The *uncertain* destiny of proprietary capitalism: Family connections and the allocation of entrepreneurial talent

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Abstract

This paper focuses on the problem of continuation of the firms, when the managerial capital of the entrepreneurs depends on the individuals innate ability and possibly on the system of relations transmitted across the generations of entrepreneurs of the same family. It argues that in societies in which the family connections have a productive value, entrepreneurs choose to transfer the control of the firms within the family not only to the highest, but also to the lowest ability heirs since the connections embedded in the family networks guarantee a minimum level of profitability to the family businesses. We develop a model consistent with the observed patterns of the firm level and aggregate performance differentials of the family and non-family firms. We show that although there is no difference in the firm level performance of the family and non-family firms operated by the entrepreneurs at the top of the ability distribution, at aggregate level the family connections lower the average performance of the family firms, since a mass of low ability heirs manages the family business without the relevant entrepreneurial competencies. Increases in the rate of the technological progress erode the productivity of the family connections and raise the return of the innate abilities, hence decreasing the mass of low ability heirs and increasing that of the high ability individuals who operate the firms. However, if the family connections are extremely productive, the economies converge to fully immobile societies, with no exit and no entry in the entrepreneurial sector as all the firms of the economy are managed by the heirs who exploit the system of relations of the families.

JEL: J24, J62, L26, O40

Keywords: Family firms, connections, allocation of talents, technological change, economic growth

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1 Introduction

According to a well-established tradition in business history, the family firm, the firm owned and controlled by the family's members, has been a crucial source of dynamism and progress in the early phases of industrialization (Mathias and Postan, 1978; Colli, 2003). However, in the mature stages of economic development, family firms have turned into a cause of halfheartedness, conservatism and vested interests which have produced either the decline of their importance in the economy, broadly confined to the sector of small and medium enterprises, or the sclerosis of the domestic industry (Landes, 1949, 1965; Chandler, 1962, 1990; Elbaum and Lazonick, 1986; Lazonick, 1991; Morikawa, 2001). This radical and inescapable evolution of proprietary capitalism, it is typically argued, is driven by the technological and institutional changes which distinguish the stages of the industrialization process. When industrialization is in its infancy, the technology is simple and slowly changing, and the social and economic institutions are weak. In similar contexts, the family would have been the appropriate unit around which to organize the business activity: the family ties would have permitted to minimize a number of information and incentive problems, by providing motivations to start up and conduct a business, by making available financial resources, technical knowledge and entrepreneurial skills, by reducing opportunistic behaviors and supporting long-term commitments to work, invest and innovate. In fact, the family firm has been proved to be the predominant form of business organization in the early stages of development of first- and late-industrializing countries, and proprietary capitalism has not been an impediment to the appearance of entrepreneurial talent and firm entry:

> there can be little doubt that the family firm was the vehicle whereby the Industrial Revolution was accomplished ... [the] belief that the British family firm has been an important engine of economic progress is incontrovertible (Payne, 1984, p. 188).

> family structures, processes and resources furthered the break-trough of industrial capitalism and helped to solve problems of (capitalist) industrialization which could hardly have been solved otherwise (Kocka, 1981, p. 54).

As the industrialization process goes on, the scale and scope of production get bigger and the pressure of international competition and technological change get stronger. Consequently, the firm's organization and its administration develop into an ever more complex and professional affair, hard to be met by the talents available within the family, and incompatible with nepotism and dynastic motivations that typically drive succession in family firms. In addition, the strengthening of social and economic institutions made (capital and labor) markets, formal education systems and the enforcement of property rights well functioning and the substitute role of the family ties of less importance. In this new environment, the destiny of proprietary capitalism is marked: transforming into managerial capitalism, by attracting outside capital and management skills – like it took place in the United States by the end of the nineteenth century, and in Germany and Japan in the twentieth century (Chandler, 1990; Morikawa, 2001) – or degenerating in crony capitalism, by producing misallocation of entrepreneurial talent, hindrance to technological innovations and barrier to social mobility – like occurred in France throughout the nineteenth century, in Great Britain between the nineteenth and twentieth centuries, and in Italy by the end of the twentieth century (Landes, 1949; Chandler, 1990; Lazonick, 1991; Pavan, 1973; Amatori, 1997; Colli, 2003).

This description of the evolution of proprietary capitalism and the role of family firms is broadly shared by economists. Starting from the influential contributions of Ben-Porath (1980) and Pollak (1980), the family has been viewed as a form of governance for economic activity to be contrasted to other possible ways to organize exchange and production on the basis of transaction costs. In this perspective, in the first phases of economic development, when markets are largely incomplete or underdeveloped, and when politics, public sector and legal institutions are untrustworthy, the family firm consents to minimize the transaction, information and agency costs related to the acquisition of human and financial resources, and the organization of production and distribution (Bhattacharya and Ravikumar, 2001, 2005; Burkart et al., 2003; Caselli and Gennaioli, 2012; Chami, 2001; Hassler and Mora, 2000). As the process of development advances, the efficiency motivations for family firms lose of importance and the adverse impact of dynastic motivations on talent allocation and social mobility in the economy becomes predominant[1]. The objective to secure the survival of the family values makes the founder willing to leave the control of the company within the network of heirs at the expense of higher financial returns. Once again, either the weight of family firms in the country reduces, limited to small and specialized market niches, or otherwise the country is destined to the economic decline and social immobility.

Although apparently persuasive, the deterministic, structural explanation of the evolution of proprietary capitalism and its inexorable destiny has been recently disputed by the historical research, which have rehabilitated the role of family firms in the development process, and is not fully consistent with present-day evidence on the importance of family-controlled corporations and the performance of inherited firms.

First, international comparisons indicate that the family firm cannot be easily blamed to be a retarding factor in economic development. For example, as Alexander Gerschenkron (1954, p. 10) noted by now, in France in the 19th century the influence and incidence of family firms were not less strong than in Germany, and in order to maintain the thesis of the degenerative character of proprietary capitalism "Landes has to relegate vast and most significant fields of French entrepreneurial endeavor ... to qualifying footnotes"

[1] An oft-quoted and vivid image of the damages produced by the *entrepreneurship by inheritance* is given by the American financier Warren Buffet according to whom "[to] pass down the ability to command the resources of the nation based on heredity rather than merit .... [is like] choosing the 2020 Olympic team by picking the eldest sons of the gold-medal winners in the 2000 Olympics" (reported in the *New York Times* article "Dozens of the wealthy join to fight estate tax repeal", D.C. Johnston, 14 February, 2001, electronic edition).

(Fridenson, 1997; Cassis, 2003). In the same vein, Roy Church (1993, p. 39) revealed that until the 1940s, "family firms persisted in Germany probably as widely as in Britain, while in Japan the family enterprise based on holding company structure was even more dominant than in either country". The existence of stories of both flourishing and blocked development in countries dominated by family firms has therefore leads many historians to view in the different institutional environments and the networks of relationships in which family firms are embedded as the way to understand the "national differences in the capabilities and behavior of family firms [and] the distinctive characteristics of personal capitalism" (Colli and Rose, 2003, p. 341)

In a similar vein, the present-day figures on the incidence of family control in large enterprises around the world and the econometric evidence on the performance of family firms return an image of proprietary capitalism which is much less clear-cut than the Chandlerian one. First, the family-controlled firm is a common and persistent form of business organization even among publicly listed companies of well developed countries. For example, in the United States, in the mid-1990s, 4 out of the top 20 listed companies by market capitalization and one third of companies included in "Standard & Poor 500" and in "Fortune 500" indexes had a family as ultimate controlling owner (Neubauer and Lank, 1998; Porta et al., 1999; Anderson and Reeb, 2007). In Europe, in the same years, Faccio and Lang (2002) report that the percentage of listed companies controlled by families were 24% in the UK, 60% in Italy and 65% in France and Germany, while with regard to eastern Asian countries the share of family-controlled firms was above 50% except than in Japan (where the percentage was 13%; Claessens et al. (2000)).

