reference book on the theory of flows and flow network.

[2] See the review articles by Dow (2000) and by De Bandt-Hartmann (2000).

[3] See Kiyotaki-Moore (2001, 2002)

We represent a financial system as a flow network – i.e., a directed and weighted graph endowed with source nodes and sink nodes – and use the properties of network flows to analyse the dynamics of the flows of losses that propagate, in a financial system, as a consequence of an external shock. Flow network theory is a branch of graph theory that, starting with the works of nineteen century physicists such as Gustav Kirchoff, has been progressively developed and applied to a vast number of fields, ranging from telecommunication to electrical and hydraulic engineering, transportation, computer networking, industrial and military logistics, etc. To the best of our knowledge, the present work is the first application of network flow analysis to economics or finance.

The paper is organized as follows. In the next section, we review the literature related to this work. In section three, we define a financial system in terms of a flow network. In section four, we model the domino effect of direct balnce-sheet contagion as a flow of losses that crosses a financial flow network, i.e., as a *propagation* function that associates, to the links of a network, the financial losses induced by an exogenous shock. Existence and uniqueness of such a propagation function are discussed in section five, where we address a known problem of indeterminacy that arises from the intercyclicity of payments in financial networks. We identify necessary and sufficient conditions for uniqueness and, in section six, we embed this result in an algorithm that, while computing a propagation function, controls for possible indeterminacies due to the interciclycity of obligations. In section seven, we investigate the relation between the structure of a financial networks – i.e., the size and the pattern of the financial obligations that form the network – and its exposition to default contagion. For some classes of networks – such as the complete, star-shaped, incomplete regular and cycle-shaped networks – we characterise the first and final thresholds of contagion, i.e., the value of the smallest shock capable of inducing default contagion and the value of the smallest shock capable of inducing the default of all agents in the network, respectively. For generic networks we show that, under a mildly restrictive condition, the exposition to default contagion of a network – both in terms of contagion thresholds and of number of defaults induced by a shock – monotonically depends on the ratio between the values of the external debt and of the intra-network obligations of the agents in the network. Conclusions are drawn in section seven. Finally, the proofs of the theorems, lemmae and corollaries presented in the paper are collected in the Appendix.

2 Related literature

The work that is most closely related to ours is Eisenberg and Noe (2001), a seminal contribution which has provided the analytical basis and the computational tool to most of the below cited papers that perform numerical simulations to study direct contagion. Their paper and the present one study the properties of the same object – a directed and wighted graph that represents a financial system – resorting to two different analytical approaches: we use flow networks while Eisenberg and Noe resort to matrix algebra and lattice theory. These authors investigate the domino effect generated by the default of firms that participate in a single payment system. In so doing, they study the existence and the uniqueness of a vector of payments that clears a network of interdependent financial claims, where the capability of an agent to repay in full his debts depends on the solvency of his own debtors which, in turn, depends on the solvency of their debtors, and so forth. They express such a vector as a function of the operating cash flows of the members of the financial network. This function is defined on a lattice, representing such a financial system, and complies with the requirements of limited liability, debt priority and pro-rata reimbursements.

Eisenberg and Noe, as well as the present paper, do not investigate agents' behaviour in a financial network and focus on the mechanics of contagion as governed by the rules of limited liability, debt priority and pro-rata reimbursements. This marks a major difference with respect to the main theoretical analyses of direct financial contagion – due to Rochet-Tirole (1996), Freixas et al. (2000), Allen and Gale (1998, 2000) – that take explicitly into account the behaviour of banks and depositors. These authors use models of contagion in interbank liquidity networks based on, or inspired to, the seminal paper by Diamond and Dybvig (1983), where the uncertainty about the timing or the location of consumers' expenditure – hence, of depositors' withdrawals – generates the risk of liquidity shortages for the banks. In order to insure against such a liquidity risk, and in absence of perfectly functioning 'ex-post' liquidity markets, each bank holds deposits in other banks forming, in so doing, an interbank network of short-term exposures. This network serves the purpose of sharing liquidity risk and of re-allocating liquidity across banks, de facto moving customers' deposits from banks in liquidity surplus towards banks in liquidity deficit. In case of default of a bank, though, the same network becomes a channel of transmission of financial losses towards the other banks in the network, creating the possibility of systemic crisis. The initial failure of one or more banks, capable of generating a widespread financial crises, can be due to exogenous causes, as it is in Allen and Gale (1998), where financial crises arise as a consequence of downturns in the economic cycle. Recessions can cause losses in the value of the assets held by banks, losses capable of rendering them insolvent. If depositors foresee the recession, they will protect themselves from possible bank defaults by withdrawing their deposits and, in so doing, they create the conditions for the occurrence of a widespread crisis. Financial contagion can also originate from liquidity crisis. In Allen and Gale (2000) the failure of a bank is due to an idiosyncratic shortage of liquidity that forces the bank to liquidate long-term assets, incurring the costs of such 'fire sales'. They show that a 'complete' network – a network where all banks are equal to one another, all have mutual bilateral obligations and of the same amount – is more robust than an incomplete network, i.e., a network with fewer links among the banks. Freixas et al. (2000) achieve similar results: in their examples the 'complete' network structure bears the smallest risk of contagion, while a 'credit chain' structure increases the fragility of the banking system.

Understanding the relation existing between the structure of the net of financial obligations among banks and the resiliency of the banking system to withstand possible liquidity and insolvency shocks, is an important issue for central banks and policy makers. In most advanced nations, monetary authorities have imposed rules, known as "large exposure rules", to limit the credit exposures of banks towards single borrowers and increase the diversification of their portfolio. Both the Basel I and the Basel II committees have recommended this sort of controls on credit risk.[4] In setting an upper limit to single loans – usually linking the size of a loan to some measure of the capital of the lending bank – these measures also imply a growth in the number of debt/credit relations existing in a financial system, i.e., a growth in the *connectivity* of the financial network. At the same time, in several countries, the authorities have encouraged mergers and acquisitions in the banking sector, leading to more concentrated systems with fewer and larger operators. In most cases this policy has reinforced, if not generated, two-tiers banking systems where few large operators act as money centers, i.e., each of them is connected to many small banks which, in turn, are not connected among themselves. Whether this policies have rendered financial systems more or less resilient to withstand systemic shocks, given the structural changes that they brought along, is a question that does not have an obvious answer.

[4]See, on the web page of the Bank for International Settlements, the documents "Principles for the Management of Credit Risk" and the paragraphs 729 and 736 of "The New Basel Capital Accord".

The above mentioned theoretical papers have investigated this issue, but only by means of stylized examples.[5] As Upper (2007, page 2 and 3) puts it "Unfortunately, analytical results on the relationship between market structure and contagion have been obtained only for a limited number of highly stylised structures of interbank markets, which are of limited use when it comes to assessing the scope for contagion in real world banking systems.[...] Given the scarcity of theoretical results, researchers have increasingly turned to computer simulations to study contagion." Upper refers to several authors who, in order to assess the robustness of different network structures, have studied the mechanics of default contagion using numerical simulations, foregoing the microeconomic behaviour of banks and depositors. Such papers – which includes the works by Sheldon and Maurer (1998), Furfine (2003), Wells (2002), Elsinger, Lehar and Summer (2006), Upper and Worms (2004), Degryse and Nguyen (2004), Blavarg and Nimander (2002), Cifuentes (2003), Mistrulli (2005,2006),Canedo and Martínez Jaramillo (2009) – have analyzed national banking systems, in most cases estimating the structure of national interbank networks,[6] using simulations to evaluate their exposure to default contagion.

Numerical simulations are also used by Shin et al. (2005) and Nier et al. (2007), who analyze generic network structures, rather than specific national ones. Shin et al. present a model where default contagion is exacerbated by the effects of 'fire sales'. They show that if the demand for illiquid assets is not perfectly elastic, the forced and untimely sale of such assets by financially distressed operators induces further reductions in their market value, feeding further contagion. Nier et al. build their model on a previous and unpublished version of the present paper.[7] Using a computing device, these authors generate random banking networks, in the fashion of the random graphs a là Erdős-Rényi, and use them to run numerical simulations aiming at evaluating the exposure to systemic risk of different network structures.

While the framework presented here lends itself to computational exercises, as shown by Nier et al., the ambition of the present work is to investigate the mechanics of financial contagion with analytical methods.

[5] As Rochet-Tirole-Parigi (2000) declare "Because of the complexity of the transfers involved in the matrix [of financial obligations], we will illustrate our findings in two symmetric extreme cases." [page 187]

[6] Apart from Mistrulli (2005, 2006) who used data about the actual interbank exposures, data in possession of Bank of Italy.

[7] That preliminary version of this paper was presented at the Bank of England in may 2004.

3 The financial flow network

The purpose of a financial system is the intermediation of the supply of funds provided by final claimants – that we will generically label as 'households', who hold shares, bonds and deposits – and the demand expressed by the final users of funds, such as companies, mortgage holders, governments, etc. Let such a system be composed by a set of financial intermediaries $\Omega = \{\omega_i\}, i = 1...n$, which are directly or indirectly connected to one another by financial obligations, namely bonds and deposits, and let $d_{ij} \in \mathbb{R}^+$ be the amount of debt, if any, that agent i owes agent j. Each agent in Ω is characterized by its own balance sheet. On the asset side, let $a_i \in \mathbb{R}^+$ be the value of the sum of *external assets* owned by ω_i, which are liabilities of agents – the final users of funds – who do not belong to Ω, let $A = \{a^k\}, k = 1...m$, be the set of external assets such that each a^k in A appears in the balance sheet of at least one operator in Ω, and let $a_i^k \in \mathbb{R}^+$ be the amount of asset k held by agent i, if any. Besides the external assets, an agent ω_i can hold *internal assets* which are liabilities of other agents in Ω, and let $c_i = \sum_j d_{ji}$ be the sum of the such assets held by agent i. On the liability side of the balance sheet, let $d_i \in \mathbb{R}^+$ be the sum of the debts that ω_i owes to households and to agents in Ω, in the possible forms of bonds, loans and deposits: $d_i = h_i + \sum_j d_{ij}$, where h_i is the *external debt* of ω_i, i.e., the amount of debt claims against ω_i held by households, and $\sum_j d_{ij}$ is the *internal debt* of agent i, i.e., the claims against ω_i held by other members of Ω. For simplicity, we assume that all debts have the same seniority. Finally, the value of the equity of the i-th agent, e_i, is set by the budget identity $e_i \equiv a_i + c_i - d_i - h_i$. We assume that the value of the external assets is set by the market and take the other balance sheet headings c_i, d_i, h_i, as well as the debts d_{ij}, at their nominal values. For the sake of simplicity, we also assume that all the shares issued by the members of Ω are held by households, i.e., there is no cross-holding of shares among the financial intermediaries.

We represent this financial system as a *multisource network*, i.e., a directed and connected graph, with some sources and two sinks, with links endowed with non-negative capacities.[8] Let $N = \left\{\Omega, A, T, H, L^\Omega, L^A, L^T, L^H, \Gamma\right\}$ be a multisource network where:

1. $\Omega = \{\omega_i\}$ is the set of n nodes that represent the above defined financial intermediaries.

