“MIRROR REVELATION” IN SECOND-PRICE TULLOCK AUCTIONS

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Abstract

A main distinguishing feature of the Vickrey sale auction is that bidding the own reserve price is a weakly dominant strategy. Therefore, although the auction can allow for multiple Nash Equilibria, the possibility of relying upon a weakly dominant strategy greatly simplifies bidders’ strategic decisions. Yet, because of the equilibrium multiplicity, the auctioneer can never be sure if players truly revealed their reserve prices. In the paper we introduce second price Tullock auctions, as a main example of lottery auctions, to see whether revelation of reserve prices could still occur in alternative second price mechanisms. With complete and private information our main finding is that in equilibrium, the auctioneer can now perfectly infer players’ reserve prices, however not because each bidder submits his maximum willingness to pay but rather because his reserve price is disclosed by his opponents. For this reason we named this “mirror revelation”. In such auctions, offering the own reserve price is not even a Nash Equilibrium.

1.Introduction

A main distinguishing feature of the second price Vickrey sale auction (Vickery, 1961; Milgrom 2004; Krishna, 2009) is that revealing the truth, bidding one's reserve price, is a weakly dominant strategy. Therefore, although the auction can admit multiple Nash Equilibria, included inefficient ones, the possibility of relying upon a weakly dominant strategy greatly simplifies bidders' strategic decisions and, moreover, allows for the auctioneer to be informed about their reserve price. Based on this remarkable property of the design, the presumption is that whenever in such an auction players submit an offer they would fully reveal their maximum willingness to pay. However, given the possibility of multiple equilibria the auctioneer, at least in principle, can never be sure whether submitted and reserve prices coincide.

Taking the above considerations as a starting point in the paper we tackle the following main issue. That is if, and under what conditions, reserve price revelation could be obtained in alternative second-price auctions mechanisms. We address the issue by considering "second-price" lottery auctions. In particular, as a main example, we do so within a generalized version of Tullock auctions (Tullock 1980), that is auctions where the winner is determined probabilistically, and the winning probability is proportional to the own submitted price. Tullock contests and associated winning probabilities, have been extensively studied in a variety of contexts (see among others Paul-Whilhite, 1990; Skaperdas 1996; Anderson et al, 1998; Corchon, 2007; Konrad, 2009; Alcaide et al, 2010; Cason et al, 2010). In particular, the Tullock mechanism has been widely adopted to model rent-seeking and lobbying activities, sport competitions, R&D incentives (Konrad, 2009). Such applications are typically formalized as first price all-pay contests, that is auctions where all players pay what they bid, and the highest price wins. A further motivation for considering these auctions is that lotteries, included National Lotteries, where the size of the prizes is determined by the amount invested by participants, can be seen as a form of all-pay (lottery) auctions, in which a bidder's value is endogenous and the winning probability is a Tullock contest function.

In the paper we introduce and investigate second-price Tullock lottery auctions, both winner only-pays and all-pay, with the following two main questions in mind. First to see whether revealing one's reserve price keeps being a weakly dominating strategy, as in the standard Vickrey auction, and then whether revelation of bidders' reserve price can be made more precise. In particular, when bidders have positive reserve values, if and where observation of positive offers could perfectly reveal their maximum willingness to pay.

In the initial, simplest, two player auction model that we discuss the answer is negative to the first question and positive to the second. As for the first question, introducing a lottery to award the object not only prevents truth telling from being a weakly dominant strategy but also to be a Nash Equilibrium. As for the second question however the set of Nash Equilibria, unlike the Vickrey mechanism, enjoys the nice property that the only equilibrium pair where both players bid positive prices perfectly reveals to the auctioneer bidders' reserve price. However, there is a main difference with respect to the Vickrey auction, since in equilibrium a player's willingness to pay is revealed by the opponent's offer and not by his own submitted price. In the paper we call this phenomenon “Mirror Revelation” (MR) and it would now be a sharp, unequivocal, identification of players’ reserve prices.

We then extend the analysis to $n > 2$ players to see if and how the above results go through. We see that MR still occurs though in a more general way, namely not only with profile of bids containing strictly positive entries but also with profiles having some zero bids. In this case, sharpness of revelation is guaranteed by the fact that no MR equilibrium profile of bids can include maximum bids. Finally, we consider a simplest two-player, two-values, incomplete information model to see that there could be a multiplicity of symmetric equilibrium, increasing and decreasing, bidding functions.