Second, the recent empirical studies measuring the accounting and market performance of family firms do not unambiguously support the prediction that family firms underperform relative to non-family firms. For example, with regard to US, Anderson and Reeb (2007) analyze the Return on Assets and Tobin's q of non-financial firms in the S&P 500 and show that on average family firms tend to overperform: the ROA is higher for family firms in which both the founder and the founder's descendant serve as CEO while the q is higher for family firms with outside and founder CEOs. Villalonga and Amit (2006) confirm that among the Fortune 500 firms those run by descendant-CEOs have a lower market performance, however this value destruction is limited to second-generation family firms. Pérez-González (2006) considers a sample of 335 successions in publicly traded family-controlled firms, finding that companies run by a family successor have a lower ROA and Market-to-Book value during the three years after the succession. However, the undeperformance result disappears if the descendant CEO has accumulated enough human capital by attending a selective college. Evidence for other countries is equally mixed: the negative impact of the family succession on performance is found by Smith and Amoako-Adu (1999) and Morck et al. (2000) for listed firms in Canada, Cronqvist and Nilsson (2003) for listed firms in Sweden, Bennedsen et al. (2007) for Danish firms and Cucculelli and Micucci (2008) for small-medium Italian firms, while it is rejected by Favero et al. (2006) for listed Italian firms, Sraer and Thesmar (2007) for listed firms in

France and Barontini and Caprio (2006) for public corporations in 11 European countries.

On the whole, therefore, two general conclusions can be drawn from historical and economic research on family firms: (1) the impossibility to identify a sole and certain destiny for proprietary capitalism in the process of economic development; (2) the coexistence in the same or in similar economic environments of well-performing and underperforming family firms.

In this paper we present a very stylized overlapping generations macroeconomic model which accommodates both the polarization of family firms into two groups with different levels of profitability and the uncertain destiny of proprietary capitalism.

2 The model

Consider an overlapping generations economy in which economic activity extends over infinite discrete time. The economy is composed by a fringe of competitive firms that produce a single homogeneous good employing efficiency units of human capital and managerial capital as inputs of the production process. The aggregate supply of the factors of production is endogenously determined by the individuals occupational choice between entrepreneurship and employment.

In each period t, a generation is born. It is populated by a continuum of individuals of mass one that differ in their innate abilities, which are distributed uniformly on the unit interval, $a_t^i \sim U[0,1]$. Individuals live for two periods. In the first period of their life (childhood), they accumulate either the managerial capital, needed to operate a firm, or the general human capital that can be supplied on the labor market. In the second period of the life (adult/parenthood), individuals allocate their unit time endowment between child investment and work. Parents make the occupational choice for their offspring. As a consequence, adult individuals either manage a firm (entrepreneur) or work as an employee, depending on the "capital" accumulated in their childhood and financed out by the parental time investment; correspondingly, they gain a payoff (residual profits or wage) and consume. Thus, in each period t, a number of entrepreneurs (n_t) and a number of workers ($1 - n_t$) coexist and are endogenously determined by the individuals occupational choices.

2.1 Production of the final good

Firms are operated by a single manager (or entrepreneur). The output of each firm is produced according to the following production function:

$$y_t^i = A_t m_t^i \left(H_t^i\right)^{1-\alpha} \qquad (1)$$

where m_t^i is the *managerial capital* of the entrepreneur i to be discussed below, H_t^i the quantity of efficiency units of human capital, A_t the aggregate technology of the economy and $\alpha \in (0,1)$. The firm level production function exhibits decreasing returns to scale in the variable factor, reflecting the limited span of control of the single manager (Lucas, 1978)[2]. Each entrepreneur (i.e. firm) chooses the quantity H_t^i such to maximize profits, taking as given the equilibrium wage rate w_t:

$$\max_{H_t^i \geq 0} \pi_t^i = A_t m_t^i \left(H_t^i\right)^{1-\alpha} - w_t H_t^i \tag{2}$$

The conditional demand function for firm i is:

$$H_t^i = \left(\frac{(1-\alpha) A_t m_t^i}{w_t}\right)^{1/\alpha} \tag{3}$$

For any given equilibrium wage rate, the maximum profits of each entrepreneur with managerial capital m_t^i are obtained by substituting (3) in (2), as:

$$\pi_t^i = \pi_t \left(m_t^i\right)^{1/\alpha} \tag{4}$$

where

$$\pi_t = \beta \left(\frac{A_t}{w_t^{1-\alpha}}\right)^{1/\alpha} \tag{5}$$

are the profits per-efficiency units of managerial capital that are common across the economy as they are determined by the level of the aggregate technology available in the economy as a whole and by the equilibrium wage rate, with $\beta \equiv \alpha (1-\alpha)^{\frac{1-\alpha}{\alpha}}$.

2.2 The accumulation of the factors of production

The managerial capital of each entrepreneur (m_{t+1}^i) originates from two sources; the innate individual ability and the investment of the parents. We describe the two channels by means of the following specification

$$m_{t+1}^i = \iota\phi \left(1 - g_{t+1}\right) \tau_t^{\phi} + a_{t+1}^i \tau_t^a \tag{6}$$

First, the managerial skills of the individuals - of generation $t+1$ - depend on the innate abilities a_{t+1}^i and on the time invested in the improvement of their entrepreneurial

[2] Following the original spirit of the Lucas (1978) paper, physical capital could be introduced by assuming a small open economy with perfect capital mobility and allowing the decreasing returns to scale to depend on a span of control parameter $\vartheta \in (0,1)$, as $y_t^i = A_t m_t^i [(K_t^i)^{\alpha} (H_t^i)^{1-\alpha}]^{\vartheta}$. In this environment, the rate of interest, and hence the dynamics of the physical capital, would be internationally fixed without affecting the results. However, the model would result complicated at the point of intractability making the analysis less transparent.

ability (τ_t^a) by the parents - generation t.

Further, the managerial capital of the children of entrepreneurs can be augmented with the connections established by the parents and possibly transmitted across generations of entrepreneurs. The anecdotal and empirical evidence reported in Section 1 corroborates the idea that part of the individual managerial capital is embedded in extended family networks that are specific of dynasties of entrepreneurs. The entrepreneurs can, indeed, employ part of their unit-time endowment to introduce their heirs into a network of connections (τ_t^ϕ) that can ensure a minimum level of profitability to the family business when the heirs will be in charge of managing it. Hence, we assume that the system of relations of the firms - the connections - is nontradeable on the markets as it can be transferred only within the families of entrepreneurs. We formalize this hypothesis with the indicator function $\iota = 0, 1$ that takes value 1 if the individuals are the descendant of entrepreneurs and value 0 if the individuals are born from parent who were workers at time t.

In addition to the individual heterogeneity in innate abilities, the accessibility of the network of relations introduces a further source of heterogeneity across families, through which we distinguish two types of firms; family (or inherited) versus non-family (or new) firms. The inherited firms are the ones transferred within the family by the generations of entrepreneurs to their heirs, while the new firms are new businesses started by the descendants of the workers. While the firms heirs can manage the family businesses exploiting both their innate ability and the connections inherited from the parents, new potential entrepreneurs can employ only their innate ability in order to start and operate new businesses. This comparative advantage that the heirs have with respect to the new entrepreneurs depends on the productivity of the family connections that is, in turn, determined by the institutional setting of the economy[3]. In particular, the parameter $\phi \geq 0$ measures the extent to which the institutional arrangement of the society allows the family connections to affect the profitability of the firms. Higher levels of ϕ are associated to institutional environments with stronger productivity of the connections transmitted throughout the generations, while $\phi = 0$ describes societies in which the family connections have no productive effects on the family business. Furthermore, the profitability of the family connections is subject to an erosion effect (Galor and Tsiddon, 1997; Galor and Moav, 2000; Hassler and Mora, 2000) that depends on the dynamism of the economy, identified by the growth rate of the aggregate technology $g_{t+1} = (A_{t+1} - A_t)/A_t$. In highly dynamic environments, increases in the rate of the technological progress make the transfer of the connections across generations more difficult and unproductive such to dilute the productivity of the connections established in the past periods. Increases in the growth rates of the technology, hence, make the time investment for the introduction of the heirs into the system of relations of the family less valuable and induce the entrepreneurs to invest in the heirs entrepreneurial ability. In technologically stationary economies, the productivity of the family connections is scaled by the amount of the steady state growth

[3]This comparative advantage can also be interpreted as an implicit fixed cost that the new generations of entrepreneurs need to pay to start up a new firm.

rates g^*; in economies stagnating at a low g^*, the entrepreneurs find relatively more rewarding to invest in the network of connections with respect to dynamic economies with higher level of g^*. Finally, the greater the ability of the heir, the greater the cost for the entrepreneur to invest in the connection factor rather than directly in the heir entrepreneurial ability.