2. $A = \{a^k\}$, is the set of m *source nodes*, i.e., nodes with no incoming links, that

[8] See Ahuja et al. (1993), sections 1 and 2, or Diestel (2000), ch. 6.

represent the external assets held by the members of Ω.

3. T is a *sink*, i.e., a terminal node with no outgoing links. This node represents the shareholders who own the equity of the agents in Ω.

4. H is a *sink* node representing the households who hold debt claims, in the form of deposits and bonds, against the agents in Ω.

5. $L^{\Omega} \subseteq \Omega^2$ is a set of ordered pairs of nodes in Ω, i.e., a set of directed links $\{l_{ij}\}$ representing the liabilities d_{ij}, where l_{ij} starts from node ω_i and ends in node ω_j, and $l_{ij} \in L^{\Omega}$ only if $d_{ij} > 0$.

6. $L^A = \left\{l_i^k\right\}$ is a set of directed links, with start nodes in A and end nodes in Ω, that connect the external assets to their owners, where $l_i^k \in L^A$ only if $a_i^k > 0$.

7. $L^T = \{l_T^i\}$ is a set of directed links, with start nodes in Ω and end node T.

8. $L^H = \{l_H^i\}$ is a set of directed links, with start nodes in Ω and end node H.

9. $\Gamma : L^{\Omega}, L^A, L^T, L^H \to \mathbb{R}^+$ is a map, called *capacity function*, that associates i) to each l_{ij} the value of the corresponding liability d_{ij}, ii) to each l_i^k the value of the corresponding asset a_i^k, iii) to each l_T^i the equity, e_i, of its start node ω_i, and iv) to each l_H^i the external debt, h_i, of its start node ω_i.

We shall refer to N as a *financial flow network* or, for brevity, as a network N, while we shall refer to a generic multisource network simply as a network.

4 Propagation of losses and defaults: the domino effect

We now use the above defined financial flow network to model the process of direct financial contagion among the agents in Ω as a flow of financial losses that crosses N. This flow is initiated by an exogenous negative shock that consists of a loss of value of some of the external exposures a^k. To define a shock, let $b^k \in [0,1]$ be a parameter that measures the fraction of the value of the asset a^k which is lost. An exogenous shock is an assignment of value to the vector $[b^k], k \in A$, where at least one of its components assumes a strictly positive value. If $b^k > 0$, then source node a^k is activated and sends to its direct descendants in Ω − i.e., to the nodes $\omega_i \in \Omega$ such that $l_i^k \in L^A$− a financial loss equal to $b^k a_i^k$. The shock, i.e., the flow of losses out of the source nodes, is a vector of scalars $[b^k a^k]$. It what follows, we distinguish between *common shock*, that affects more than an agent in Ω, from *idiosyncratic shocks,* i.e., shocks born by a single node only.

As a shock occurs, the involved source nodes release a flow of losses into the network. The propagation of these losses across N is governed by the rules of limited liability, debt priority and pro-rata reimbursement of creditors. When a node ω_i suffers a loss, this loss is first absorbed by the net worth of the node. Only the residual loss, if any, is passed over to other nodes in Ω. The losses that are offset by the equity of the agents in Ω are born by households, in their capacity as shareholders, thus they exit from the flow of losses that circulate across Ω to end up directly into the sink T. To represent this property we introduce, for each node in Ω,an *absorption function*

$$\beta_i(\lambda_i) = \min\left(\frac{\lambda_i}{e_i}, 1\right) \tag{1}$$

where λ_i is the total loss born by the i-th node, received from source nodes and/or from other nodes in Ω. The variable $\beta_i \in (0,1)$ measures the share of net worth lost by a node. If a node ω_i receives a positive flow of losses, it sends to the sink an amount of its own equity equal to $\beta_i e_i$.

The equity of a financial intermediary measures its *absorption capacity.* If the losses suffered by ω_i are larger than its net worth, then this node is insolvent and sends the residual loss, $\lambda_i - e_i$, to its creditors. For each node in Ω, let

$$b_i(\lambda_i) = \max\left(0, \frac{\lambda_i - e_i}{d_i}\right) \tag{2}$$

be its *insolvency function.* The variable $b_i \in [0,1]$ assumes a value of zero if the i-th operator is solvent, while it assumes a strictly positive value if the operator defaults. In the latter case, the assets of the insolvent node are liquidated and its creditors get a *pro rata* refund. We assume that this is done without delays and without incurring in bankruptcy costs.[9] The creditors fall into two categories: the direct descendants of ω_i in Ω – i.e., the nodes $\omega_j \in \Omega$ such that $l_{ij} \in L^{\Omega}$, also said *children* nodes of ω_i – and the households who own claims, in the form of bonds and/or deposits, against ω_i. The variable b_i measures the fraction of the i-th agent's debt that is not recovered through liquidation, i.e., the *loss-given-default* ratio of the failing agent. When the i-th agent becomes insolvent, households receive a loss equal to $b_i h_i$ (if $h_i > 0$), that ends into the sink H, while a node ω_j which is a creditor of node ω_i receives from the latter a loss equal to $b_i d_{ij}$. The loss born by a

[9]Bankruptcy costs can be introduced in the model by adding extra sources of losses that get activated in case of defaults. These extra losses would (obviously) make the system more prone to widespread crisis without substantially altering the results presented below.

financial intermediary in Ω is the sum of the losses, if any, received from its external and internal exposures:

$$\lambda_i = \sum_k b_k a_i^k + \sum_j b_j d_{ji}.$$

As the occurrence of a shock causes an inflow of losses into the system, the absorption and insolvency functions govern the propagation of such losses across the network by assigning a positive real value to each link in N.

Definition 1 *Let $f : L^A, L^\Omega, L^T, L^H \to \mathbb{R}^+$ be a map such that: $f(l_i^k) = b^k a_i^k$, $f(l_{ij}) = b_i d_{ij}$, $f(l_H^i) = b_i h_i, f(l_T^i) = \beta_i e_i$, and call this function a* propagation *in a network N.*

Such a propagation function is a *flow* in N. A flow over a generic network is a vector valued function, defined over the links of the network, such that: i) for all the links in the network, the scalar associated to a link does not exceed its capacity; and ii) for all the nodes in the network which are neither a source node nor a terminal node, the *divergence* − i.e., the difference between the total flow arriving at a node and the total flow departing from such a node − is null.

Definition 2 *Let $G = (\Omega, L, s, t)$ be a network where: Ω is a set of nodes, $L \subseteq \Omega^2$ is a set of directed links, and s and t are the source and the sink node, respectively. Let $L^+(\omega_i)$ ($L^-(\omega_i)$) be the set of the outgoing (incoming) links of a node $\omega_i \in \Omega$. A function $\varphi : L \to \mathbb{R}^+$ is a* flow *in G if it satisfies the following conditions:*

a. $\varphi(l) \leq \Gamma(l)$, for all l in L; (Capacity constraint)

b. $\sum_{L^+(\omega_i)} \varphi(L) = \sum_{L^-(\omega_i)} \varphi(l)$, for all $\omega_i \in \Omega$; (Flow conservation)

Theorem 1 *The above defined propagation function is a* flow *in a network N.*

A flow out of the sources of a network is feasible, also said *legitimate*, – i.e. it exists – if it entirely reaches the sink. In the next section we first show that any propagation in N is feasible and then we pin down sufficient and necessary conditions for the uniqueness of a propagation.

5 Existence and uniqueness of a propagation function in a financial flow network

5.1 Network capacity and feasibility of a propagation

Every network has an upper bound to its overall capacity to carry a flow. The carrying capacity of a network is equal to the value of the largest flow out of the sources that can cross the network and be entirely absorbed by the sink, i.e., the largest feasible flow. In general, the carrying capacity of a network is smaller or equal to the absorbing capacity of its sink. Finding the feasible flow of maximum value, for a given network, is a fundamental problem in the study of networks – known, in fact, as the *maximum flow problem.* This problem has been addressed by the celebrated result of Ford and Fulkerson (1956), known as the *minimum cut-maximum flow* theorem. Before presenting this theorem, we need to introduce the notions of a cut and of its capacity.

A *cut* in a network N is a partition $\left\{U, \overline{U}\right\}$ of $\{A \cup \Omega \cup T \cup H\}$, where U and $\overline{U}$ are two non-empty sets such that $A \subseteq U$ and $(T, H) \in \overline{U}$. Let $L(U)$ be the set of links that cross such a partition, i.e., the union of the set of *forward links* going from U into $\overline{U}$, $L^{+}(U) := \left\{l_i^k \in L^A \mid k \in A, \omega_i \in \overline{U}\right\} \cup \left\{l_{ij} \in L^{\Omega} \mid \omega_i \in U, \omega_j \in \overline{U}\right\} \cup \left\{l_{it} \in L^T \mid \omega_i \in U\right\}$, and of the set of *backward links* going in the opposite direction, $L^{-}(U) := \left\{l_{ij} \in L^{\Omega} \mid \omega_i \in \overline{U}, \omega_j \in U\right\}$. The *capacity* of a cut is the sum of the capacities of its forward links. The maximum carrying capacity of a network is set by the cut which has the smallest capacity among all possible cuts of the network:

Theorem 2 *(Ford and Fulkerson, 1956) In every network, the largest value of a feasible flow equals the capacity of a cut of smallest capacity.*

This upper bound is always attainable in flow networks which are somehow administrated to the end of maximising the flow that goes from the sources to the sinks. This is the case of flow networks such as pipeline systems or electrical networks, where the flows are centrally controlled and the networks themselves are designed to achieve this end. More specifically, the achievement of the above defined maximum flow is possible only if there are no flows crossing the minimum cut backward, i.e., from the sink towards the sources.[10]

[10] In that case it is the net flow, i.e. the forward flow less the backward flow, that crosses the minimum cut and reaches the sink.

This requisite implies that there are no cycle flows crossing the cut. Such a condition is not guaranteed at all in a financial system, where the flows of losses follow the rules of bankruptcy in a predetermined and decentralised fashion. Nonetheless, in a financial network the upper bound never binds a flow of losses: the largest possible flow out of the source nodes always reaches the sink. To establish this, we consider the scenario that is most unfavourable to the to forward transmission of a flow: the case where, for all cuts in a network, all backward links are filled to capacity. We then look at the *net capacity* of the cuts in N, that is the residual forward capacity (if any) of a cut when its backward flow is maximal.

Definition 3 *The* net capacity *of a cut,* $\Gamma\left\{U,\overline{U}\right\}$, *is the sum of the capacities of its forward links less the sum of the capacities of its backward links:* $\Gamma\left\{U,\overline{U}\right\}=\sum\limits_{L^{+}(U)}\Gamma(x)-\sum\limits_{L^{-}(U)}\Gamma(x)$.

In a network N, the budget identities of the nodes in Ω imply that the net capacity of all cuts is the same and equals the total value of the external assets:

Lemma 1 *In a financial flow network* N, *the net capacity of all cuts* $\left\{U,\overline{U}\right\}$ *equals the capacity of the cut* $\{A,(\Omega,H,T)\}$.