2 The Model with Complete and Private Information

We begin by setting the general framework. Consider one indivisible object on sale in a sealed bid auction, with $n = 2,3, \dots$ competing bidders. Players have private values and complete information about their reserve prices $v_i \in [0, \bar{v}]$, with $\bar{v} > v_{i-1} \geq v_i > 0$ for all $i = 2, .., n$. Though finite, we assume $\bar{v}$ to be “much larger” than v_1, more explicitly greater than nv_1. From now on $b = (b_1, .., b_n) = (b_i, b_{-i})$, as usual, will indicate the n-dimensional profile of bids, as expressed by the bid b_i of player i and the $n - 1$ dimensional profile of bids of his opponents b_{-i}. The object is awarded according to the following mechanism. Each player i submits his sealed bid $b_i \geq 0$. Upon having received them the auctioneer publicly opens the envelopes and assigns the object randomly, with probabilities proportional to the price bid. In particular, player $i's$ winning probability is the one adopted in Tullock contests and defined by

$$p_i(b_1, \dots, b_n) = \begin{cases} \dfrac{b_i}{b_1 + .. + b_n} & if \max b_i > 0 \\ \dfrac{1}{n} & if \max b_i = 0 \end{cases}$$

It is important to anticipate here however that the main results of the paper will be valid for more general winning probability functions, as long as they satisfy the following properties of the above Tullock probabilities:

i) $\frac{\partial p_i(b_1, \dots, b_n)}{\partial b_i} > 0 \qquad if \max b_i > 0$

ii) $p_i(b_1, \dots, b_n) = \frac{1}{n} \quad if \max b_i = 0$

iii) $p_i(b_1, .., b_i = 0, \dots, b_n) = 0 \qquad if \max b_{j \neq i} > 0$

After having assigned the object the auctioneer excludes the price submitted by the winner and proceeds with a second random draw from the remaining prices. The outcome of the second draw will be the price paid by the winner.

Therefore, given the random nature of the object assignment and price determination, there is no guarantee that the highest submitted price will win and that the second highest price will be paid. Indeed, though offering a high price increases the chance of winning, there could still be positive probability for the auctioneer of assigning the object to the lowest submitted bid and be paid the highest price. As we shall see below such randomization mechanism, in deciding winner and paid price, will meaningfully affect bidders' behavior.

We now introduce the idea of a Mirror Revelation Equilibrium (MRE) in the auction game.

Definition (Mirror Revelation Equilibrium) *A profile of bids b is a MRE if it is a Nash Equilibrium of the game and if for each $i = 1,..,n$ there exists a function $f_i(b_{-i}) = v_i$.*

In words, a MRE is a Nash Equilibrium where the reserve price of each player is revealed by some function of the bids submitted by his opponents. In what follows we shall see that such functions can take various forms, depending upon the auction type and the number of players.

3 The Second Price Winner-Only-Pays Tullock Auction (SWOPTA) with Two Bidders

To gain the fundamental insights of the model, in this section we begin considering its simplest version, with $n = 2$ bidders. In this case, player $1's$ payoff is given by

$$\Pi_1(b_1) = \begin{cases} v_1 - b_2 & with\, probability\ \dfrac{b_1}{b_1 + b_2} \\ 0 & with\, probability\ \dfrac{b_2}{b_1 + b_2} \end{cases}$$

and the expected payoff by

$$\mathrm{E}\Pi_1(b_1) = (v_1 - b_2)\,\frac{b_1}{b_1 + b_2}$$

Therefore, the best reply correspondence $B_1(b_2)$ for player 1 is given by

$$B_1(b_2) = \begin{cases} b_1 \in (0, \bar{v}] & if\ b_2 = 0 \\ b_1 = \bar{v} & if\ b_2 \in (0, v_1) \\ b_1 \in [0, \bar{v}] & if\ b_2 = v_1 \\ b_1 = 0 & if\ b_2 \in (v_1, \overline{v}] \end{cases}$$

and symmetrically for the other player. Hence, the following result holds.

Proposition 1 *In a complete information SWOPTA with two players the set of pure strategies Nash Equilibria is given by the following pairs* $(b_1 \in (v_2, \bar{v}], b_2 = 0)$*;* $(\,b_1 = v_2, b_2 = v_1)$; $(\,b_1 = 0, b_2 \in (v_1, \bar{v}])$.

Proof From the best reply correspondences observe that the pairs ($b_1 = \bar{v}$, $b_2 \in (0, v_1)$) cannot be equilibria. Indeed, if $b_1 = \bar{v}$ then player $2's$ best reply would be $b_2 = 0$. The rest of the proposition follows immediately.