Alternatively, agents can work as employees supplying on the labor market their efficiency units of general human capital h^i_{t+1} that depend on the individual innate ability and on the time investment of parents τ^h_t[4]:

$$h^i_{t+1} = a^i_{t+1}\tau^h_t \tag{7}$$

2.3 Optimization

At each time t, two types of old individuals coexist in the economy (parents born at time $t-1$); the entrepreneurs and the workers. Individuals' preferences are defined over second-period consumption and the potential income of their children; they are represented by the utility function

$$u^i_t = \gamma \ln c^i_t + (1-\gamma) \ln I^i_{t+1} \tag{8}$$

where c^i_t is the consumption of the household and I^i_{t+1} is the income of the children, which depends on the career choice that the parents make for them.

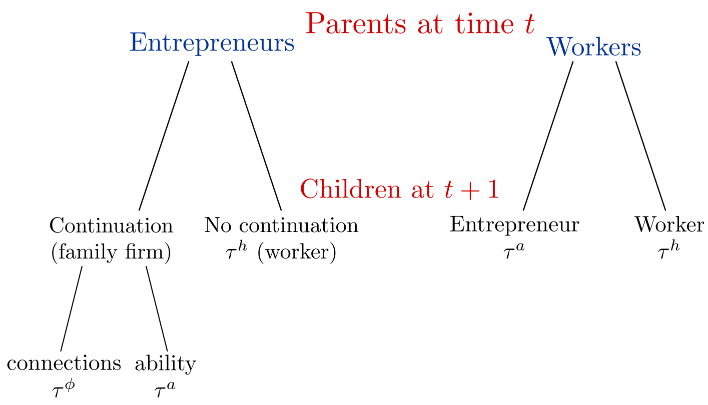
The entrepreneurs have to decide whether to retain the control of the firm within the family or not. Conditional on choosing to continue the firm within the family, they must further allocate their optimal fraction of unit time between the introduction of the heirs in a network of connections that can ensure monetary nontradeable benefits (τ^ϕ_t) and the heir entrepreneurial ability (τ^a_t). Alternatively, they can shut the firm down and invest part of their unit time in the accumulation of the general human capital of the children (τ^h_t) who, then, work for a wage.

Correspondingly, the workers can invest in the entrepreneurial ability of their descendants who set up a new firm; alternatively, they can invest in the general human capital of the children who work as employees.

Figure 1 shows the timing of the model. First, we analyze the optimal allocation of the unit time of parents between work and investment in the children. Then, the occupational choices that the parents make for their offspring are analyzed by comparing the maximum utilities received along each possible career option.

[4]More sophisticated functional forms that would allow, for instance, for the ability of the father or for decreasing marginal returns of the time investment do not change the results of the paper.

Figure 1: Timing



2.3.1 The Entrepreneurs

Continuation Conditional on choosing to continue the firm within the family, the entrepreneurs determine the income of their heirs by investing in their managerial capital; formally, the income of the heirs is given by:

$$I_{t+1}^i = \pi_{t+1}^i = \pi_{t+1} \left(m_{t+1}^i\right)^{1/\alpha} = \pi_{t+1} \left(\phi \left(1 - g_{t+1}\right) \tau_t^\phi + a_{t+1}^i \tau_t^a\right)^{1/\alpha} \tag{9}$$

where $\iota = 1$ holds. The entrepreneurs share their unit-time endowment between the accumulation of the heir managerial capital and the operation of the firm so that their budget constraint is given by

$$\pi_t^i \left(1 - \tau_t^\phi - \tau_t^a\right) = c_t^i \tag{10}$$

Substituting (9) in (8), the problem of the entrepreneurs can be restated as the choice:

$$\begin{aligned}\{\tau_t^\phi, \tau_t^a\} = \arg\max \Big\{ \gamma \ln \left[\pi_t^i \left(1 - \tau_t^\phi - \tau_t^a\right)\right] + \\ + (1-\gamma) \ln \left[\pi_{t+1} \left(\phi \left(1 - g_{t+1}\right) \tau_t^\phi + \tau_t^a a_{t+1}^i\right)^{1/\alpha}\right] \Big\}\end{aligned} \tag{11}$$

subject to

$$\tau_t^\phi + \tau_t^a \leq 1$$

The solutions are

$$
\left(\tau^{\phi}, \tau^{a}\right) = \begin{cases} \left(\frac{1-\gamma}{1-\gamma(1-\alpha)}, 0\right) & \text{if } a^{i}_{t+1} < \bar{a}_{t+1} \equiv \phi\left(1-g_{t+1}\right) \\ \\ \left(0, \frac{1-\gamma}{1-\gamma(1-\alpha)}\right) & \text{if } a^{i}_{t+1} > \bar{a}_{t+1} \equiv \phi\left(1-g_{t+1}\right) \end{cases} \tag{12}
$$

The entrepreneurs allocate always a constant fraction of their unit-time toward the introduction of the heir into the management of the firm, but the type of investment depends on the innate ability of the children. If the heir is of low ability type $\left(a^{i}_{t+1} < \bar{a}_{t+1}\right)$, parents prefer to invest this fraction of time for the presentation of the heir into the network of connections, because the time cost of "developing the managerial skills of the son" is too high. Conditional on the will to retain the firm within the family, the entrepreneurs, hence, choose to invest in a system of relations that can guarantee a minimum level of profits to the heirs, regardless of their ability. If the heirs are of high ability type $\left(a^{i}_{t+1} > \bar{a}_{t+1}\right)$, instead, the monetary cost (losses of future profits) of not training them is too high, and hence the parents invest directly in the heirs entrepreneurial ability.

The threshold level of ability that determines this choice depends on the net productivity of the connections $(\bar{a}_{t+1} \equiv \phi\left(1-g_{t+1}\right))$. If either the institutional setting does not allow the family connections to play any role in the productivity of the firm $(\phi = 0)$ or the growth rate of the economy is extremely high $(g_{t+1} > 1)$ or both and such that the net connection productivity is lower than zero $(\bar{a}_{t+1} \leq 0)$, it is never profitable for the parents to invest their time to introduce the heirs in the network of connections[5].

For any $\bar{a}_{t+1} > 0$, two cases must be distinguished. If $\bar{a}_{t+1} \in (0,1)$, namely if $g_{t+1} > \bar{\phi} = (\phi - 1)/\phi$, the maximum utility that the entrepreneurs obtain by continuing the firm within the family is described by the piecewise function υ^{e}_{e}[6]:

$$
\upsilon^{e}_{e} = \begin{cases} \delta + \gamma \ln \pi^{i}_{t} + (1-\gamma)\ln\left[\pi_{t+1}\left(\phi\left(1-g_{t+1}\right)\right)^{1/\alpha}\right] \equiv \upsilon^{e}_{\phi} & \text{if } a^{i}_{t+1} < \bar{a}_{t+1} \\ & \text{and } g_{t+1} > \bar{\phi} \\ \delta + \gamma \ln \pi^{i}_{t} + (1-\gamma)\ln\left[\pi_{t+1}\left(a^{i}_{t+1}\right)^{1/\alpha}\right] \equiv \upsilon^{e}_{a} & \text{if } a^{i}_{t+1} > \bar{a}_{t+1} \end{cases} \tag{13}
$$

where we made use of (4) and $\delta \equiv \gamma \ln \gamma + \gamma \ln \alpha + \frac{(1-\gamma)}{\alpha}\ln(1-\gamma) - \frac{(1-\gamma(1-\alpha))}{\alpha}\ln(1-\gamma(1-\alpha))$. The maximum utility of the parents is function of the type of investment in their heirs managerial capital; hence, υ^{e}_{ϕ} and υ^{e}_{a} represent the maximum utility that the entrepreneurs receive by investing respectively either in the connections or in the entrepreneurial ability of the heir, conditional on continuing the firm within the family.