In other words, the net capacity of all cuts in N is equal to the the total exposure, of the financial system as a whole, towards the final users of funds: $\sum_{A}a^{k}$. Lemma 6, coupled with the maximum flow-minimum cut theorem, delivers the following proposition:

Theorem 3 *The largest value of a feasible propagation defined in a network* N *is equal to the largest possible flow out of the source nodes, i.e., the largest possible shock.*

This means that, in a financial network N, the budget identities of the agents in Ω guarantee the existence of all possible propagations, i.e., the propagations induced by all possible exogenous shocks.

5.2 Cycles and nominal indeterminacy of a propagation

The interdependence of obligations that constitutes the fabric of a financial network, can create problems of indeterminacy to the propagation function defined above: under some conditions the propagation induced by a given shock is not unique. In this section we pin down the conditions that create such indeterminacy and asses its scope and implications.

The problem of non-uniqueness of payment flows in a financial network was first pointed out by Eisenberg and Noe (2001). These authors explain the possible indeterminacy of the vector of payments that clears a network of interdependent financial claims with the following example: "Suppose the system contains two nodes, 1 and 2, both without any operating cash flows. Moreover, each node has nominal liabilities of 1.00 to the other node.[...] In this example, any vector $p_t = t(1,1)$, $t \in [0,1]$, is a clearing vector of the system" [Eisenberg and Noe, op.cit., page 249]. In this case, the flow of payments that goes from node 1 towards node 2 depends only on the payments that node 1 receives from node 2, and vice versa, therefore they can reimburse each other with any payment comprises between zero and unity.

The origin of this indeterminacy lies in the joint and simultaneous determination of the losses of the agents that belong to a *cycle* of defaulting agents or, more precisely, to a *strongly connected component* (henceforth SCC) of defaulting agents.[11] If a set of defaulting nodes is strongly connected, the losses that these nodes pass to one another are cyclically interdependent, and their insolvency functions of are simultaneously determined, like in the above example of a cycle of two defaulting agents. This simultaneity can generate indeterminacy: Under the conditions that we identify below, the value taken on by a propagation in a SCC of defaulting agents is not uniquely defined. Such a simultaneity does not arise at all if the propagation unfolds only along simple paths (as opposed to cycles). A propagation that does not generate cycle flows – as it is always the case for a propagation that takes place in an acyclic network N – does not pose problems of non-uniqueness:

Lemma 2 *A propagation in N is uniquely defined if it does not embed any cycle flow, i.e., if it does not entail any SCC of defaulting agents.*

[11] A *cycle* in a directed graph is a directed path such that its start node and end node coincide. A *directed path* is a sequence of nodes, with a start node and an end node, such that for any two consecutive nodes, i and $i+1$, these is a link going from i to $i+1$. A directed graph is said to be *strongly connected* if there exists a directed path going from each node to every other node in the graph. A subgraph that is strongly connected is called a *strongly connected component*. In other words, two nodes, i and j, are in the same strongly connected component if and only if there exists a directed path from i to j and there exists a directed path from j to i.

It is the occurrence of SCC's of defaulting nodes that generates the cyclical interdependence of payments which, in turn, can render a propagation indeterminate. For our purposes, we distinguish between *closed* and *open* SCC's:

Definition 4 *Let* $\mathbf{S} = (S, L(S))$, *where* $S \subseteq \Omega$ *and* $L(S) \subseteq S^2 \subseteq L^\Omega$, *be a strongly connected component of a network* N. *We say that* $\mathbf{S}$ *is* ***open*** *if there exists at least one link in* L *starting from a node in* S *and ending in* H *or in a node in* $\Omega \backslash S$. *Conversely, we say that* $\mathbf{S}$ *is* ***closed*** *if there is no link in* L *starting from a node in* S *and ending in* H *or in a node in* $\Omega \backslash S$.

In other words, the members of a closed SCC are indebted only among themselves. Conversely, in an open SCC, at least one member of such a component is indebted to the households H or to nodes in Ω that do not belong to the SCC. We now proceed to show that a) a propagation is not uniquely defined if and only if it entails closed SCC's of insolvent nodes, b) the indeterminacy is confined to such closed SCC's, and c) the emergence of closed SCC's of defaulting nodes in a propagation can be unambiguously detected.

The cyclical interdependence of obligations, that arises in any SCC of defaulting agents, renders indeterminate the flow of losses which is passed around among such agents if and only if they form a closed SCC. Conversely, open SCC's of defaulting agents do not generate any indeterminacy of the propagation function:

Lemma 3 *Let* $\mathbf{S} = (S, L(S))$ *be a SCC in* N *and let* f *be a propagation in* N. *The value of the propagation on the links in* $L(S)$ *– hence the value taken on by the insolvency function* $b_i(\lambda_i)$ *of the nodes* $\omega_i \in S$ *– is not uniquely defined if and only if (a)* $\mathbf{S}$ *is closed, and (b) all nodes in* S *default.*

This indeterminacy, if it arises, has a limited scope. Any positive flow of pro-rata reimbursements, in a closed SCC of failed agents, is simply a clearing transaction among its members, with no consequences on their own financial conditions. Moreover, and most important, this possible indeterminacy is confined to closed SCC's of defaulting nodes, it never affects the values taken on by the propagation in the rest of the network. This is due to the fact that a closed SCC is a *cul-de-sac*: the losses that reach the nodes in such a SCC do not come out it, these losses are born by their shareholders only, ending up entirely into the sink T.

Definition 5 *Let $\Theta = \{\overline{\mathbf{S}}\}$ be the (possibly empty) set of* ***closed*** *SCC's of nodes in N, and let $L^{\Theta} \subseteq L^{\Omega}$ be the set of links connecting pairs on nodes that belong to members of Θ.*

Theorem 4 *Let f be a propagation in a network N. Then: 1) f is uniquely defined on the links in $L \backslash L^{\Theta}$, and 2) f is indeterminate on the links of a closed SCC $\overline{\mathbf{S}} = (\overline{S}, L(\overline{S})) \in \Theta$ if and only if all nodes in $\overline{S}$ default.*

This theorem refines the analogous result put forward by Eisenberg and Noe (2001). These authors demonstrate that, for a clearing payment vector to be uniquely defined, it is sufficient that all *risk orbits* in the network are *surplus sets*; where the risk orbit of a node is its set of descendants, and a surplus set is a set of nodes such that "no node in the set has any obligation to any node outside the set and the set has positive operating cash flows" [Eisenberg and Noe, op. cit., page 241]. The authors also show that, for any clearing vector of payments, it is impossible for all nodes in a surplus set to have zero equity value, i.e., at least one node in the set does not default. In the light of the above theorem, we can replace this condition with a less restrictive one and state that, for a propagation in a network N to be uniquely defined, it is necessary and sufficient that all closed SCC's in N, if any, are surplus sets.

A useful consequence of the above theorem is that the occurrence of closed SCC's of defaulting agents in N is unequivocally revealed by the value taken on by a propagation function on the links in $L \backslash L^{\Theta}$. The above theorem implies that the flow of losses received by a closed SCC is uniquely defined. Moreover, to cause the failure of all the agents that form a closed SCC, the flow of losses that reach such a SCC must be maximal – i.e., it must be equal to the total exposure, of the SCC as a whole, towards the rest of the network.

Corollary 1 *Let $\overline{\mathbf{S}} = (\overline{S}, L(\overline{S})) \in \Theta$ be a closed SCC in N. All nodes in $\overline{S}$ default if and only if the flow of losses that reaches $\overline{\mathbf{S}}$ from the rest of the network – i.e., the flow across the partition $\{(A, \Omega \backslash \overline{S}), \overline{S}\}$ – is maximal, i.e.:*

$$\overrightarrow{f}\ \{(A, \Omega \backslash \overline{S}), \overline{S}\} = \sum_{k \in A} \sum_{i \in \overline{S}} b_k a_i^k + \sum_{j \notin \overline{S}} \sum_{i \in \overline{S}} b_j d_{ji}.$$

This implies that, in computing a propagation, the occurrence of the conditions that cause the above described indeterminacy can be detected unambiguously by monitoring the flow that reaches the closed SCC's in N, as it is done in the algorithm presented below.

6 Computing a propagation

An algorithm that computes a propagation f in N must perform two tasks: calculate f and check for possible indeterminacies, i.e., monitor the occurrence of closed SCC's of insolvent agents. This can be done as follows.

In calculating f, we add a superscript $t = 1, 2, 3, ...$ to the variables involved in the computation – namely $\lambda_i^t, b_i^t, \beta_i^t$ – to indicate the value taken on by these variables at each iteration of the algorithm. Recall that $\lambda_i = \sum_k b_k a_i^k + \sum_j b_j d_{ji}$ and let

$$[\lambda_i]_{1\times n} = [b_k]_{1\times m} \left[a_i^k\right]_{m\times n} + [b_j]_{1\times n} [d_{ji}]_{n\times n}$$

be the vector of the losses born by the agents in Ω. Then:

1. For a given value assignment of the vector $[b^k]$, compute $[\lambda_i^t] = [b_k] \left[a_i^k\right] + \left[b_j^{t-1}\right] [d_{ji}]$, starting with $t = 1$ and setting $b_j^0 = 0$;

2. compute $[\beta_i^t] = [\beta_i(\lambda_i^t)]$ and $[b_i^t] = [b_i(\lambda_i^t)]$ according to (1) and (2);

3. if $\sum_\Omega \beta_i^t e_i + \sum_\Omega b_i^t h_i < \sum_A b_k a_k$, then start again from point 1; if $\sum_\Omega \beta_i^t e_i + \sum_\Omega b_i^t h_i = \sum_A b_k a_k$, then move to step 4;

4. define the set of insolvent agents induced by f, $\underline{\Omega} = \{\omega_i \in \Omega | b_i > 0\}$, and the subgraph composed by such nodes, $\underline{\mathbf{\Omega}} = (\underline{\Omega}, L(\underline{\Omega}))$;

5. search for SCC's $\mathbf{S} = (S, L(S))$ in $\underline{\mathbf{\Omega}}$. If there is none, then stop; if there is at least one $\mathbf{S}$ in $\underline{\mathbf{\Omega}}$, then move to step 6;

6. for every $\mathbf{S}$ in $\underline{\mathbf{\Omega}}$, define the matrix of the coefficient of the system of equations used in part (1) of the proof of lemma 10:

$$[d(\mathbf{S})]_{m\times m} = \begin{bmatrix} d_1 & -d_{21} & \cdots & \cdots & \cdots & -d_{m1} \\ -d_{12} & d_2 & -d_{32} & \cdots & \cdots & \vdots \\ \vdots & -d_{23} & d_3 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ -d_{1(m-1)} & \vdots & \vdots & \vdots & \ddots & -d_{m(m-1)} \\ -d_{1m} & \cdots & \cdots & \cdots & -d_{(m-1)m} & d_m \end{bmatrix}$$

where $m = |S|$;

7. for every $\mathbf{S}$ in $\underline{\mathbf{\Omega}}$, compute the sum of the rows of $[d(\mathbf{S})]_{m \times m}$. For every $\mathbf{S}$ in $\underline{\mathbf{\Omega}}$ such that this sum is equal to zero, label $\mathbf{S}$ as 'closed' and the b_i's of the nodes in S as 'indeterminate'.