Some comments are in order. The equilibrium pairs in the above proposition are a subset of the Nash Equilibria in the standard Vickery auction, which could be seen as selected by the Tullock randomization device, according to a "behavioral discontinuity" induced by the mechanism.

The intuition behind such equilibrium selection is simple. If player 2, and symmetrically for player 1, thinks the opponent will bid above his reserve price then, because of the randomized assignment, it would be best for him to offer a zero price since by submitting a positive price, however small, he would run the risk of being drawn as winner and make negative profits. Alternatively, if player 2 thinks the opponent bids below v_2 then he will want to maximize his chance of winning by offering a price above v_1, which player 1 will not want to outbid. Finally, if player 2 thinks that $b_1 = v_2$ then his profit will be zero whether or not he wins. Therefore, any offer could be a best reply, included $b_2 = v_1$.

The only pair with strictly positive components fully reveals the players' reserve price, however each player does not reveal the own price but his opponent's, and so it is a MRE. Moreover, notice also that the MRE would not be in weakly dominant strategies and that truth revealing is not even a Nash Equilibrium. Of course, for a MRE to be effective in practice the auctioneer has to count on players being well informed about each other reserve price. In this sense, exact revelation may turn out to be more difficult in this context.

As for the auctioneer's revenue, the following corollary points out how the SWOPTA Nash Equilibria compare to the weakly dominant equilibrium of the Vickrey auction.

Corollary 1 *For the auctioneer the MRE of a complete information SWOPTA with two players is preferable to the equilibrium in weakly dominant strategies of the Vickrey auction, while the other two SWOPTA equilibria are not.*

Proof Immediate. Indeed, the equilibrium pair ($b_1 = v_2, b_2 = v_1$) provides an expected revenue of $\frac{2v_2v_1}{v_1+v_2}$ to the auctioneer which, since $v_1 \geq v_2$, will be no lower than his payoff in the weakly dominant equilibrium of the Vickrey auction. The same would not occur with the other two equilibrium pairs, since both provide the auctioneer with zero expected payoff.

Finally, the following proposition clarifies that with complete information no other equilibria can exist.

Proposition 2 *In a complete information model with $n = 2$ bidders there is no mixed strategy Nash equilibrium.*

Proof In a mixed strategy equilibrium players expected payoff must be constant over their support. However, it is immediate to see that player 1 expected payoff $E\Pi_1(b_1) = (v_1 - b_2)\frac{b_1}{b_1+b_2}$

is increasing for $v_1 - b_2 > 0$ and decreasing for $v_1 - b_2 < 0$, which implies that there cannot be a support (except for $b_2 = v_1$) for a mixed strategy equilibrium of player 2, where $E\Pi_1(b_1)$ is constant. An analogous reasoning holds for player 1 and the result follows.

4. The SWOPTA with $n > 2$ Bidders

In this section we extend the model to $n > 2$ bidders and discuss to what extent the previous results generalize. We'll see if and how the main findings for $n = 2$ could carry through. In this case, player $i's$ payoff, with $i = 3, .., n$ is defined by

$$\mathrm{E}\Pi_i(b_i) = \begin{cases} v_i - b_k & with\ probability\ \dfrac{b_i}{(\sum_{j=1}^{n} b_j)}\dfrac{b_k}{(\sum_{j\neq i}^{n} b_j)} \\ 0 & with\ probability\ \dfrac{\sum_{j\neq i}^{n} b_j}{\sum_{j=1}^{n} b_j} \end{cases}$$

where $k \neq i$. Hence the expected payoff becomes

$$\mathrm{E}\Pi_i(b_i) = (v_i - Eb_{-i})\ \frac{b_i}{(\sum_{j=1}^{n} b_j)}$$

where

$$Eb_{-i} = \sum_{k\neq i}^{n} b_k \frac{b_k}{(\sum_{j\neq i}^{n} b_j)} = \sum_{k\neq i}^{n} \frac{b_k^2}{(\sum_{j\neq i}^{n} b_j)}$$

is the expected value of the second price drawn, conditionally to having selected b_i in the first draw. Therefore, player $i's$ best reply correspondence would now be given by

$$B_i(b_{-i}) = \begin{cases} b_i \in (0, \bar{v}] & if\ Eb_{-i} = 0 \\ b_i = \bar{v} & if\ Eb_{-i} \in (0, v_i) \\ b_i \in [0, \bar{v}] & if\ Eb_{-i} = v_i \\ b_i = 0 & Eb_{-i} \in (v_i, \bar{v}] \end{cases}$$