[5] In this case, the model is isomorphic to the Lucas (1978) approach that we can consider as the benchmark model, following which the selection process implies that a mass of high ability individuals would be entrepreneurs, while the remaining part of the population would work as employee.

[6] The superscript indicates the type of the parent, while the subscript that of the children, with $\{e, w\}$ corresponding respectively to entrepreneurs and workers.

If $g_{t+1} < \bar{\phi}$, namely $\bar{a}_{t+1} \equiv \phi(1 - g_{t+1}) \geq 1$, it is never profitable for the parents to invest in the entrepreneurial ability of the heirs. The entrepreneurs who decide to transfer the firm within the family invest always in the connections, regardless of the ability of the heirs; hence, their indirect utility function is given by

$$\upsilon_e^e = \upsilon_\phi^e \quad \text{if } g_{t+1} < \bar{\phi} \tag{14}$$

No continuation Conditional on the entrepreneur choosing to not continue the firm within the family, the income of the children is given by the wage earned on the labor market, $I_{t+1}^i = w_{t+1}h_{t+1}^i$, which is determined by the equilibrium wage rate w_{t+1} and the amount of efficiency units of human capital supplied h_{t+1}^i. Parents, hence, have to choose how to allocate their time between the operation of the firm and the accumulation of the general human capital of the children, so that the budget constraint is given by:

$$\pi_t^i\left(1 - \tau_t^h\right) = c_t^i \tag{15}$$

The problem of the entrepreneur can be restated as the choice of τ_t^h:

$$\tau_t^h = \arg\max\left\{\gamma \ln\left[\pi_t^i\left(1 - \tau_t^h\right)\right] + (1-\gamma)\ln\left(w_{t+1}a_{t+1}^i\tau_t^h\right)\right\} \tag{16}$$

subject to

$$\tau_t^h \leq 1$$

The solution is $\tau^h = 1 - \gamma$ and the indirect utility function is given by:

$$\upsilon_w^e = \eta + \gamma \ln \pi_t^i + (1-\gamma)\ln\left(w_{t+1}a_{t+1}^i\right) \tag{17}$$

where $\eta \equiv \gamma \ln \gamma + (1-\gamma)\ln(1-\gamma)$ and υ_w^e represents the maximum utility that the entrepreneurs obtain by shutting the firm down and choosing for their descendants the employment sector.

2.3.2 The workers

At each time t, workers choose the type of education, and hence the occupation, of their children, selecting between either a general or a managerial education. The problem of the parent who decides to provide the children an amount of efficiency units of general human capital is defined as the choice of τ_t^h:

$$\tau_t^h = \arg\max \left\{ \gamma \ln c_t^i + (1-\gamma) \ln \left(w_{t+1} a_{t+1}^i \tau_t^h \right) \right\} \tag{18}$$

subject to

$$w_t^i \left(1 - \tau_t^h\right) = c_t^i$$
$$\tau_t^h \leq 1$$

The solution is $\tau^h = 1 - \gamma$ and the associated indirect utility function is given by:

$$\upsilon_w^w = \eta + \gamma \ln w_t^i + (1-\gamma) \ln \left(w_{t+1} a_{t+1}^i \right) \tag{19}$$

where υ_w^w represents the maximum utility of the workers when their descendants work also for a wage.

Correspondingly, conditional on selecting a managerial career for their offspring, parents choose the optimal amount of unit time to invest in the children entrepreneurial ability; applying $\iota = 0$ in (6), the problem of the parents can be formulated as the choice:

$$\tau_t^a = \arg\max \left\{ \gamma \ln c_t^i + (1-\gamma) \ln \left[\pi_{t+1} \left(a_{t+1}^i \tau_t^a \right)^{1/\alpha} \right] \right\} \tag{20}$$

subject to

$$w_t^i \left(1 - \tau_t^a\right) = c_t^i$$
$$\tau_t^a \leq 1$$

The solution is $\tau^a = \frac{1-\gamma}{1-\gamma(1-\alpha)}$, with the maximum utility given by

$$\upsilon_e^w = \delta + \gamma \ln w_{i,t} + (1-\gamma) \ln \left[\pi_{t+1} \left(a_{t+1}^i \right)^{1/\alpha} \right] \tag{21}$$

where υ_e^w represents the maximum utility that the workers obtain by choosing to invest in the entrepreneurial ability of their descendants in order to allow them to start new firms.

2.4 Occupational choice

The occupational choice that the parents make for their offspring is regulated by the extent to which the institutional setting of the society allows the connections established by the generations of entrepreneurs to be profitable for the family businesses. Throughout, we distinguish two societies according to the degree of the parameter ϕ as follows:

Definition 1. *Let's define:*

- *Entrepreneurial society: an economy in which family connections are not extremely productive; specifically, a society in which the institutional setting is described by $\phi > 0$ such that $\bar{a}_{t+1} \equiv \phi(1 - g_{t+1}) \in (0, 1)$, or:*

$$g_{t+1} > \bar{\phi} = \frac{\phi - 1}{\phi}$$

- *Crony society: an economy in which family connections are extremely productive; specifically, a society in which the institutional setting is described by $\phi > 0$ high enough and such that $\bar{a}_{t+1} \equiv \phi(1 - g_{t+1}) \geq 1$, or:*

$$g_{t+1} \leq \bar{\phi} = \frac{\phi - 1}{\phi}$$

2.4.1 Entrepreneurial society

The entrepreneurs The entrepreneurs compare the two indirect utility functions υ_e^e (13) and υ_w^e (17), choosing to continue the firm within the family if $\upsilon_e^e \geq \upsilon_w^e$ and, conversely, to shut it down if $\upsilon_e^e \leq \upsilon_w^o$. Eqs. (13) and (17) identify two threshold levels of ability $a_{t+1}^{\phi} \equiv a_{t+1}^i : \upsilon_{\phi}^e = \upsilon_w^e$ and $a_{t+1}^a \equiv a_{t+1}^i : \upsilon_a^e = \upsilon_w^e$ such that for any $a_{t+1}^i < (>)\, a_{t+1}^{\phi} \Rightarrow \upsilon_{\phi}^e > (<)\, \upsilon_w^e$ and for any $a_{t+1}^i > (<)\, a_{t+1}^a \Rightarrow \upsilon_a^e > (<)\, \upsilon_w^e$; using (5), (13) and (17), a_{t+1}^{ϕ} and a_{t+1}^a are defined as

$$a_{t+1}^{\phi} = \theta \bar{a}_{t+1}^{1/\alpha} \left(\frac{A_{t+1}}{w_{t+1}} \right)^{1/\alpha} \tag{22}$$

and

$$a_{t+1}^a = \left(\frac{w_{t+1}}{\theta^{\alpha} A_{t+1}} \right)^{\frac{1}{1-\alpha}} \tag{23}$$

where θ is a collection of parameters[7]. The thresholds in (22) and (23), and hence the equilibrium choices, depend on the expected wage (w_{t+1}) that the entrepreneurs require to shut the firm down and to choose the employment sector for their children; higher equilibrium wage rates are associated, *ceteris paribus*, with lower a_{t+1}^{ϕ} and higher a_{t+1}^a, implying weaker incentives of the entrepreneurs to transmit the firm within the family due

[7] Appendix C contains all the collection of parameters of the model.

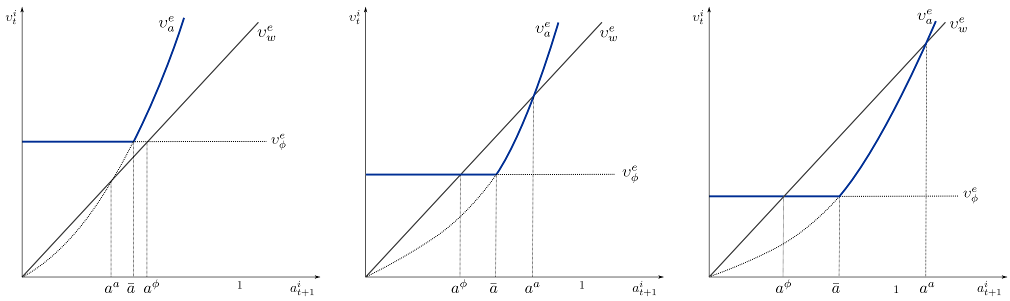
to the stronger outside option of the employment sector. Proposition 1 collects the three possible configurations of the equilibrium choices that are also graphically represented in Figure 2[8].