The first three steps of this algorithm calculate, for a given shock vector $[b_k]$, the value of the propagation f through the iterated application – node by node, along the directed paths of N – of the absorption and insolvency functions, $\beta_i(\lambda_i)$ and $b_i(\lambda_i)$, defined above. The values of the vectors $[\lambda_i^t], [\beta_i^t], [b_i^t]$ computed in step 1 and 2, are strictly increasing in t as long as there are nodes in Ω with strictly positive divergence, i.e., as long as there exists at least one $i \in \Omega$ s.t. $\lambda_i^t > \beta_i^{t-1} e_i + b_i^{t-1} d_i$, which, in turn, implies that $\sum_{\Omega} \beta_i^t e_i + \sum_{\Omega} b_i^t h_i < \sum_A b_k a_k$. Conversely, the repeated iteration of the algorithm yields stationary values of the vectors at hand once the flow out of the sources has been entirely absorbed by the sinks, i.e., when $\sum_{\Omega} \beta_i^t e_i + \sum_{\Omega} b_i^t h_i = \sum_A b_k a_k$. A feasible flow, in a flow network, ends up entirely into the sinks, and the feasibility of any propagation in a financial network N is guaranteed by theorem 7. Hence, the equality condition in step 3 is eventually achieved, then the divergence of all nodes in Ω is null and neither the losses arriving at a node nor the losses departing from a node can grow anymore: the computation of f stops and the algorithm delivers the pair of n dimensional vectors $\{[\beta_i], [b_i]\}$ which identify the propagation caused by the shock vector $[b_k]$.

Each iteration of steps 1-3 computes the passing of losses from a set of nodes in N to their children nodes. In absence of SCC's of defaulting agents, the length of the longest possible path in N is equal to n and so is the largest possible number of iterations of this first part of the algorithm. Conversely, in presence of SCC's of defaulting agents – that generate cycle flows – the algorithm converges asymptotically and monotonically to the final values $\{[\beta_i], [b_i]\}$ by computing progressively smaller augmentations of the cycle flows. Since the values at hand are sums of money, this problem can be easily overcome by setting an approximation of, say, one cent of a euro. Discretizing, in this fashion, the variables at hand, the stationary values $\{[\beta_i], [b_i]\}$ are obtained in a finite number of iterations of steps 1-3.

Steps 4-7 of the above algorithm control for the occurrence of closed SCC's of insolvent agents and signal the indeterminacy of the insolvency parameters in $[b_i]$ of the agents in such SCC's. Finding the SCC's of a directed graph, as step 5 requires, is a known basic issue in computer science and the literature provides several algorithms for it.[12] Steps 6 and

[12] The most known are the ones due to R. Tarjan and to H. Gabow. See Cormen et al. (2001), chapter

7 identify the closed SCC's of insolvent agents induced by f, resorting to the singularity condition of the matrix $[d(\mathbf{S})]_{m\times m}$. This condition, in turn, is necessary and sufficient to cause indeterminacy of the insolvency functions of a set of insolvent agents, as it has been argued above in the proof of lemma 11.

It is the case to point out that calculating f – by running steps 1-3 – on closed SCC's in N, is a waste of computing time if such closed SCC's turn out to be composed entirely of insolvent agents. This possible inefficiency is avoided by resorting to the following modified version of the above algorithm:

1. Search for SCC's $\mathbf{S} = (S, L(S))$ in N. If there is none, then proceed to compute f with steps 1-3 of the above algorithm. If there is at least one $\mathbf{S}$ in N, then proceed to step 2;

2. for every $\mathbf{S}$ in N, define $[d(\mathbf{S})]_{m\times m}$ and compute the sum of its rows. Then label as 'closed' SCC's $\overline{\mathbf{S}}$ the $\mathbf{S}$ in N such that the sum of the rows of $[d(\mathbf{S})]_{m\times m}$ is null and define the set $\Theta = \{\overline{\mathbf{S}}\}$;

3. let $|\Theta| = Y$. If $Y = 0$, proceed to compute f with steps 1-3 of the above algorithm . If $Y \geq 1$, then let $\Theta = \{\overline{\mathbf{S}}_y | y = 1, ..., Y\}$ and construct a new network $N\backslash\Theta$: for every $\overline{\mathbf{S}}_y \in \Theta$, i) replace $\overline{\mathbf{S}}_y$ with a sink node T_y, ii) direct the links across the cut $\{(A, \Omega), (\overline{S}_y, H, T)\}$ of N into the sink T_y.

4. run steps 1-3 of the above algorithm on the so modified network $N\backslash\Theta$.

5. for every $y \in \Theta$, compare the flow $\overrightarrow{f}\{(A, \Omega), T_y\}$ that ends into the sink T_y, with the total equity of the nodes in $\overline{S}_y$. If $\overrightarrow{f}\{(A, \Omega), T_y\} = \sum_{i\in\overline{S}} e_i$, then, for all nodes $\omega_i \in \overline{S}_y$, set $\beta_i = 1$, label b_i as 'indeterminate' and stop. If $\overrightarrow{f}\{(A, \Omega), T_y\} < \sum_{i\in\overline{S}} e_i$, then run steps 1-3 of the above algorithm on $\overline{\mathbf{S}}_y$.

The search for closed SCC's is here done at the beginning of the algorithm, by steps 1 and 2. If any closed SCC is found in N, then step 3 modifies the network, by replacing such components with sink nodes, and step 4 calculates the values taken on by f on the links in $L\backslash L^{\Theta}$. The legitimacy of this operation is guaranteed by part (1) of theorem 4. Part (2) of this theorem is applied in step 5, where the closed SCC's of insolvent agents are identified by comparing the flow of losses that enters each closed SCC $\overline{\mathbf{S}}_y$ in N, i.e.,

22, and the literature cited therein.

the flow across the partition $\{(A, \Omega), T_y\}$ of $N \backslash \Theta$, with the total absorbing capacity of the nodes in $\overline{\mathbf{S}}_y$. Finally, f is computed on the closed SCC's in N where f is uniquely defined, the ones with at least one solvent agent.

On the one hand, this second algorithm saves computing time by avoiding the calculation of f on the SCC's of N where f is indeterminate, if f induces any closed SCC's of insolvent agents. This gain grows with the number and the size of such SCC's. On the other hand, this algorithm requires more time than the first one above for a) the identification of the closed SCC's in N, because $\underline{\Omega} \subseteq \Omega$, and b) the transformation of N into $N \backslash \Theta$. The choice between these two algorithms will ultimately depend, case by case, on the expectations of the analyser with respect to the occurrence of closed SCC's of defaulting agents. For instance, in working on networks composed mostly of banks that hold customer deposits, the analyser can reasonably expect a very limited presence of closed SCC's and, therefore, choose the first algorithm. Conversely, the presence of closed SCC's is more likely to occur in networks with a large number of financial intermediaries who do not have obligations (bonds and deposits) towards the households in H. In this case, the second algorithm is potentially more efficient than the first one.

7 Contagion in different network structures

Different networks propagate losses in different fashions. The effects of a shock on a network N depend on the two elements that form its *structure*: a) the shape of the network, i.e. the pattern formed by the links in L^{Ω}, and b) the values of the assets and liabilities of the agents in the network, i.e., the capacities of the links in L^{Ω}. We study the effects of these two determinants of a propagation separately, beginning with the former.

In this section we focus our attention to the effects of external shocks that cause the default of some nodes in Ω while leaving the value of the external assets of the other nodes unaffected. The purpose of this restriction is to isolate the contagion caused by the domino effect from the contagion caused by common exposures to exogenous shocks. Let $\sigma = [\widehat{b}^k]$ be a shock vector and let D be the set of agents in Ω who default as a consequence of this shock, $D = \left\{\omega_i \in \Omega \mid \sum_k \widehat{b}^k a_i^k + \sum_j b_j d_{ji} \geq e_i\right\}$. Let D' be the set of *primary* defaults, i.e. the set of agents that suffer a loss of value of their external assets (the initial shock) large enough to cause their default , i.e. $D' = \left\{\omega_i \in \Omega \mid \sum_k \widehat{b}^k a_i^k \geq e_i\right\}$. We assume that $\sum_k \widehat{b}^k a_i^k = 0$ for all nodes $\omega_i \in \Omega \setminus D'$. Let $D'' = D \setminus D'$ be the set of *secondary* defaults, i.e. the set of agents who would be solvent if they had not received losses from their debtors in Ω. There is no default contagion if the set D'' is empty.

7.1 Thresholds of default contagion

To evaluate and compare the contagiousness of differently shaped networks, we look at two characteristics of a network: the first and the final thresholds of contagion.

Definition 6 *The* ***first threshold*** *of contagion of a network* N, $\tau_1(N)$, *is the magnitude of the smallest shock that is large enough to cause secondary defaults. Correspondingly, the* ***final threshold*** *of contagion of a network,* $\tau_2(N)$, *is the value of the smallest shock that is capable of inducing the failure of all nodes in the network.*

7.1.1 Complete networks

A network where each agent lends to every other agent in the network (and, therefore, everybody borrows from everybody else) is said to be *complete*. Let a complete financial network $N^c = \{\Omega, A, T, H, L^\Omega, L^A, L^T, L^H, \Gamma\}$ be such that its set of links L^Ω is maximal, i.e., $L^\Omega = \{l_{ij} | i \neq j; i, j = 1, 2, ..., n\}$. Moreover we assume that: i) all agents in N^c are equal to one another; ii) all links in L^Ω have the same weight, i.e., the debt of each node towards any other node in $N^c$is equal to d_{ij} for all $l_{ij} \in L^\Omega$.

Theorem 5 *In a complete network* N^c *the first threshold of contagion* τ_1^c *and the final threshold of contagion* τ_2^c *coincide and are equal to*

$$\tau^c = ne_i + e_i \frac{h_i}{d_{ij}}. \tag{3}$$

This result shows that the complete network, on one hand, is entirely resilient to relatively small shocks, i.e. faces no defaults for shocks smaller than τ^c. On the other hand, for large enough shocks – larger than or equal to τ^c – this network induces a complete system melt down. The same applies to the star-shaped network, if the central node is in the set of primary defaults, as shown below.

7.1.2 Star-shaped networks

A star-shaped network is composed by a central node, ω_c, that borrows from and lends to each of the peripheral nodes ω_p, $p = 1, 2, ..., n-1$,which, in turn, have no financial obligations among themselves. Let a star-shaped financial network $N^s = \{\Omega, A, T, H, L^\Omega, L^A, L^T, L^H, \Gamma\}$

be such that $L^{\Omega} = \{l_{pc}, l_{cp} | p = 1, 2, ..., n-1\}$ .We assume that all links in L^{Ω} have the same weight, i.e., $d_{cp} = d_{pc} = d_p$, for all links in L^{Ω}.