Thus, the following proposition holds

Proposition 3 *In a complete information SWOPTA with $n > 2$ bidders the MRE profiles solve the system of n equations $f_i(b_{-i}) = Eb_{-i} = v_i$, with $i = 1, .., n$. Moreover, there always exists an MRE.*

Proof First notice that at a MRE, for all $i = 1, .., n$, it must be $\sum_{j\neq i}^{n} b_j > 0$ since otherwise $\sum_{j\neq i}^{n} b_j = 0$ would imply $Eb_{-i} = 0 \neq v_i > 0$, contradicting MRE. Then, from the best reply correspondence, it follows immediately that for all $i = 1, .., n$, at any MRE the equation

$$Eb_{-i} = \sum_{k\neq i}^{n} \frac{(b_k)^2}{(\sum_{j\neq i}^{n} b_j)} = v_i$$

must be solved. Since the domain of bid profiles b is $B = [0, \bar{v}]^n$, hence compact and convex, Brouwer's fixed point theorem guarantees that the following non-linear, continuous, n-variables mapping $g(b): B \to B$ defined as

$$g_i(b) = \frac{\sum_{k=1}^{n} (b_k)^2 - b_{n-i+1}^2}{v_{n-i+1}} - \sum_{k\neq i}^{n} b_k + b_{n-i+1}$$

where $g_i(b)$ is the $i-th$ component of the mapping and $i = 1,..,n$, has a fixed point. This proves the existence of an MRE since a fixed point $g_i(b) = b_i$ exists if an only if $Eb_{-(n-i+1)} = v_{n-i+1}$.

The next result clarifies that reserve prices will be offered only under specific circumstances.

Corollary 2 *Consider a complete information SWOPTA with $n > 2$ bidders where at least two players have different reserve value. Then there is no Nash Equilibrium where each reserve price will be offered by some player. Therefore there is no equilibrium where each player bids his own reserve price.*

Proof Immediate by observing that if at least two reserve prices are different and if all reserve prices are bid by some player then $Eb_{-1} < v_1$ which would imply $b_1 = \bar{v} > v_k$, with $k = 1,..,n$, contradicting the initial assumption.

Proposition 2 and Corollary 2 suggest that MRE can still occur, though not at individual level, but considering averages of bids. Moreover, for specific values an MRE profile may contain some zero components, as the following example with $n = 3$ illustrates. The system of equations now becomes

$$(b_2)^2 + (b_3)^2 = (b_2 + b_3)v_1$$

$$(b_2)^2 + (b_1)^2 = (b_2 + b_1)v_3$$

$$(b_1)^2 + (b_3)^2 = (b_1 + b_3)v_2$$

It is easy to check that there cannot be two zero bids since otherwise one of the equations could not be solved. The system would however allow for a solution vector with a zero component. In particular provided that $v_2 = \frac{(v_1)^2+(v_3)^2}{v_1+v_3}$ the vector $b = (b_1 = v_3, b_2 = 0, b_3 = v_1)$ is the only MRE profile of bids solving the system and such that $v_1 \geq v_2 \geq v_3$. Notice however that non MRE vectors could not have only a b_2 zero component, hence the auctioneer could not mistake them for an MRE. Indeed, suppose $b_2 = 0$ is the only zero component of a non MRE. Then if $b_1 > v_3$ it must be $b_3 = 0$. Alternatively, if $b_1 < v_3$ then $b_3 = \bar{v} > v_1$ and so $b_1 = 0$. Both conclusions would contradict the initial assumption. Therefore no MRE can be "mistaken" for non MRE and, even with zero bids components, mirror revelation can still be the case.

Moreover, for example, from the second equation of the above system it follows that

$$b_1 = \frac{v_3 \mp \sqrt{v_3^2 + 4v_3 b_2 - 4b_2^2}}{2}$$

The largest of the two solutions is maximized when the expression $v_3^2 + 4b_2(v_3 - b_2)$ inside the square root, seen as function of b_2, is maximum that is when $b_2 = \frac{v_3}{2}$, leading to

$$b_1 \le \frac{v_3(1+\sqrt{2})}{2} < 2v_3$$

By a similar reasoning it also follows that $b_2 < 2v_3$ and $b_3 < 2v_2$. That is, at a MRE all bids are bounded above by $v_2 < \overline{v}$. Therefore, with three players the only MRE where all components are positive have bid components strictly below $\overline{v}$. But at a non MRE with all positive bids, the system of weak inequalities