Proposition 1 (Equilibrium choice)**.** *If $g_{t+1} > \bar{\phi}$, there exist two thresholds of the equilibrium wage rate $\hat{w}$ and $\tilde{w}$, with $\hat{w} < \tilde{w}$, such that the equilibrium choices of the entrepreneurs are defined by the following possible configurations:*

1. *No-Exit. For any $w_{t+1} \leq \hat{w}$: $\upsilon^e_\phi > \upsilon^e_w$ and $\upsilon^e_\phi > \upsilon^e_a$ contemporaneously hold if $a^i_{t+1} < \bar{a}_{t+1}$, while $\upsilon^e_a > \upsilon^e_w$ and $\upsilon^e_a > \upsilon^e_\phi$ contemporaneously hold if $a^i_{t+1} > \bar{a}_{t+1}$.*
2. *Exit and polarization. For any $w_{t+1} \in (\hat{w}, \tilde{w})$: $\upsilon^e_\phi > \upsilon^e_w > \upsilon^e_a$ if $a^i_{t+1} < a^\phi_{t+1}$, $\upsilon^e_a > \upsilon^e_w > \upsilon^e_\phi$ if $a^i_{t+1} > a^a_{t+1}$, while $\upsilon^e_w > \upsilon^e_\phi$ and $\upsilon^e_w > \upsilon^e_a$ contemporaneously hold for $a^i_{t+1} \in (a^\phi_{t+1}, a^a_{t+1})$.*
3. *Exit. For any $w_{t+1} > \tilde{w}$: $\upsilon^e_\phi > (<)\, \upsilon^e_w$ if $a^i_{t+1} < (>)\, a^\phi_{t+1}$.*

Proof. It results from the comparison of the maximum utilities in (13) and (17). □

Figure 2: Equilibrium choice: entrepreneurs



(a) No-exit **(b)** Exit and polarization **(c)** Exit

The wage w_{t+1}, to be endogenously determined as an equilibrium outcome of the labor market, shapes the trade-off of the parents in choosing the career path of their descendants. If it is low enough (i.e. $w_{t+1} \leq \hat{w}$[9]), entrepreneurs have never the incentive to shut the firms down, and hence the control of all the firms is retained within the family; the outside option of working for a wage is, indeed, always dominated by the potential profits that the heirs can gain by continuing the management of the family business. As a consequence, the distribution of the abilities of the heirs across the family firms depends only on the net profitability of the family connections, $\bar{a}_{t+1}$ (Fig. 2a). The heirs with an ability level lower than $\bar{a}_{t+1}$ continue operating the family businesses taking advantage of the system

[8]In order to make the solutions clearer, Figure 2 depict the following monotonic transformation of the indirect utility functions: $\tilde{\upsilon}^o_\phi = \theta \bar{a}_{t+1}^{\frac{1}{\alpha}} \left(A_{t+1} w_{t+1}^{-1}\right)^{\frac{1}{\alpha}}$, $\tilde{\upsilon}^o_a = \theta \left(A_{t+1} w_{t+1}^{-1} a^i_{t+1}\right)^{\frac{1}{\alpha}}$, $\tilde{\upsilon}^o_w = a^i_{t+1}$. These are obtained, dividing (13) and (17) by $(1-\gamma)\, w_{t+1} \left(\pi^i_t \gamma\right)^\gamma$ and taking all to the power of $1/(1-\gamma)$.

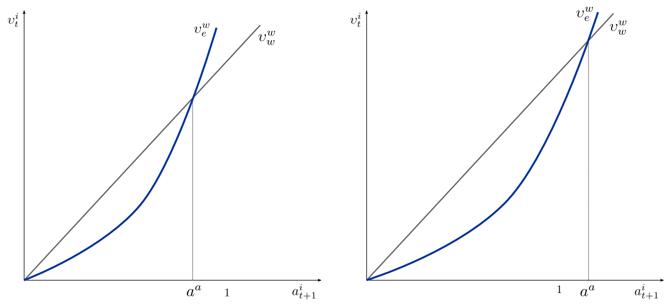
[9]The threshold $\hat{w} = \theta^\alpha A_{t+1} \bar{a}_{t+1}^{1-\alpha}$ is defined as that wage $\hat{w} \equiv w_{t+1} : a^\phi_{t+1} = \bar{a}_{t+1} = a^a_{t+1}$, with $a^a_{t+1} < \bar{a}_{t+1} < a^\phi_{t+1}$ if $w_{t+1} < \hat{w}$ and $a^\phi_{t+1} < \bar{a}_{t+1} < a^a_{t+1}$ if $w_{t+1} > \hat{w}$.

of relations inherited by their parents, while those with an ability level higher than $\bar{a}_{t+1}$ receive an entrepreneurial education that they employ in the management of the family firms.

For higher values of the wage (i.e. $w_{t+1} > \hat{w}$), the exit from the family business becomes a rewarding option. In particular, for wage rates not extremely high (i.e. $w_{t+1} \in (\hat{w}, \tilde{w})$[10]), a polarization of the family firms, in terms of the ability of the managing heirs, does emerge (Fig. 2b); heirs with an ability level lower than a^{ϕ}_{t+1} as well as heirs with an ability level higher than a^{a}_{t+1} continue operating the family business, while those with an ability level $a^{i}_{t+1} \in (a^{\phi}_{t+1}, a^{a}_{t+1})$ leave the family business to work for a wage. In this case, indeed, low ability agents ($a^{i}_{t+1} < a^{\phi}_{t+1}$) earn a wage on the labor market lower than the profits they gain as managers of the family firms due to the private benefits accruing from the network of connections inherited by their parents; conversely, high ability heirs ($a^{i}_{t+1} > a^{a}_{t+1}$) are selected by their parents to continue the family business since the potential profits they can generate by using their entrepreneurial ability are greater than the wage earned as employees. If the wage is high enough (i.e. $w_{t+1} > \tilde{w}$), even the individuals in the upper tail of the ability distribution leave the family business to join the employment sector (Fig. 2c) so that all the family firms are managed by the low ability heirs through the system of the family relations.

The workers Old generations of workers compare the indirect utility functions υ^{w}_{w} (19) and υ^{w}_{e} (21), choosing for their offspring the entrepreneurial education rather than the general human capital if $\upsilon^{w}_{e} \geq \upsilon^{w}_{w}$, and otherwise if $\upsilon^{w}_{e} \leq \upsilon^{w}_{w}$. Since workers have no access to the connections of the dynasties of entrepreneurs, their descendants can set up a new firm and become entrepreneurs only if their innate ability is high enough. As follows from (19) and (21), $\upsilon^{w}_{e} \geq (\leq)\, \upsilon^{w}_{w}$ if $a^{i}_{t+1} \geq (\leq)\, a^{a}_{t+1}$ so that the offspring with an ability level higher than a^{a}_{t+1} can become entrepreneurs, while those with an ability level lower than a^{a}_{t+1} receive a general human capital and work for a wage (Fig. 3).

Figure 3: Equilibrium choice: workers



(a) Entry **(b)** No entry

The effective creation of new firms, and hence the effective entry of new entrepreneurs in the market, depends on the equilibrium wage rate w_{t+1}. If $w_{t+1} < \tilde{w}$ (i.e. $a^{a}_{t+1} < 1$),

[10]The threshold $\tilde{w} = \theta^{\alpha} A_{t+1}$ is defined as that wage $\tilde{w} \equiv w : a^{a}_{t+1} = 1$, and $a^{a}_{t+1} < (>)\, 1$ if $w_{t+1} < (>)\, \tilde{w}$.

the highest ability descendants of workers receive a managerial education and become entrepreneurs by starting new businesses (Fig. 3a); otherwise, if $w_{t+1} \geq \tilde{w}$, all the offspring of the workers work as employees since the wage earned on the labor market is greater than the profits generated by operating new firms (Fig. 3b).