In a star-shaped network, the contagion thresholds depend on the distribution of the initial shock between the center and the periphery of the network. We obtain results for the three possible cases: 1) the shock is idiosyncratic and borne by the central node alone: $D' = \omega_c$; 2) the shock is borne by ω_c and by some peripheral nodes, and 3) the shock is borne by peripheral nodes only:

Theorem 6 *The first threshold of contagion, τ_1^s, and the last threshold of contagion, τ_2^s, of a star-shaped network N^s are the following:*

1. *(a) if $D' = \omega_c$ and (b) if $D' = \{\omega_c, \omega_p | \text{for some } p \in (1, ..., n-1)\}$, then the first and the last threshold coincide and are equal to*

$$\tau^s = (n-1)e_p + e_c + e_p \frac{h_c}{d_p};$$

2. *if $D' = \{\omega_p | \text{for some } p \in (1, ..., n-1)\}$ and $\omega_c \notin D'$, then the first threshold is equal to*

$$\tau_1^s = me_p + e_c \left(1 + \frac{h_p}{d_p}\right)$$

where m is the minimum number of peripheral defaults which is sufficient to induce the default of the central node, i.e. m is such that $\sum_{p=1}^{m} d_p = e_c$, while the final threshold is equal to

$$\tau_2^s = \left[(n-1)e_p + e_c + e_p \frac{h_c}{d_p}\right] \left(1 + \frac{h_p}{d_p}\right).$$

7.1.3 Incomplete regular networks

A generic incomplete network is a network such that the set of links L^{Ω} is not maximal. Nothing can be established, with the present analytic and non-stochastic methodology, about the contagion thresholds of incomplete networks, unless some restrictions are imposed on their structure – as it is done above with the star-shaped networks which are, obviously, incomplete networks themselves. Often, in the literature, incomplete networks are assumed to be 'regular', in the sense that each node in the network has the same indegree and outdegree, i.e., the same number of incoming and outgoing links. Let an incomplete regular network of degree r be $N^r = \left\{\Omega, A, T, H, L^\Omega, L^A, L^T, L^H, \Gamma\right\}$, let $|in_i|$ and $|out_i|$ be, respectively, the number of incoming and outgoing links of node ω_i, then: $L^\Omega = \{l_{ij} || in_i| = |out_i| = r, \forall i = \Omega\}$. As above, we assume that all links in L^Ω have the same weight d_r, and all nodes have the same balance sheet values, $a_r + rd_r = e_r + h_r + rd_r$. Under this restrictions, it is straightforward to characterise the first threshold of an idiosyncratic shock:

Theorem 7 *The first threshold of contagion, τ^r, of an idiosyncratic shock in an incomplete regular network of degree r, N^r, is equal to:*

$$\tau_1^r = (r+1)e_r + e_r \frac{h_r}{d_r}$$

We failed to identify the first and final thresholds induced by a generic shock on a network N^r, as well as the final threshold of an idiosyncratic shock. The rationale of this impasse lies in the fact that the unfolding of a propagation in an incomplete network, beyond the first line of defaults induced by a generic shock, remains ambiguous unless strong restrictions are imposed (as we do below by setting $r = 1$). Indeed, shocks of equal magnitude have different effects on a network N^r, depending on i) the position of the nodes in D', and ii) the distribution of external losses across such nodes.[13]

7.1.4 Cycle-shaped networks

An incomplete regular network with degree equal to unity forms a cycle. Formally, a cycle-shaped financial network $N^o = \left\{\Omega, A, T, H, L^\Omega, L^A, L^T, L^H, \Gamma\right\}$ is such that $L^\Omega = \{l_{ij} | i = 1, 2, ..., n;\ j = i + 1 \text{ for } i = 1, ..., n-1, \text{ and } j = 1 \text{ for } i = n\}$. As above, assume that all links in L^Ω have the same weight, i.e., $d_{ij} = d_i$, and all nodes have the same balance sheet, $a_i + c_i = e_i + h_i + d_i$. In this network, the effects of an external shock that involves more than one agent, $|D'| > 1$, crucially depend on the position that such defaulting nodes have on the cycle network. In order not to resort to implausible restrictions on this issue, we content ourselves with the analysis of the impact of idiosyncratic shocks.

[13] For instance, it can be shown that, for any incomplete regular network, it is possible to find a set of initially defaulting agents, D', and a distribution of losses among them, such that the first and the final thresholds are the same as the ones of a complete network N^c. This is the case for a cycle-shaped network, as defined below, hit by a shock such that: i) D' is composed by half of the nodes in the network; ii) all nodes in D' suffer an external loss of equal amount; iii) each node in D' is adjacent to two nodes in $\Omega \backslash D'$.

Theorem 8 *The first threshold of contagion, τ_1^o, and the last threshold of contagion, τ_2^o, of an idiosyncratic shock in a cycle-shaped network N^s are the following:*

1.

$$\tau_1^o = 2e_i + e_i \frac{h_i}{d_i};$$

2.

$$\tau_2^o = 2e_i + e_i \frac{h_i}{d_i} + e_i \left(1 + \frac{d_i}{h_i}\right) \left[\left(1 + \frac{h_i}{d_i}\right)^{n-2} - 1 \right].$$

7.1.5 Comparing the contagion thresholds of different network structures

For the sake of comparability, we set the four types of networks considered here to be composed by the same number of agents, n, and to be endowed with the same total stock of equity, $E = \sum_{i \in \Omega} e_i$, and with the same total external debt, $H = \sum_{i \in \Omega} h_i$. In order to isolate the effects that the shape of a network has on its contagion thresholds from the effects that the balance sheet ratios e_i/h_i and h_i/d_i have on such thresholds, we set these ratios to be the same for all agents in all networks.[14] The effects of this latter restriction are discussed below.

Under these conditions we have the following

Corollary 2 *1. The contagion thresholds of the star-shaped and of the complete networks are such that:*

$$\tau_1^s < \tau^s < \tau^c < \tau_2^s.$$

2. The first contagion thresholds of the cycle-shaped network, τ_1^o, and of the incomplete regular network, τ_1^r, are such that:

$$\tau_1^o < \tau_1^r < \tau^c.$$

3. The last contagion thresholds of the cycle-shaped network and of the incomplete regular network are both larger than the threshold of the complete network τ^c.

[14] This restriction implies that i) the banks in the complete, incomplete and cycle-shaped networks all have the same balance sheet $a_i + c_i = e_i + h_i + d_i$, and ii) in the star-shaped network, the balance sheet of the peripheral nodes is equal to $a_p + c_p = e_p + h_p + d_p$ while the balance sheet of the central nodes is equal to $a_c + c_p(n-1) = e_c + h_c + d_p(n-1)$, where $e_c = (n-1)e_p$ and $h_c = (n-1)h_p$.

7.2 Value of balance sheets headings and contagion thresholds

In the network structures analysed above, the stock of equity e and the h/d ratio between internal and external debt are the only headings of the balance sheets of the agents in a network that determine its contagion thresholds. Moreover, all the above characterised thresholds are increasing in the equity endowments, e, and in the h/d ratio. The protective role played by the equity stock is not surprising: the larger the equity of the members of a network, the larger the amount of losses that can be absorbed by those agents, the higher the contagion thresholds of the network (and, of course, the smaller the set of defaults induced by any given shock).

The relevance of the h/d ratio, in turn, lies in the fact that this ratio governs the allocation of the flow of losses, released by defaulting nodes, between external creditors (households) and internal ones (other nodes in Ω). The smaller this ratio between external and internal debt, the smaller the portion of losses that, at each default, is sent into the sink H, and the larger the flow of losses that continues to circulate among the nodes in Ω, and vice versa. Therefore, the smaller the h/d ratio: i) the larger the portion of an external shock that overflows from the primary defaults towards the rest of the network; ii) the smaller the smallest shocks capable of causing secondary defaults (the contagion thresholds), and iii) the larger the number of defaults induced by a shock. This is true, under a mildly restrictive condition, for all financial flow networks. In order to establish this result, we restrict the attention to networks where each agent holds an amount of internal exposures, c_i, equal to its internal debt, d_i, and we vary the h/d ratio by varying proportionally the value of all intra-network obligations – the weights of the links in Ω – while keeping constant the value of the other balance sheet headings. The scope of these restrictions is discussed below.

Definition 7 *Let N be a financial flow network such that $c_i = d_i$, for all $i \in \Omega$, and let $\{N^\epsilon\}$ be the set of financial flow networks, indexed by $\epsilon \in \mathbb{R}^+$, such that: i) all networks in $\{N^\epsilon\}$ are equal to N in everything but the weights of the links in L^Ω;ii) for all d_{ij} in N, the corresponding d_{ij}^ε in N^ϵ is equal to $d_{ij}^\varepsilon = (1+\epsilon)d_{ij}$.*

Within this class of networks, and for any given shock, the flow of losses that a defaulting agent passes to his creditors in Ω is increasing in ϵ, i.e. it grows as the h/d ratio diminishes.[15]

[15] As can be checked by inspecting the argument of the proof of the lemma below, the losses that defaulting nodes pass to other nodes in Ω do not grow if the h/d ratio remains constant.

Lemma 4 *Let σ be an external shock to the networks in $\{N^\epsilon\}$ and let D' be the set of primary defaults induced by σ. Let $L^{D'} = \{l_{ij} | i \in D', j \in \Omega\}$ be the set of the outgoing links of the nodes in D'. Then the flow of losses that crosses a link $l_{ij} \in L^{D'}$ in a network N^ϵ, as a consequence of a shock, is increasing in ϵ.*

This result stems from the pro-rata allocation of losses among the debtors of defaulting agents: the relative growth of the internal debts with respect to the external debts of defaulting agents, transfers part of the losses from their external to their internal debtors. As a consequence, the amount of losses that the primary defaults send to the other nodes in Ω (the 'contagious' flow), as well as the amount of losses that circulate among the nodes in D', grow with ϵ, while the flow of losses sent to the sink H correspondingly diminishes with ε. This means that the larger ϵ, the larger the flow carried by each link across the cut $(D', \Omega \backslash D')$. Clearly this implies that the losses received by each creditor of the defaulting nodes in D', for any given shock, grow as ϵ increases. Thus, as ϵ grows, progressively smaller shocks are sufficient to induce contagion. For the same reason, the larger ϵ, the larger the number of defaults induced by a given shock. In sum, a proportional growth of the value of the intra-network obligations d_{ij}'s, while the extra-network obligations h_i's remain fixed, renders a network increasingly exposed to default contagion, both in terms of thresholds and of scope of contagion:

Theorem 9 *1. Let $\{\tau'_\epsilon\}$ and $\{\tau''_\epsilon\}$ be, respectively, the sets of first and of final contagion thresholds of the networks in $\{N^\epsilon\}$ and let such sets be indexed by $\epsilon \in \mathbb{R}^+$. Then τ'_ϵ and τ''_ϵ are decreasing in ϵ.*

2. Let σ be an external shock to the set of networks $\{N^\epsilon\}$ and let $\{D^\epsilon\}$ be the set, indexed by $\epsilon \in \mathbb{R}^+$, composed of the sets of defaults induced by σ in the networks in $\{N^\epsilon\}$. Then the number of defaults, i.e. the cardinality of D^ϵ, is increasing in ϵ.