$$(b_2)^2 + (b_3)^2 \le (b_2 + b_3)v_1$$

$$(b_2)^2 + (b_1)^2 \le (b_2 + b_1)v_3$$

$$(b_1)^2 + (b_3)^2 \le (b_1 + b_3)v_2$$

must be satisfied (because otherwise one component would be zero) with at least one of them being satisfied as strict inequality (since otherwise it would be an MRE). This implies that one of the bid components has to be equal to $\overline{v}$. As a consequence, when observing a profile of non-zero bids the auctioneer knows that the offers are revealing if no component is equal to $\overline{v}$. The above observations generalize to any finite number of players as follows

Proposition 4 *For each MRE of a complete information SWOPTA, with $n > 2$, no positive component of the equilibrium profile of bids could be equal to $\overline{v}$.*

Proof Start considering MRE with only positive components. At such equilibria the following system of equalities must be satisfied

$$\sum_{i\neq k}^{n} b_i^2 = v_k\left(\sum_{i\neq k}^{n} b_i\right) \quad for\ all\ k = 1,..,n \qquad (1)$$

Take, for simplicity, $k > 1$ and express (1) as a quadratic equation in b_1 as follows

$$b_1^2 - v_k b_1 + \left(\sum_{i\neq(1,k)}^{n} b_i^2 - \sum_{i\neq(1,k)}^{n} b_i\right) = 0 \quad (2)$$

Solving (2) we obtain

$$b_1 = \frac{v_k \mp \sqrt{v_k^2 + \sum_{i\neq(1,k)}^{n} b_i^2 - \sum_{i\neq(1,k)}^{n} b_i}}{2}$$

where the largest solution cannot be greater than $\frac{v_k(1+\sqrt{n-1})}{2} < \overline{v}$. Since this holds for all $k = 1,..,n$ it follows that no MRE profile can contain bids equal to $\overline{v}$. Instead, at least one $\overline{v}$ will have to be contained by a non MRE profile with all positive components.

Consider now a MRE with $k = 1,..,n-2$ zero components. Indeed, there could not be $n-1$ zero components otherwise the remaining nth component would be associated to a zero value, contradicting the assumption that all values are positive. Moreover, by Z and NZ we indicate the subset of components with, respectively, zero and non-zero bids. Hence, at an MRE it is

$$\sum_{i\neq k}^{n} b_i^2 = \sum_{i\in NZ}^{n} b_i^2 = v_k(\sum_{i\in NZ}^{n} b_i) = v_k(\sum_{i\neq k}^{n} b_i) \quad for\ all\ k \in Z$$

and

$$\sum_{i\neq k}^{n} b_i^2 = \sum_{i\in NZ-\{k\}}^{n} b_i^2 = v_k(\sum_{i\in NZ-\{k\}}^{n} b_i) = v_k(\sum_{i\neq k}^{n} b_i) \quad for\ all\ k \in NZ$$

Hence, by a similar reasoning it is immediate to check that also in this case no $\overline{v}$ can appear in the bids profile while, instead, at least one of them must appear in a non MRE profile of bids.

5 The Second Price All-Pay Tullock Auction (SAPTA) with Two Bidders

We now consider the more common version of the Tullock contest, that is the all-pay auction, where players pay what they submit, whether or not they win the object. However in this case the standard all-pay mechanism must be adjusted to take account of the randomizing awarding device as well as of the second price mechanism. Hence, if bids represent sunk investments, players here have an additional strategic component to deal with, namely that they have to calibrate their offers considering the risk of over-investment. That is, the risk of offering a price higher than the one they would end up paying, and in so doing waste resources. A way the auctioneer could take account of this could be simply to compensate the winner of the difference between the price offered and the price received, in case the latter is smaller than the former. This is indeed the mechanism we shall consider in this section though, of course, there could be alternative ways to formalize the point, such as defining the price paid by the winner as the maximum between the first and the second price drawn.