2.4.2 Crony society

If $g_{t+1} < \bar{\phi}$, namely $\bar{a}_{t+1} \equiv \phi\,(1 - g_{t+1}) \geq 1$, it is never profitable for the entrepreneurs to transfer the control of the firm within the family by investing in their heirs entrepreneurial ability; given that $\upsilon_{\phi}^{e} > \upsilon_{a}^{e}$ holds throughout the ability distribution of the young individuals, the indirect utility functions governing the equilibrium choices are only υ_{ϕ}^{e} (14) and υ_{w}^{e} (17). If the wage is low enough (i.e. for $w_{t+1} \leq \hat{\hat{w}}$[11]) there is never exit from the family business as $\upsilon_{\phi}^{e} > \upsilon_{w}^{e}$ holds for any a_{t+1}^{i}; all the heirs, regardless of their innate ability, retain the control of the family firms due to the network of connections inherited by the parents. Otherwise, if the wage is high enough (i.e. for $w_{t+1} > \hat{\hat{w}}$), the exit from the family business becomes a rewarding option for some of the highest ability individuals as $\upsilon_{\phi}^{e} > (\leq)\, \upsilon_{w}^{e}$ if $a_{t+1}^{i} < (\geq)\, a_{t+1}^{\phi}$. In this case, indeed, assigning the control of the family firms to all the heirs with an ability level lower than a_{t+1}^{ϕ} ensures a payoff greater than that guaranteed by the employment sector due to the private benefits accruing from the network of connections; only if the children are of high ability type (i.e. $a_{t+1}^{i} \geq a_{t+1}^{\phi}$), it becomes worthwhile for them to work for a wage and hence to invest in their general human capital.

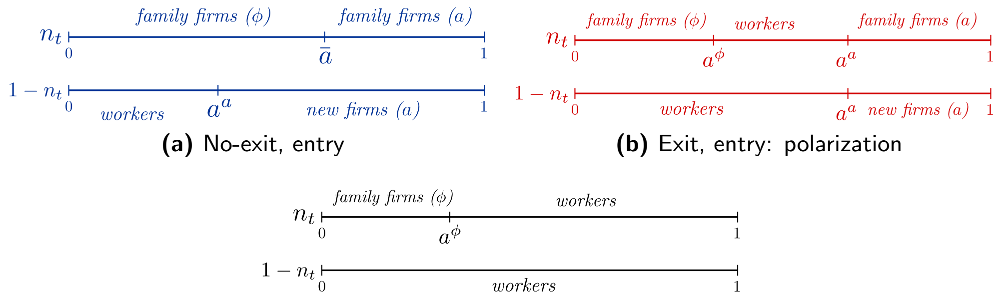
Correspondingly, workers choose to offer to their offspring the managerial education, and hence to select for them the entrepreneurial career, only if the wage is low enough to guarantee that starting a new business is more profitable than working for a wage; formally, if $w_{t+1} < \tilde{w}$ (i.e. $a_{t+1}^{a} < 1$), (19) and (21) imply that $\upsilon_{e}^{w} \geq (\leq)\, \upsilon_{w}^{w}$ if $a_{t+1}^{i} \geq (\leq)\, a_{t+1}^{a}$. Otherwise (i.e. for $w_{t+1} \geq \tilde{w}$), it is never profitable to opt for the entrepreneurial sector since $\upsilon_{e}^{w} < \upsilon_{w}^{w}$ for any a_{t+1}^{i}.

3 Macroeconomic equilibrium

The aggregate demand and supply of human capital as well as the aggregate managerial capital, and therefore the output per capita, are endogenously determined by the occupational choices that parents make for their descendants. To begin with, Figures 4 and 5 offer an aggregate representation of the distribution of (abilities across) firms and workers as implied by the occupational choices of any given initial mass of entrepreneurs (n_t) and workers ($1 - n_t$).

[11] The threshold $\hat{\hat{w}} = \theta^{\alpha} A_{t+1} \bar{a}_{t+1}$ is defined as that wage $\hat{\hat{w}} \equiv w : a_{t+1}^{\phi} = 1$, and $a_{t+1}^{\phi} < (\geq)\, 1$ if $w_{t+1} > (\leq)\, \hat{\hat{w}}$, with $\hat{\hat{w}} > \tilde{w}$. Notice also that $\tilde{w} < \hat{w} < \hat{\hat{w}}$ holds for $\bar{a}_{t+1} \geq 1$, while $\hat{\hat{w}} < \hat{w} < \tilde{w}$ if $\bar{a}_{t+1} \in (0, 1)$.

Figure 4: Distribution of abilities across firms and workers: entrepreneurial society



(a) No-exit, entry

(b) Exit, entry: polarization

(c) Exit, no-entry

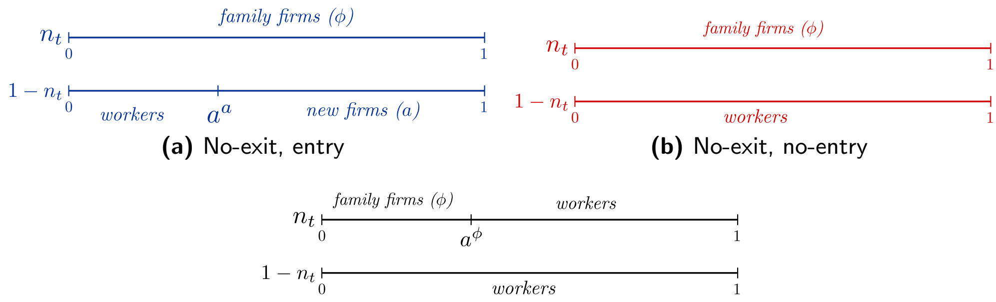
For any expected equilibrium wage rate low enough[12], there is never exit from the family business as well as there is some entry of newly founded firms both in the entrepreneurial and in the crony society (Fig. 4a and 5a). While in both societies there is an analogous distribution of the descendants of the workers, as only the highest ability individuals become entrepreneurs (i.e. those with $a^i_{t+1} > a^a_{t+1}$), the sorting of the firms heirs is affected by the different weight of the family connections. Due to the strong impact of the net connection productivity, in the crony society all the heirs retain the control of the family business exploiting the connections inherited from the parents; conversely, the entrepreneurial society stimulates some of the highest ability heirs to accumulate entrepreneurial skills in order to continue operating the family business (i.e. those with $a^i_{t+1} > \bar{a}_{t+1}$). As the expected equilibrium wage rate increases, a clear differentiation emerges in the distribution of the abilities and of the type of firms across the two societies (Fig. 4b and 5b). In the entrepreneurial society, higher wage rates (i.e. $w_{t+1} \in (\hat{w}, \tilde{w})$) induce some exit from the family businesses causing a polarization in the abilities of the managing heirs. As Figure 4b illustrates, both heirs with an ability level lower than a^ϕ_{t+1} and higher than a^a_{t+1} continue operating the family firms, exploiting respectively the connections inherited from their parents and their entrepreneurial ability, while those with intermediate innate ability (i.e. $a^i_{t+1} \in (a^\phi_{t+1}, a^a_{t+1})$) exit from the family business to join the employment sector. Furthermore, the entry in the entrepreneurial sector is still a valuable option for the highest ability offspring of the workers, despite the greater wages. On the other side, the crony society converts into a fully immobile economy[13], with no exit of the heirs from the family businesses and no entry in the entrepreneurial sector (Fig. 5b). Finally, for equilibrium wage rates high enough[14], the two types of societies turn back to similar distribution structures (Fig. 4c and 5c), according to which not only the descendants of the workers decide to not start new firms, but also the mass of high ability firms heirs exit from the family businesses; as a consequence, in both the economies only family firms persist and they are all managed by the low ability heirs.

[12]For any $w_{t+1} < \hat{w}$ in the entrepreneurial society and for any $w_{t+1} < \tilde{w}$ in the crony one (see footnote 11).