These results are obtained under two restrictions: $c_i = d_i$, for all $i \in \Omega$, and the proportionality of the above considered changes in the value of intra-network obligations. The former restriction is merely a convenient way to ensure that the change in the value of the intra-network obligations is compatible with the balance sheet constraints. This restriction can be replaced without altering the above results.[16] Conversely, the latter

[16] For instance, it could be replaced by assuming that, as ϵ grows, the nodes s. t. $c_i > d_i$ sell external assets to the nodes s. t. $c_i < d_i$ preserving, in so doing, the equality between assets and liabilities in the balance sheets of the agents in Ω.

restriction – that keeps the proportions among the intra-network obligations fixed while varying their values – is a necessary condition for lemma 21 because a non proportional change of such obligations may shift losses from poorly capitalized agents towards highly capitalized ones (or, with similar effects, from defaulting nodes with a low h/d ratio to nodes with a high h/d ratio).[17]

8 Conclusions

In this paper we represent a financial network as a flow network and model the diffusion of losses and defaults, originated by an exogenous shock, as a flow that crosses such a network. Using some properties of network flows, we obtain three results. First, we address a know problem of indeterminacy that arises, in payment systems, from the possible existence of strongly connected components of defaulting agents. We establish necessary and sufficient conditions for the uniqueness of clearing intercyclical payments and use these conditions in an algorithm that, while computing the contagion process, controls for the occurrence of possible indeterminacies. Second, we investigate the relation between the shape of a financial networks and its exposure to default contagion. We characterise first and final contagion thresholds (i.e., the value of the smallest shock capable of inducing default contagion and the value of the smallest shock capable of inducing the default of all agents in the network, respectively) for different network shapes, namely the complete, star-shaped, incomplete regular, and cycle-shaped networks. We find that: i) First and final thresholds coincide in complete and in star-shaped networks (when the center is in the set of primary defaults) because, in both cases, the losses that overflow from the primary defaults are evenly spread among all remaining nodes; and ii) the first (final) thresholds of incomplete regular and cycle-shaped networks are both smaller (larger) than the ones of complete networks. This implies that the class of incomplete regular networks (which includes the cycle-shaped ones), compared to the classes of complete and star-shaped networks, is more exposed to episodes of contagion due to shocks of small magnitude and scope, and less

[17] This is best explained by an example. Consider a network with just three nodes, ω_1, ω_2 and ω_3, where ω_1 is indebted with both ω_2 and ω_3, and $e_2 > e_3$. Suppose that ω_1 defaults. In this case, an increase of the value of d_{12}, while d_{13} is kept constant, implies a shift of losses from ω_3 towards ω_2 and, therefore, implies an increase of the first (and last) contagion threshold of this simple network as well as, for some shock values, a decrease of the number of defaults. Thus, a non-proportional increase of intra-network obligations does not necessarily imply a larger exposition to default contagion.

exposed to the risk of complete system melt-downs. Third, we find that the ratio between the external debt of the agents in a network (i.e. the debt towards claimants who do not belong to the network, such as households) and their internal debt (i.e. the debt towards other agents in the network) determines the exposition to contagion of the network. Ceteris paribus, the larger the ratio between the intra-network exposures and the external debt of the agents in a network, the more the network is exposed to default contagion, both in terms of scope and of thresholds of contagion.

Appendix: proofs of theorems, lemmae and corollaries

Proof of theorem 1. 1: The capacity constraint is satisfied because i) $f(l_i^k) = b^k\Gamma(l_i^k)$ for all l_i^k in L^A, $f(l_{ij}) = b_i\Gamma(l_{ij})$ for all l_{ij} in L^Ω, $f(l_H^i) = b_i\Gamma(l_H^i)$ for all l_H^i in L^H, and $f(l_T^i) = \beta_i\Gamma(l_T^i)$ for all l_{it} in L^T; and ii) $b^k, b_i, \beta_i \in [0,1]$, for all $i \in \Omega$ and all $k \in A$. 2: The budget identity of the balance sheets of the agents in Ω, together with the rules of limited liability and debt priority $-$ encoded in (1) and (2) $-$ ensure that any flow of losses that arrives in a node is redirected first towards the sink and, for the residual part, towards the node's descendants in Ω. In notation: $\sum_{X^-(\omega_i)}\varphi(x) = \lambda_i = \beta_i(\lambda_i)e_i + b_i(\lambda_i)d_i = \sum_{X^+(\omega_i)}\varphi(x)$, for all $\omega_i \in \Omega$. ■

Proof of lemma 1. Let $\{U_i, \overline{U}_i\}$ be a cut in N and let $\{U_{i-1}, \overline{U}_{i-1}\}$ be another cut in N such that $U_{i-1} = U_i \backslash \omega_i$; $\omega_i \in U_i$. The set of forward links of U_i is $L^+(U_i) = L^+(U_{i-1}) + \{l_{ij} \in L^\Omega \mid \omega_i \in U_i, \omega_j \in \overline{U}_i\} + l_{it} - \{l_i^k \in L^A \mid a^k \in A\} - \{l_{ji} \in L^\Omega \mid \omega_j \in U_{i-1}\}$, while the set of backward links of U_i is $L^-(U_i) = L^-(U_{i-1}) + \{l_{ji} \in L^\Omega \mid \omega_j \in \overline{U}_i\} - \{l_{ij} \in L^\Omega \mid \omega_j \in U_{i-1}\}$. Thus we can express the capacity of $\{U_i, \overline{U}_i\}$ as

$$\begin{aligned}
\Gamma\{U_i, \overline{U}_i\} &= \Gamma\{U_{i-1}, \overline{U}_{i-1}\} + \Gamma\{l_{ij} \in L^\Omega \mid \omega_j \in \overline{U}_i\} + \Gamma(l_{it}) \\
&\quad -\Gamma\{l_i^k \in L^A \mid a^k \in A\} - \Gamma\{l_{ji} \in L^\Omega \mid \omega_j \in U_{i-1}\} \\
&\quad -\Gamma\{l_{ji} \in L^\Omega \mid \omega_j \in \overline{U}_i\} + \Gamma\{l_{ij} \in L^\Omega \mid \omega_j \in U_{i-1}\}
\end{aligned}$$

Since: i) $\Gamma\{l_{ij} \in L^\Omega \mid \omega_j \in \overline{U}_i\} + \Gamma\{l_{ij} \in L^\Omega \mid \omega_j \in U_{i-1}\} = d_i$; ii) $\Gamma\{l_{ji} \in L^\Omega \mid \omega_j \in U_{i-1}\} + \Gamma\{l_{ji} \in L^\Omega \mid \omega_j \in \overline{U}_i\} = c_i$; iii) $\Gamma\{l_i^k \in L^A \mid a^k \in A\} = a_i$ and iv) $\Gamma(l_{it}) = e_i + h_i$, by the budget identity $a_i + c_i \equiv e_i + d_i + h_i$ we obtain that $\Gamma\{U_i, \overline{U}_i\} = \Gamma\{U_{i-1}, \overline{U}_{i-1}\}$. This procedure can be iterated for all pairs of cuts $\{U_i, \overline{U}_i\}$ $\{U_{i-1}, \overline{U}_{i-1}\}$ in N, starting from $\{U_{i-1}, \overline{U}_{i-1}\} = \{A, (\Omega, H, T)\}$. ■

Proof of lemma 2. Let $P^1(\omega_i) = \left\{\omega_j | l_{ji} \in L^\Omega\right\}$ be the set of parent nodes of ω_i, let $P^2(\omega_i)$ be the set of the parent nodes of the parent nodes of ω_i and so forth for $P^3(\omega_i)$, $P^4(\omega_i), ..., P^n(\omega_i)$.The union of such sets, $P(\omega_i) = \cup_{j=1}^n P^j(\omega_i)$, forms the set of the *ancestors* of ω_i, i.e. the set of nodes ω_j in Ω s.t. there exists a directed path from ω_j to ω_i. For each $i \in \Omega$, β_i and b_i are both uniquely defined functions of λ_i which, in turn, is a uniquely defined function of the insolvency functions of the defaulting nodes in $P^1(\omega_i)$ which, in turn, are uniquely defined functions of the losses suffered by the nodes in $P^1(\omega_i)$ which, in turn, are uniquely defined functions of the values taken on by the insolvency functions of the defaulting nodes in $P^2(\omega_i)$, and so forth up to the source nodes of N. In sum, λ_i, β_i and b_i are functions of the values taken on by the insolvency functions of the defaulting nodes in $P(\omega_i)$. In absence of cycles of defaulting agents, we have that no node in Ω,belongs to the set of its own ancestors. This implies that λ_i, β_i and b_i are obtained through the non-recursive iteration of uniquely defined functions, thus they are uniquely defined as well. ■

Proof of lemma 3. To show that conditions (a) and (b) are individually necessary and jointly sufficient to generate the indeterminacy of a propagation in a SCC, we first assume that all nodes in S default and discuss the implications of $\mathbf{S}$ being closed or open. Then we assume that $\mathbf{S}$ is closed and show that all nodes in S must default for the indeterminacy to arise.

1) Let us assume that all nodes in S default. Each node ω_i in S receives losses equal to $\lambda_i = \sum_{k \in A} b_k a_i^k + \sum_{j \notin S} b_j d_{ji} + \sum_{j \in S} b_j d_{ji}$, where $\sum_{k \in A} b_k a_i^k$ is the flow of losses (if any) that ω_i receives directly from the source nodes; $\sum_{j \notin S} b_j d_{ji}$ is the flow of losses that it receives through its defaulting parent nodes that do not belong to S (if any); and $\sum_{j \in S} b_j d_{ji}$ is the flow of losses that ω_i receives from its defaulting parent nodes that belong to S. For the time being, we resort to a simplifying assumption. We assume that f does not induce any cycle flow along the directed paths that connect the source nodes in A to the members of S, i.e., there is no SCC of defaulting nodes laying between the source nodes and S. By Lemma 8, this assumption implies that the flow of losses received by the nodes in S from nodes in $\Omega \backslash S$ is uniquely defined. On this basis we take, for all ω_i in S, $\sum_{j \notin S} b_j d_{ji}$ as a datum and express $b_i(\lambda_i)$ as a linear function of the insolvency functions, the b_j's, of the other members of S :

$$b_i = \frac{\sum_{k \in A} b_k a_i^k + \sum_{j \notin S} b_j d_{ji} - e_i}{d_i} + \frac{\sum_{j \in S} b_j d_{ji}}{d_i}$$

rewritten as

$$b_i d_i - \sum_{j \in S} b_j d_{ji} = \sum_{k \in A} b_k a_i^k + \sum_{j \notin S} b_j d_{ji} - e_i$$

For a SCC composed of m nodes, we have a system composed of m of such linear equations:

$$\begin{bmatrix} d_1 & -d_{21} & \cdots & \cdots & \cdots & -d_{m1} \\ -d_{12} & d_2 & -d_{32} & \cdots & \cdots & \vdots \\ \vdots & -d_{23} & d_3 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ -d_{1(m-1)} & \vdots & \vdots & \vdots & \ddots & -d_{m(m-1)} \\ -d_{1m} & \cdots & \cdots & \cdots & -d_{(m-1)m} & d_m \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{bmatrix} = \begin{bmatrix} \sum_{k \in A} b_k a_1^k + \sum_{j \notin S} b_j d_{j1} - e_1 \\ \sum_{k \in A} b_k a_2^k + \sum_{j \notin S} b_j d_{j2} - e_2 \\ . \\ . \\ . \\ \sum_{k \in A} b_k a_m^k + \sum_{j \notin S} b_j d_{jm} - e_m \end{bmatrix}$$

The solution of this system – i.e., the vector of unknowns $[b_1, b_2, ..., b_m]$, hence the flow values assigned by f to the links in L^S – is indeterminate if and only if the matrix of the coefficients of the system is singular.[18] The components of such a matrix have the following properties:

(1) $d_i \geq \sum_{j \in S} d_{ij}$, for every $i \in \{1, 2, ..., m\}$;

(2) for every $i, j \in \{1, 2, ..., m\}$, if $i \neq j$, then there exists a sequence of indexes $i_1, i_2, ..., i_k$, where $i = i_1$ and $j = i_k$, such that $d_{i_1 i_2} \cdot d_{i_2 i_3} \cdot \cdots \cdot d_{i_{k-1} i_k} \neq 0$.