Therefore, in the simplest model with two players bidder 1's payoff function is given by

$$\Pi_1(b_1) = \begin{cases} v_1 - b_2 & with\ probability\ \dfrac{b_1}{b_1+b_2} \\ -b_1 & with\ probability\ \dfrac{b_2}{b_1+b_2} \end{cases}$$

with the expected payoff now defined as

$$\mathrm{E}\Pi_1(b_1) = (v_1 - 2b_2)\ \frac{b_1}{b_1 + b_2}$$

Hence, the best reply function $B_1(b_2)$ for player 1 is given by

$$B_1(b_2) = \begin{cases} b_1 \in (0, \bar{v}] & if\ b_2 = 0 \\ b_1 = \bar{\bar{v}} & if\ b_2 \in (0, \frac{v_1}{2}) \\ b_1 \in [0, \bar{v}] & if\ b_2 = \frac{v_1}{2} \\ b_1 = 0 & if\ b_2 \in (\frac{v_1}{2}, \bar{\bar{v}}] \end{cases}$$

and symmetrically for the other player. Since $B_1(b_2)$ now has the same structure, as in the winner-only-pays auction, the next result follows immediately..

Proposition 5 In SAPTA with two players the pure strategies Nash Equilibria are given by the following pairs $(b_1 \in (\frac{v_2}{2}, \bar{v}], b_2 = 0)$; $\left(b_1 = \frac{v_2}{2}, b_2 = \frac{v_1}{2}\right)$; $\left(b_1 = 0, b_2 \in (\frac{v_1}{2}, \bar{v}]\right)$

The all-pay structure affects only the bids level but not the nature of the equilibria. Indeed, as in Proposition 1, the only equilibrium with strictly positive bids is an MRE where each player offers half the opponent's reserve price. As a consequence, the auctioneer expected profit $\frac{v_2 v_1}{v_1+v_2}$ is lower than v_2. Therefore, the SAPTA is always less desirable than the truth revealing equilibrium of the Vickrey auction.

6 The SAPTA with $n > 2$ Bidders

With more than two players bidder $i's$ profit would be defined as follows

$$\Pi_i(b_i) = \begin{cases} v_i - b_k & with\ probability\ \frac{b_i}{(\sum_{j=1}^n b_j)} \frac{b_k}{(\sum_{j\neq i}^n b_j)} \\ -b_i & with\ probability\ \frac{\sum_{j\neq i}^n b_j}{\sum_{j=1}^n b_j} \end{cases}$$

and the expected payoff given by

$$\mathrm{E}\Pi_i(b_i) = (v_i - Eb_{-i} - \sum_{j\neq i}^{n} b_j)\ \frac{b_i}{(\sum_{j=i}^n b_j)}$$

Consequently, bidder $i's$ best reply function now is

$$B_i(b_{-i}) = \begin{cases} b_i \in (0, \bar{v}] & if\ Eb_{-i} + \sum_{j \neq i}^{n} b_j = 0 \\ b_i = \bar{v} & if\ Eb_{-i} + \sum_{j \neq i}^{n} b_j \in (0, v_i) \\ b_i \in [0, \bar{v}] & if\ Eb_{-i} + \sum_{j \neq i}^{n} b_j = v_i \\ b_i = 0 & if\ Eb_{-i} + \sum_{j \neq i}^{n} b_j \in (v_i, \bar{\bar{v}}] \end{cases}$$

and the following proposition would hold

Proposition 6 *In a complete information SAPTA with* $n > 2$ *bidders the MRE profiles solve the system of* n *equations* $Eb_{-i} + \sum_{j \neq i}^{n} b_j = v_i$, *with* $i = 1, .., n$. *Moreover, MRE always exist.*

Proof Analogous to that of Proposition 2, except that now the fixed point theorem applies to the function $h(b): B \to B$ defined as

$$h_i(b) = \frac{\sum_{k=1}^{n} (b_k)^2 - b_{n-i+1}^2 + (\sum_{k=1}^{n} b_k + b_{n-i+1})^2}{v_{n-i+1}} - \sum_{k \neq i}^{n} b_k + b_{n-i+1}$$

Similar considerations as in Proposition 4, hold as for MRE. Only equlibria without $\bar{v}$ components could be

7 The SWOPTA with Two Bidders and Incomplete Information

In this section we discuss incomplete information on bidder's values, in the simplest SWOPTA two-players, two-values game. In particular, we assume v_1 and v_2 to be independently and identically distributed random variables that can take only two values, $0 < x < y < \bar{v}$, with (respectively) probability $0 < p < 1$ and $1 - p$.