[13]For any equilibrium wage rate for $w_{t+1} \in (\tilde{w}, \hat{\hat{w}})$.

[14]For any $w_{t+1} > \tilde{w}$ in the entrepreneurial society and for any $w_{t+1} > \hat{\hat{w}}$ in the crony one.

Figure 5: Distribution of abilities across firms and workers: crony society



(a) No-exit, entry **(b)** No-exit, no-entry **(c)** Exit, no-entry

The *competitive equilibrium* of the economy is, thus, defined by a wage rate w_{t+1} and an allocation of individuals between work and entrepreneurship such that at any time t: for any expected wage w_{t+1}, a) parents choose optimally the occupations of their descendants (Section 2.4); b) firms maximize profits (4); and c) the labor market - the only market of this economy - clears. Using (3) and (7) and integrating the optimal choices of each type of parent (entrepreneur, worker) over the ability distribution of the young individuals (a^i_{t+1}), the aggregate supply, H^S_{t+1}, and demand of human capital, H^D_{t+1}, are the followings:

$$H^S_{t+1}(w_{t+1}) = \int_{\{\mathcal{W}\}} h^i_{t+1} da^i_{t+1} = \int_{\{\mathcal{W}\}} a^i_{t+1} \tau^h da^i_{t+1} \tag{24}$$

and

$$H^D_{t+1}(w_{t+1}) = \int_{\{\mathcal{N}\}} H^i_{t+1} da^i_{t+1} = \int_{\{\mathcal{N}\}} \left(\frac{(1-\alpha) A_{t+1} m^i_{t+1}}{w_{t+1}} \right)^{1/\alpha} da^i_{t+1} \tag{25}$$

where $\{\mathcal{N}\}$ and $\{\mathcal{W}\}$ are the relevant sets of firms and workers at time $t+1$ implied by the occupational choices of the parents at time t.

Proposition 2 (Competitive equilibrium)**.** *For any possible n_t, it exists a unique competitive equilibrium defined by the tuple $\left\{w_{t+1}, H^S_{t+1}, H^D_{t+1}\right\}$ such that*

$$H^S_{t+1}(w_{t+1}) = H^D_{t+1}(w_{t+1})$$

The equilibrium wage rate w_{t+1} is a monotonically increasing function of the proportion of firms n_t; $w'_{t+1}(n_t) > 0$.

Proof. See Appendix A for the complete characterization of the equilibrium. □

The equilibrium wage rate w_{t+1} depends on the size and the composition of the two sectors. The entrepreneurial sector, whose size is equal to the proportion of firms n_{t+1}, is composed of a fraction of pre-existing firms transmitted across generations of entrepreneurs (family firms) and, for the remaining part, of new firms established by the descendants of the workers (non-family firms). Likewise, the employment sector, of size $1 - n_{t+1}$, is composed by the individuals of the generation $t+1$, whose parents were either entrepreneurs or workers. Hence, the equilibrium wage rate for the generation $t+1$ depends, *ceteris paribus*, also on the allocation of the parents between work and entrepreneurship at time t, as reflected in the proportion of firms n_t. This correspondence leads to the following Corollary

Corollary 1. *At each point in time t and for each type of society, there exist two threshold levels of the proportion of firms n_t such that*

- *Entrepreneurial society:*

$$w_{t+1} \begin{cases} \leq \hat{w} & \text{if } n_t \leq \hat{n}_t \\ \in (\hat{w}, \tilde{w}) & \text{if } n_t \in (\hat{n}_t, \tilde{n}_t) \\ \geq \tilde{w} & \text{if } n_t \geq \tilde{n}_t \end{cases}$$

- *Crony society:*

$$w_{t+1} \begin{cases} \leq \tilde{w} & \text{if } n_t \leq \underset{\sim}{n}_t \\ \in \left(\tilde{w}, \hat{\hat{w}}\right) & \text{if } n_t \in \left(\underset{\sim}{n}_t, \hat{\hat{n}}\right) \\ \geq \hat{\hat{w}} & \text{if } n_t \geq \hat{\hat{n}} \end{cases}$$

Proof. See Appendix A. □

At the competitive equilibrium, the per capita output of the economy is determined by the allocation of the individuals across the two occupations and by the growth rate of the technology as follows

$$Y_{t+1} = \int_{\{\mathcal{N}\}} y^i_{t+1} da^i_{t+1} = \Pi_{t+1} + w_{t+1} H^S_{t+1} \tag{26}$$

The output per capita, obtained by integrating the firm level productions over all the firms of the economy (26), is fully exhausted between the profits of the entrepreneurs, $\Pi_{t+1} \equiv \int_{\{\mathcal{N}\}} \pi^i_{t+1} da^i_{t+1}$, and the wages of the workers, $w_{t+1} H^S_{t+1}$. It is a function of the proportion of the firms and of the distribution of the entrepreneurial skills of the economy as well as of the growth rate of the technology; substituting (3) in (1), it follows from (26) that

$$Y_{t+1} = (1-\alpha)^{\frac{1-\alpha}{\alpha}} \left(\frac{A_{t+1}}{w^{1-\alpha}_{t+1}} \right)^{\frac{1}{\alpha}} \int_{\{\mathcal{N}\}} \left(m^i_{t+1} \right)^{\frac{1}{\alpha}} da^i_{t+1} \tag{27}$$

From Corollary 1 and using the optimal occupational choices established in Section 2.4, the output per capita can finally by written as

$$Y^E_{t+1} = \begin{cases} \frac{\theta_8}{1+\alpha}\left(\frac{A_{t+1}}{w^{1-\alpha}_{t+1}}\right)^{\frac{1}{\alpha}}\left\{n_t \bar{a}^{\frac{1+\alpha}{\alpha}}_{t+1} + \alpha\left[1-(1-n_t)\left(a^a_{t+1}\right)^{\frac{1+\alpha}{\alpha}}\right]\right\} & \text{if } n_t \leq \hat{n}_t \\ \theta_8\left(\frac{A_{t+1}}{w^{1-\alpha}_{t+1}}\right)^{\frac{1}{\alpha}}\left\{n_t a^{\phi}_{t+1}\bar{a}^{\frac{1}{\alpha}}_{t+1} + \frac{\alpha}{1+\alpha}\left[1-\left(a^a_{t+1}\right)^{\frac{1+\alpha}{\alpha}}\right]\right\} & \text{if } n_t \in (\hat{n}_t, \tilde{n}_t) \\ \theta_8\left(\frac{A_{t+1}}{w^{1-\alpha}_{t+1}}\right)^{\frac{1}{\alpha}}\left[n_t a^{\phi}_{t+1}\bar{a}^{\frac{1}{\alpha}}_{t+1}\right] & \text{if } n_t \geq \tilde{n}_t \end{cases} \tag{28}$$

if $g_{t+1} > \bar{\phi}$ (entrepreneurial society), and

$$Y^C_{t+1} = \begin{cases} \theta_8\left(\frac{A_{t+1}}{w^{1-\alpha}_{t+1}}\right)^{\frac{1}{\alpha}}\left\{n_t \bar{a}^{\frac{1}{\alpha}}_{t+1} + \frac{\alpha}{1+\alpha}(1-n_t)\left[1-\left(a^a_{t+1}\right)^{\frac{1+\alpha}{\alpha}}\right]\right\} & \text{if } n_t \leq \underset{\sim}{n}_t \\ \theta_8\left(\frac{A_{t+1}}{w^{1-\alpha}_{t+1}}\right)^{\frac{1}{\alpha}}\left[n_t \bar{a}^{\frac{1}{\alpha}}_{t+1}\right] & \text{if } n_t \in \left(\underset{\sim}{n}_t, \hat{\hat{n}}\right) \\ \theta_8\left(\frac{A_{t+1}}{w^{1-\alpha}_{t+1}}\right)^{\frac{1}{\alpha}}\left[n_t a^{\phi}_{t+1}\bar{a}^{\frac{1}{\alpha}}_{t+1}\right] & \text{if } n_t \geq \hat{\hat{n}} \end{cases} \tag{29}$$

if $g_{t+1} < \bar{\phi}$ (crony society), where θ_8 is a collection of parameters. Eqs. (28) and (29) clarify the contributions of the allocation of the entrepreneurial skills and of the growth rate of the technology to the output per capita. The effect of the distribution of the abilities of the entrepreneurs who manage the proportion of firms n_{t+1} of the economy is sensitive to the incentive of the past generations of entrepreneurs to transmit the firm within the family to their heirs and, hence, to the weight of the family connections. Furthermore, changes in the growth rate of the technology affect the output per capita through two channels. On a side increases in the rate of technological progress shift directly the level of the technology A_{t+1}. On the other side, they affect the output per capita by influencing the composition of the entrepreneurial sector through the erosion effect.