Property (1) holds with the equality sign for the nodes in S that are indebted only to other nodes in S. Thus, in a closed SCC we have that $d_i = \sum_{j \in S} d_{ij}$, for every $i \in \{1, 2, ..., m\}$. Conversely, in an open SCC we have that $d_i > \sum_{j \in S} d_{ij}$ for at least one $i \in \{1, 2, ..., m\}$. Property (2) is a formal expression of strong connectivity: for every ordered pair $(\omega_i, \omega_j) \in S$ there exists a directed path that starts in ω_i and ends in ω_j.

Now demonstrating the *if* part of the lemma is straightforward. If **S** is a closed SCC, then $d_i = \sum_{j \in S} d_{ij}$, for every $i \in \{1, 2, ..., m\}$. In this case the sum of the rows of the

[18] I am indebted to Paola Cellini for her generous help in characterizing the singularity conditions of such a matrix.

coefficient matrix is null and so is its determinant.

In order to establish the *only if* part of the Lemma, we suppose that the determinant of the matrix is null and show that, in such a case, $d_i = \sum_{j \in S} d_{ij}$ for every $i \in \{1, 2, ..., m\}$, which means that **S** is closed. We use the fact that the determinant of a matrix is null if and only if its rows are linearly dependent. Thus we suppose that there exist m real numbers $\gamma_1, \gamma_2, ..., \gamma_m$ not all null and such that, for every $i, j \in \{1, 2, ..., m\}$,

$$\gamma_i d_i = \sum_{j \neq i} \gamma_j d_{ij}.$$

We can suppose, re-ordering the indexes if necessary, that $|\gamma_m| \geq |\gamma_i|$ for all $i < m$, thus $\gamma_m \neq 0$, therefore

$$d_m = \sum_{j \neq m} \frac{\gamma_j}{\gamma_m} d_{mj}$$

where $\frac{\gamma_j}{\gamma_m} \leq \left| \frac{\gamma_j}{\gamma_m} \right| \leq 1$. Then condition (1) implies that $\gamma_m = \gamma_j$ for every j such that $d_{mj} \neq 0$, thus it implies that $d_m = \sum_{j \neq m} d_{mj}$.

Consider now the set

$$E = \left\{ j \in \{1, 2, ..., m\} \, | \gamma_j = \gamma_m \right\}$$

If E coincides with $\{1, 2, ..., m\}$, we obtain that condition (1) holds with the equality sign for all $i \in \{1, 2, ..., m\}$, which is our thesis. Seeking a contradiction, let us suppose that E does not coincide with $\{1, 2, ..., m\}$. Then we can suppose, re-ordering the indexes if necessary, that there exists an index $h > 1$ such that $E = \{h, ..., m\}$, i.e., such that $\gamma_i = \gamma_m$ if and only if $i \geq h$. The strong connectivity of **S**, as expressed by condition (2), implies that, for at least one index $i \in \{h, ..., m\}$, there exists an index $k < h$ such that $d_{ik} > 0$. Let then be $d_{ik} > 0$—with $i \in \{h, ..., m\}$ and $k < h$—and repeat the above reasoning with the index i in place of m : the relation

$$d_i = \sum_{j \neq i} \frac{\gamma_j}{\gamma_i} d_{ij}$$

together with condition (1) and with the fact that $|\gamma_i|$ is maximal, implies that $\gamma_i = \gamma_j$ for every j such that $d_{ij} \neq 0$ and, therefore, that $\gamma_i = \gamma_m = \gamma_k$, contradicting the hypothesis that $k < h$. This proves that, if the determinant of the above coefficient matrix is null, then all γ_i are equal among themselves and, therefore, $d_i = \sum_{j \neq i} d_{ij}$, for every $i, j \in \{1, 2, ..., m\}$. Finally, we can remove the above made simplifying assumption because i) by definition, no closed SCC can lie along the paths that go from A to S, and ii) the presence of an open SCC of defaulting agents along such paths does not cause any indeterminacy, as it is shown above.

2) Suppose that $\mathbf{S}$ is closed and that only part of the nodes in S default. Let $\widetilde{S} \subset S$ be the set of defaulting nodes in S and let $L(\widetilde{S})$ be the set of the links that connect such failing agents among themselves. Then the subgraph $(\widetilde{S}, L(\widetilde{S})) \subset \mathbf{S}$ may or may not be a SCC. If it is not, no indeterminacy arises, as implied by lemma 8. If $(\widetilde{S},L(\widetilde{S}))$ is strongly connected, such a SCC of defaulting agents is open because $(\widetilde{S}, L(\widetilde{S})) \subset \mathbf{S}$ and $\mathbf{S}$ is strongly connected. Hence, also in this case the propagation is uniquely defined on the links in $L(\widetilde{S})$. ■

Proof of theorem 4. 1) Recall that $L = L^A \cup L^\Omega \cup L^H \cup L^T$. Then: i) the values taken on by f on the links in L^A are exogenously set by the external shock; ii) by definition, the nodes in a closed SCC $\overline{\mathbf{S}}$ have no debts towards the nodes in $\Omega \backslash \overline{\mathbf{S}}$ or the households in H. Thus, any possible indeterminacy of the values of the insolvency functions of the nodes in a $\overline{\mathbf{S}} \in \Theta$ has no effects on the flow of losses received by the nodes in $\Omega \backslash \overline{\mathbf{S}}$, which unfold along the links in $L^\Omega \backslash L(\overline{S})$, and by the households, which are defined on the links in L^H. Lemma 8 and 10 establish that a propagation that unfolds along simple paths and open SCC's of N is uniquely defined. Therefore the propagation is uniquely defined on all links in L^H and in $L^\Omega \backslash L^\Theta$; iii) The absorption functions $\beta_i(\lambda_i)$ of the nodes in a $\overline{\mathbf{S}} \in \Theta$, that set the values taken on by f on the links going from $\overline{S}$ to T, are i) uniquely defined if at least one node in $\overline{\mathbf{S}}$ is solvent, and ii) all equal to unity, if all nodes in $\overline{\mathbf{S}}$ default. Thus, a propagation is uniquely defined on the links in L^T.

2) It follows from lemma 10. ■

Proof of corollary 1. The total loss received by the nodes in $\overline{S}$ from the rest of the network is: $\sum_{k \in A} \sum_{i \in \overline{S}} b_k a_i^k + \sum_{i \in \overline{S}} \sum_{j \notin \overline{S}} b_j d_{ji}$. Aggregating the balance sheets of the members of $\overline{S}$ we obtain: $\sum_{k \in A} \sum_{i \in \overline{S}} a_i^k + \sum_{i \in \overline{S}} \sum_{j \notin \overline{S}} d_{ji} = \sum_{i \in \overline{S}} e$, the debts cross-held by the members of $\overline{S}$ net out and the exposures are entirely backed by the total equity in $\overline{S}$. Thus, if the sum of the losses born by the agents in $\overline{S}$ is maximal – i.e., if $\sum_{k \in A} \sum_{i \in S} b_k a_i^k + \sum_{i \in S} \sum_{j \notin S} b_j d_{ji} = \sum_{k \in A} \sum_{i \in S} a_i^k + \sum_{i \in S} \sum_{j \notin S} d_{ji}$ – then all agents in $\overline{S}$ fail. Conversely, if the sum of the losses born by the agents in $\overline{S}$ is less than maximal – i.e., if $\sum_{k \in A} \sum_{i \in S} b_k a_i^k + \sum_{i \in S} \sum_{j \notin S} b_j d_{ji} < \sum_{k \in A} \sum_{i \in S} a_i^k + \sum_{i \in S} \sum_{j \notin S} d_{ji}$ – then at least one of such agents has a strictly positive residual equity, i.e., it does not default. ■

In demonstrating theorem 5, 6 and 8 below, we resort to a known property of network flows: for a flow defined in a flow network, the value of the net forward flow that crosses a cut is the same for all the cuts of the network. Applying this property to a financial flow network N, we obtain a convenient feature of a propagation f in N: the value of the net forward flow that crosses all cuts of N equals the value of the exogenous shock, i.e., it is equal to the flow that crosses the cut $\{A, (\Omega, H, T)\}$.

As above, let $\{U, \overline{U}\}$ be a cut of a financial flow network N, $L^+(U)$ be the set of forward links going from U into $\overline{U}$, and $L^-(U)$ be the set of backward links going from $\overline{U}$ into U. Let $f[L^+(U)] = \sum_{l \in L^+(U)} f(l)$ be the *forward* flow that crosses $\{U, \overline{U}\}$ going from nodes in U into nodes in $\overline{U}$, and let $f[L^-(U)] = \sum_{l \in L^-(U)} f(l)$ be the *backward* flow that crosses the cut at hand in the opposite direction. The *net flow* that crosses $\{U, \overline{U}\}$ is equal to $\overrightarrow{f}\{U, \overline{U}\} = f[L^+(U)] - f[L^-(U)]$.