In what follows we fully characterize symmetric Bayes-Nash equilibrium bidding functions $0 \leq \beta_1(v) = \beta(v) = \beta_2(v) \leq \bar{v}$ and observe that there is a multiplicity of them. To do so start considering player $i = 1,2$ who, upon having observed $v_i = x$, bids $b_i(x)$. Assuming player $j \neq i = 1,2$ to adopt $\beta_j(v)$, player $i's$ expected payoff function is given by

$$E\Pi_i(b_i(x)) = p\left(x - \beta_j(x)\right)\frac{b_i(x)}{b_i(x) + \beta_j(x)} + (1-p)\left(x - \beta_j(y)\right)\frac{b_i(x)}{b_i(x) + \beta_j(y)} \quad (3)$$

Differentiating (3) with respect to $b_i(x)$ leads to

$$\frac{dE\Pi_i(b_i(x)}{db_i(x)} = p\left(x - \beta_j(x)\right)\frac{\beta_i(x)}{(b_i(x) + \beta_j(x))^2} + (1-p)\left(x - \beta_j(y)\right)\frac{\beta_j(y)}{(b_i(x) + \beta_j(y))^2} \quad (4)$$

Similarly, if player i observes $v_i = y$ then bids $b_i(y)$, with his expected payoff function given by

$$E\Pi_i(b_i(y)) = p\left(y - \beta_j(x)\right)\frac{b_i(y)}{b_i(y) + \beta_j(x)} + (1-p)\left(y - \beta_j(y)\right)\frac{b_i(y)}{b_i(y) + \beta_j(y)} \quad (5)$$

which differentiated with respect to $b_i(y)$ leads to

$$\frac{dE\Pi_i(b_i(x)}{db_i(x)} = p\left(x - \beta_j(x)\right)\frac{\beta_j(x)}{(b_i(x) + \beta_j(x))^2} + (1-p)\left(x - \beta_j(y)\right)\frac{\beta_j(y)}{(b_i(x) + \beta_j(y))^2} \quad (6)$$

Hence, we can formulate the main finding of this section

Proposition 7 *There are multiple symmetric Bayes-Nash equilibria such that $\beta(x) < x$ and $\beta(y) > y$ or $\beta(x) > y$ and $\beta(y) < x$*

Proof Notice first that there cannot be a symmetric equilibrium where both $\beta(x)$ and $\beta(y)$ are either lower than x or higher than y. Indeed, if $\beta_j(x), \beta_j(y) > y$ then (4) and (6) are negative and for player i would be optimal to choose $\beta_i(x) = 0 = \beta_i(y)$. Analogously, if $\beta_j(x), \beta_j(y) < x$ then (4) and (6) are positive and it is optimal to choose $\beta_i(x) = \overline{v} = \beta_i(y)$, which prove the claim. By a similar reasoning, there cannot exist symmetric equilibria with $\beta(x) < x < \beta(y) < y$, $\beta(y) < x < \beta(x) < y$, $x < \beta(x) < y < \beta(y)$, $x < \beta(y) < y < \beta(x)$. Finally, from (3) and (5) it follows also that at a symmetric equilibrium $\beta(x)$ and $\beta(y)$ must differ from x and y.

Hence, still from (4) and (6), at such equilibrium it must either be $(x - \beta(x)) > 0$ and $\left(y - \beta(y)\right) < 0$ or $\left(y - \beta(x)\right) < 0$ and $\left(x - \beta(y)\right) < 0$.

Consider first $(x - \beta(x)) > 0$ and $\left(y - \beta(y)\right) < 0$. Because at a symmetric equilibrium it is $b_i(x) = \beta(x) = \beta_j(x)$, rearranging (4) we obtain

$$\frac{(x - \beta(x))}{(\beta(y) - x)} = \frac{(1-p)4\beta(x)\beta(y)}{p(\beta(x) + \beta(y))^2} \quad (7)$$

and, again, by symmetry $b_i(y) = \beta(y) = \beta_j(y)$ and rearranging (6) it follows that

$$\frac{(y - \beta(x))}{(\beta(y) - y)} = \frac{(1-p)(\beta(x) + \beta(y))^2}{p4\beta(x)\beta(y)} \quad (8)$$

Hence, from (7) and (8) it is

$$\frac{4\beta(x)\beta(y)}{(\beta(x) + \beta(y))^2} = \frac{p(x - \beta(x))}{(1-p)(\beta(y) - x)} = \frac{(1-p)(\beta(y) - y)}{p(y - \beta(x))}$$

and so

$$\left(x-\beta(x)\right)\left(y-\beta(x)\right)=\left(\frac{(1-p)}{p}\right)^2(\beta(y)-x)(\beta(y)-y) \qquad (9)$$