Lemma 5 *Let f be a propagation in a network N and let $\{U, \overline{U}\}$ be a cut of N. The net flow across a cut $\{U, \overline{U}\}$ in N is equal to the flow out of the source nodes: $\overrightarrow{f}\{U, \overline{U}\} = \overrightarrow{f}\{A, (\Omega, H, T)\} = \sum_A b^k a^k$.*

Proof. See Diestel (2000), page 126, for a proof that refers to generic flow networks. ■

Proof of theorem 5. Lemma 5 ensures that the value of an exogenous shock, i.e., the flow that crosses the cut $\{A, (\Omega, H, T)\}$, is equal to the forward flow that crosses the cut $\{(A, D'), (\Omega \backslash D', T, H)\}$, which is also the net flow across this cut, since no flow crosses it in the opposite direction. Let m be the number of primary defaults caused by a shock $[\widehat{b^k}]$, $m = |D'|$. Each of node ω_i in D' sends 1) to the sink T a flow equal to its own equity e, 2) to the sink H a flow equal to $b_i h$, and 3) a flow equal to $b_i d$ to each of its $(n-m)$ creditors in $\Omega \backslash D'$. The shock that comes out of the source nodes, $\overrightarrow{f}(A, (\Omega, H, T))$, is then equal to

$$me + \sum_{i=1}^{m} b_i h + \sum_{i=1}^{m} b_i d(n-m) \tag{4}$$

where 1) me is the value of the flow $\overrightarrow{f}(D', T)$ going from D' to T, 2) $\sum_{i=1}^{m} b_i h$ is the flow $\overrightarrow{f}(D', H)$ that goes from D' to H, and 3) $\sum_{i=1}^{m} b_i d(n-m)$ is the flow $\overrightarrow{f}(D', \Omega \backslash D')$ going from D' to $\Omega \backslash D'$. In a complete network N^c, each node in $\Omega \backslash D'$ receives, from its defaulting debtors, a flow of losses equal to $\sum_{i=1}^{m} b_i d$. For default contagion to occur, this flow of losses must be larger than or equal to the absorbing capacity of a node: $\sum_{i=1}^{m} b_i d \geq e$. The value of an exogenous shock that is exactly large enough to cause such a condition to be fulfilled, i.e., such that $\sum_{i=1}^{m} b_i d = e$, constitutes both the first and the final threshold of contagion of a network N^c : all nodes in $\Omega \backslash D'$ default together if such a threshold is reached. This condition for contagion requires that $\sum_{i=1}^{m} b_i = e/d$ and, substituting this value in (4), we obtain the first and final contagion thresholds of a network N^c. ■

Proof of theorem 6. As above, lemma 5 ensures that the value of a shock, $\overrightarrow{f}\{A, (\Omega, H, T)\}$, is equal to the forward flow that crosses the cut $\{(A, D'), (\Omega \backslash D', T, H\}$. Then, with respect to the two cases listed in the theorem, we have that:

1a) if $D' = \omega_c$, the flow that crosses the cut $\{(A, \omega_c), (\Omega \backslash \omega_c, H, T)\}$ is equal to

$$e_c + b_c h_c + b_c d_p (n-1),$$

where $b_c(\lambda_c)$ is the insolvency function of ω_c. Contagion occurs for any shock such that $b_c d_p (n-1) \geq e_p (n-1)$. The smallest of such shocks is the one that causes $b_c d_p (n-1) = e(n-1)$, hence $b_c = e_p/d$.This condition characterises both the first and the final threshold of contagion: if $b_c = e_p/d_p$, all agents in N^s default. Substituting $b_c = e_p/d_p$ into the above equation delivers the result.

1b) if $D' = \{\omega_c, \omega_p | \text{for some } p \in (1, ..., n-1)\}$, the flow that crosses the cut $\{(A, D'), (\Omega \backslash D', H, T)\}$ is equal to

$$(m-1)e_p + e_c + \sum_{p \in D' \backslash \omega_c} b_p h_p + b_c h_c + b_c d_p (n-m),$$

where: $m = |D'|$, $(m-1)e_p + e_c = \overrightarrow{f}(D', T)$, $\sum_{p \in D' \backslash \omega_c} b_p h_p + b_c h_c = \overrightarrow{f}(D', H)$ and $b_c d_p (n-m) = \overrightarrow{f}(D', \Omega \backslash D')$. As above, both first and complete contagion occur for any shock s.t. $b_c \geq e_p/d_p$. The smallest of such shocks are the ones s.t. $b_c = e_p/d_p$ and $b_p = 0$. When both these conditions are met, the value of shock is equal to the total stock of equity $(n-1)e_p + e_c$ plus the losses send by the central node into the sink H, equal to $b_c h_c$. Conversely, any shock s.t. $b_p > 0$ sends an amount of losses into the sink H equal to $b_c h_c + (m-1) b_p h_p$ wich is (obviously) larger than $b_c h_c$.

In other words, a shock s.t. $b_c = e_p/d_p$ and $b_p = 0$ is sufficiently large to cause the default of all nodes in the network, filling up the sink node T, while inflicting on bondholders and depositors in H the minimum possible amount of losses. Thus, the smallest shock that can cause the default of all nodes – if some peripheral nodes are in D' along with the central node – is equal to

$$\begin{aligned}\tau_1^s &= \tau_2^s = (m-1)e_p + e_c + \frac{e_p}{d_p}h_c + e_p(n-m) \\ &= (n-1)e_p + e_c + \frac{e_p}{d_p}h_c. \qquad (5)\end{aligned}$$

2) if $D' = \{\omega_p | \text{for some } p \in (1, ..., n-1)\}$ and $\omega_c \notin D'$, the flow that crosses the cut $\{(A, D'), (\Omega \backslash D', H, T)\}$ is equal to

$$me_p + \sum_{p=1}^{m} b_p h_p + \sum_{p=1}^{m} b_p d_p$$

where me_p and $\sum_{p=1}^{m} b_p h_p$ are the flows that D' sends into T and H, respectively, and $\sum_{p=1}^{m} b_p d_p$ is the flow that the central node ω_c receives from the defaulting nodes in D'. The condition for the first threshold of contagion is: $\sum_{p=1}^{m} b_p d_p = e_c$, hence $\sum_{p=1}^{m} b_p = e_c/d_p$ and, substituting this into the above equation, we obtain that $\tau_1^s = me_p + e_c(1 + h_p/d_p)$;

The second and final threshold of contagion, is set by the flow that crosses the cut $\{(A, D', \omega_c), (\Omega \backslash (D', \omega_c), H, T)\}$ which is equal to

$$me_p + e_c + \sum_{p=1}^{m} b_p h_p + b_c h_c + b_c d_p (n-m-1)$$

where: $me_p + e_c = \overrightarrow{f}((D', \omega_c), T)$, $\sum_{p=1}^{m} b_p h_p + b_c h_c = \overrightarrow{f}((D', \omega_c), H)$, and $b_c d_p(n-m-1) = \overrightarrow{f}(\omega_c, \Omega \backslash (D', \omega_c))$. All nodes in $\Omega \backslash (D', \omega_c)$ default if the central node sends to each of them a flow larger than or equal to e_p. The final threshold of contagion is equal to the smallest of such shocks: $b_c d_p = e_p$; hence: $b_c = e_p/d_p$ and

$$\tau_2^s = (n-1)e_p + e_c + e_p \frac{h_c}{d_p} + \sum_{p=1}^{m} b_p h_p \qquad (6)$$

As above, to obtain $\sum_{p=1}^{m} b_p$, we resort to the fact that the flow that enters the central node is equal to the flow that exits from it:

$$\begin{aligned}\sum_{p=1}^{m} b_p d_p &= e_c + b_c d_p (n-1) + b_c h_c \\ &= (n-1)e_p + e_c + e_p \frac{h_c}{d_p}.\end{aligned}$$

thus

$$\sum_{p=1}^{m} b_p = (n-1)\frac{e_p}{d_p} + \frac{e_c}{d_p} + e_p \frac{h_c}{(d_p)^2}.$$

Substituting this value in (6), we obtain the above result. ■

Proof of theorem 7. The proof is trivial and is omitted. ■

Proof of theorem 8. 1. The proof is trivial and is omitted.

2. Let N^o be composed by a cycle of nodes $(\omega_1, \omega_2, ..., \omega_n)$ and suppose that $N^o$is hit by an idiosyncratic shock borne by node ω_1. $((A, \Omega \backslash \omega_n), (\omega_n, H, T))$ is the cut of N^o corresponding to the final threshold of contagion of such a shock . The flow across this cut is equal to

$$(n-1)e_i + \sum_{i=1}^{n-2} b_i h_i + b_{n-1} h_i + b_{n-1} d_i.$$

The smallest shock sufficient to cause the failure of the n-th node, is such that: $b_{n-1} d_i = e_i$, hence: $b_{n-1} = e_i / d_i$. The final threshold at hand is then equal to

$$\tau_2^o = n e_i + \sum_{i=1}^{n-2} b_i h_i + e_i \frac{h_i}{d_i}. \tag{7}$$

To obtain $\sum_{i=1}^{n-2} b_i$, we use the flow conservation property: $b_{i-1} d_i = \lambda_i = e_i + b_i h_i + b_i d_i$, for all $i = 2, ..., n$, hence:

$$b_{i-1} = \frac{e_i}{d_i} + b_i (1 + \frac{h_i}{d_i})$$

and

$$\begin{aligned} b_i &= b_{n-1} \left(1 + \frac{h_i}{d_i}\right)^{n-1-i} + \frac{e_i}{d_i} \frac{(1 + h_i/d_i)^{n-1-i} - 1}{(1 + h_i/d_i) - 1} \\ &= \frac{e_i}{d_i} \left(1 + \frac{h_i}{d_i}\right)^{n-1-i} + \frac{e_i}{h_i} \left(1 + \frac{h_i}{d_i}\right)^{n-1-i} - \frac{e_i}{h_i} \end{aligned}$$

thus

$$\begin{aligned} \sum_{i=1}^{n-2} b_i &= \frac{(1 + h_i/d_i)^{n-2} - 1}{(1 + h_i/d_i) - 1} \left(\frac{e_i}{d_i} + \frac{e_i}{h_i}\right) - (n-2) \frac{e_i}{h_i} \\ &= \left[\left(1 + \frac{h_i}{d_i}\right)^{n-2} - 1\right] \left(\frac{e_i}{h_i} + \frac{e_i d_i}{h_i^2}\right) - (n-2) \frac{e_i}{h_i} \end{aligned}$$

and finally

$$\sum_{i=1}^{n-2} b_i h_i = \left[\left(1 + \frac{h_i}{d_i}\right)^{n-2} - 1\right] \left(e_i + \frac{e_i d_i}{h_i}\right) - (n-2) e_i$$

that, substituted in (7), yields the above result. ■

Proof of corollary2. Part (1) can be checked by inspection, considering that ■

Proof of lemma 4. The flow that crosses an outgoing link $l_{ij} \in L^{D'}$ in a network N^{ϵ} is equal to $(\lambda_i - e_i)\left[(1+\epsilon)d_{ij}/\left(h_i + (1+\epsilon)\,d_i\right)\right]$ and is increasing in ϵ for the following reason. As long as $h_i > 0$, the term within square brackets is strictly increasing in ϵ. In turn, $(\lambda_i - e_i)$ is non-decreasing in ϵ because, for a given shock, the losses received by a node $\omega_i \in D'$, λ_i, grow if there is at least one node ω_j in D' such that: i) $h_j > 0$, and ii) there exists a directed path from ω_j to ω_i where all nodes along such a path are insolvent as well. If either (i) or (ii) does not hold, then λ_i remains constant as ϵ changes. Thus, $(\lambda_i - e_i)\left[(1+\epsilon)d_{ij}/\left(h_i + (1+\epsilon)\,d_i\right)\right]$ is increasing in ϵ and such monotonic relation is strict for all links $l_{ij} \in L^{D}$ departing from nodes ω_i for which: (1) both (i) and (ii) hold, and\or (2) $h_i > 0$. ■

Proof of theorem 9. Both 1 and 2 are direct consequences of lemma 4, which implies that, for any given shock, the losses received by each creditor of the defaulting nodes are increasing (at least weakly) in ϵ. The formal proof is trivial and is omitted. ■

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