Left and right hand side of (9) are both convex, quadratic, equations respectively in $\beta(x)$ and $\beta(y)$, crossing the horizontal axis at x and y. Hence, for increasing bidding functions $\beta(x) < x$ and $\beta(y) > y$ it easy to check that there would be multiple solutions of the kind

$$\beta(x;k)=\frac{(x+y)-\sqrt{(y-x)^2+4k}}{2};\ \beta(y;k)=\frac{(x+y)+\sqrt{(y-x)^2+\left(\frac{p}{(1-p)}\right)^2 4k}}{2} \qquad (10)$$

where $0 < k \le xy$ is the value that the two quadratic functions in (9) can take, with the upper bound $k = xy$ solving $\beta(x;k) = 0$.

Similarly, if $\left(y-\beta(x)\right) < 0$ and $\left(x-\beta(y)\right) > 0$ the two solutions would be

$$\beta(x;k)=\frac{(x+y)+\sqrt{(y-x)^2+4k}}{2};\ \beta(y;k)=\frac{(x+y)-\sqrt{(y-x)^2+\left(\frac{p}{(1-p)}\right)^2 4k}}{2} \qquad (11)$$

which completes the proof.

It is worth noticing that the two solutions in (10) are asymmetric unless the probability distribution is uniform, that is $p=\frac{1}{2}$. In this case $\beta(x)+\beta(y)=x+y$, and when $k = xy$ the two solutions take the particular simple form of $\beta(x) = 0$ and $\beta(y) = x + y$. Observe also that when p becomes small $\beta(y;k)$ tends to y, that is to truthful revelation, while as p goes to one then $\beta(y;k)$ gets large. That is, for any given $0 \le k \le xy$ the factor $\left(\frac{p}{(1-p)}\right)^2$ is what induces asymmetry, by affecting the scale of the quadratic on the right had side of (9), so that whenever the quadratic on the left hand side is equal to k the one on the right hand side is equal to $\left(\frac{p}{(1-p)}\right)^2 k$.

Of course, solutions (10) could have been formulated as

$$\beta(x;k)=\frac{(x+y)-\sqrt{(y-x)^2+\left(\frac{1-p}{p}\right)^2 4k}}{2};\ \beta(y;k)=\frac{(x+y)+\sqrt{(y-x)^2+4k}}{2} \qquad (12)$$

that is in such a way that $0 < k \le xy$ would now fix the value of the quadratic on the right hand side, with the factor $\left(\frac{1-p}{p}\right)^2$ this time affecting the scale of the quadratic on the right hand side. In this case, for any fixed $0 < k \le xy$ as p tends approaches one $\beta(x;k)$ tends to x while as p gets small $\beta(x;k)$ tends to zero.

Similar considerations hold for (11) except that now bids would tend to a form of "probabilistic" mirror revelation, that is it would not tend to reveal the associated value but the other one. We specify probabilistic because, for example, if one players observes x it is with probability $1-p$ that the opponent would observe y. Hence, if p tends to one, then the probability of mirror revelation is small while with higher probability both would reveal nobody's

8 Conclusions

In the paper we introduced the Tullock second price mechanism to investigate whether second price auctions, alternative to the Vickery Auction, could also enjoy the property that revealing one's reserve price is a weakly dominant strategy. Since the Vickrey Auction has multiple Nash Equilibria the auctioneer, in principle, can never be sure whether players truly revealed their reserve prices. Therefore, we further asked whether in alternative second price mechanisms revelation could be more precise than in the Vickery Auction, namely if the auctioneer by observing a certain profile of offers could immediately infer whether or not players' maximum willingness to pay is disclosed. We found that in second price Tullock Auctions, with complete information revelation of reserve prices can indeed be made unequivocal for the auctioneer, however neither because bidders choose weakly dominating strategies nor because players offer their own prices.

Indeed, the first main finding of the paper is that revelation of one's reserve price is made by his opponents, not by the player himself, which for this reason we named "Mirror Revelation". Though somewhat surprising at first, the intuition behind it is simple. For example, with two players revelation occurs only when both bidders are uncertain to win, and offering the opponent's price guarantees zero expected profit and prevents losses. A similar reasoning extends to more than two players.

The second main finding is that, in equilibrium, reserve prices can be exactly inferred by the auctioneer so that, with respect to the Vickrey Auction, in principle revelation can be made more precise. Finally, due to the lottery structure of the model not only truth telling is not a weakly dominant strategy but is not even a Nash Equilibrium.

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