4 Dynamics

4.1 Exogenous growth

In this section, we analyze the long-run evolution of the economy. As follows from (28) and (29), the evolution of the per capita output is governed by the changes in the size and in the composition of the entrepreneurial sector as well as by the changes in the rate of the technological progress. In order to stress the role of the family connections on the distribution of the entrepreneurial skills of the economy and, thus, to characterize the composition of the family firms of the society, the analysis is initially conducted under the assumption that technological change is exogenous. If the growth rate of the aggregate technology is stationary at its long-run level $g_{t+1} = g^*$, the time path of the per-capita output $\{Y_t\}_{t=0}^{\infty}$ depends only on the dynamics of the proportion of the firms of the economy, $\{n_t\}_{t=0}^{\infty}$, which is determined by the occupational choice that the parents make for their descendants[15].

4.1.1 Entrepreneurial society

If $g^* > \bar{\phi}$, the dynamical system governing the evolution of the size and the composition of the entrepreneurial sector is the following:

$$n_{t+1} = \begin{cases} 1 - a^{a}_{t+1}(1 - n_t) \equiv n^E_1(n_t) & \text{if } n_t \leq \hat{n}_t \\ n_t a^{\phi}_{t+1} + 1 - a^{a}_{t+1} \equiv n^E_2(n_t) & \text{if } n_t \in (\hat{n}_t, \tilde{n}_t) \\ n_t a^{\phi}_{t+1} \equiv n^E_3(n_t) & \text{if } n_t \geq \tilde{n}_t \end{cases} \tag{30}$$

The total number of firms of the economy at each time $t + 1$, n_{t+1}, derives from the decision of the parents to allocate their descendants between entrepreneurship and work (see Fig. 4). The mass of firms is composed of the pre-existing firms that the entrepreneurs choose to transfer to their heirs and of the new firms established by the individuals born from parents who were workers at time t. The distribution of the entrepreneurial skills over the mass of firms of the economy depends on the incentive of the entrepreneurs, who choose to continue the firms within the family, to invest either in the introduction of their heirs in the system of relations that the family firms built over the generations or in the heirs entrepreneurial skills. Using Corollary 1, if $n_t \leq \hat{n}_t$ the total number of firms at time $t+1$ is described by the law of motion in $n^E_1(n_t)$, which is determined by the mass of family firms, n_t, that are distributed for a fraction $n_t\bar{a}$ to the heirs who exploit the connections of the family and for a fraction $n_t(1 - \bar{a})$ to the heirs that accumulate entrepreneurial skills. Otherwise, if $n_t \geq \hat{n}_t$, not all the businesses are continued within the family. In particular, for any $n_t \in (\hat{n}_t, \tilde{n}_t)$ the dynamics of the total number of firms of the economy are described by the function $n^E_2(n_t)$ since only the fractions $n_t a^{\phi}_{t+1}$ and $n_t\left(1 - a^{a}_{t+1}\right)$ of family firms are continued by the heirs; the fraction $n_t a^{\phi}_{t+1}$ are managed by the lowest ability heirs that exploit the system of relations built up by their parents, while the remaining part are managed by the highest ability heirs who employ their entrepreneurial skills in the operation of the family businesses. In addition, as long as $n_t \leq \tilde{n}_t$ the total number of firms is increased with the constitution of new firms of the descendants of workers for a proportion equal to $(1 - n_t)\left(1 - a^{a}_{t+1}\right)$. If, instead, $n_t \geq \tilde{n}_t$, there is no entry of new entrepreneurs in the market and all the firms of the economy at time $t + 1$ are businesses continued within the family by the lowest ability heirs $n_t a^{\phi}_{t+1}$ such that the dynamics of the size and the composition of the firms of the economy are described by the function

[15] As shown below, the initial distribution of the individuals across the two sectors does not affect the steady state level distribution of firms, entrepreneurial skills and output per capita.

$n_3^E(n_t)$.

Lemma 1 (Properties of the dynamical system). *If $g^* > \bar{\phi}$, the dynamical system in* (30) *is characterized by the following properties:*

1. $n_1^E(0) > 0$; $n_1^E(\hat{n}_t) > \hat{n}_t$; $n_1^{E'}(n_t) > 0$; $n_1^{E''}(n_t) > 0$.
2. $n_2^E(\hat{n}_t) = n_1^E(\hat{n}_t) > \hat{n}_t$; $n_2^E(\tilde{n}_t) = n_3^E(\tilde{n}_t) < \tilde{n}_t$; $n_2^{E'}(n_t) > 0$.
3. $n_3^{E'}(n_t) > 0$; $n_3^{E''}(n_t) < 0$; $n_3^E(1) < 1$.

Proof. See Appendix B. □

Proposition 3. *If $g^* > \bar{\phi}$, it exists a unique globally stable steady state number of firms*

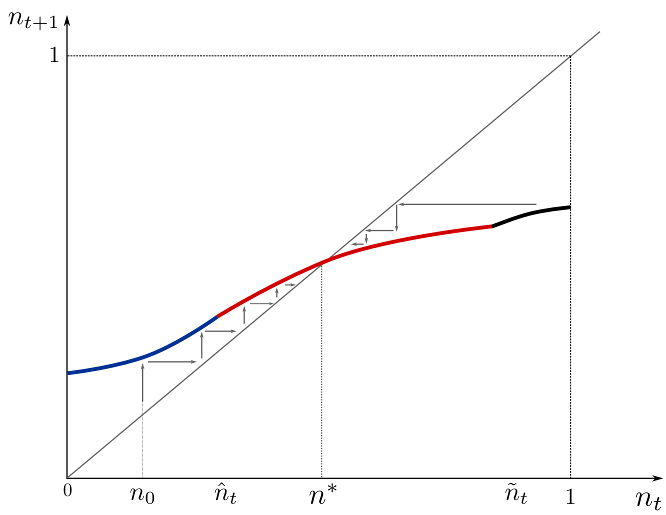
$$n^* = \frac{1 - a^a}{1 - a^\phi}$$

At the steady state, the economy features positive entry and exit of firms.

Proof. It follows from Lemma 1; see also Appendix B. □

The long-run equilibrium is marked by a positive rate of mobility across the sectors and the polarization in the distribution of the abilities of the managing heirs (Fig. 4b). In the steady state equilibrium, a proportion $n^*\left(a^a - a^\phi\right)$ of family firms is shut down so that the descendants of the generations of entrepreneurs join the employment sector, while a proportion $(1 - n^*)(1 - a^a)$ of individuals, born from parents who were workers, enters the entrepreneurial sector by setting up new firms. The surviving family firms are operated by the proportions $n^* a^\phi$ of the lowest and $n^*(1 - a^a)$ of the highest ability heirs.

Figure 6: Dynamics: entrepreneurial society



Starting, for instance, from a small proportion of firms n_0, the expected equilibrium wage rate w_{t+1} would be low enough such that current generations of entrepreneurs would choose to not shut their firms down but to bequeath them to their heirs; likewise, the highest ability descendants of the workers would set up new firms as the expected profits are greater than the low wage rate possibly earned on the market (Fig. 4a). As a consequence, the number of firms increases over time (Fig. 6), implying also an increase in the expected wage rate; as follows from (22) and (23), the increases in the expected wage rate do affect the distribution of the operating entrepreneurs since they bring forth, respectively, a decrease in a^{ϕ}_{t+1} and an increase in a^{a}_{t+1}. These changes in the size and the composition of the firms of the economy continue until the number of firms, and hence the expected wage rate, increases enough to induce a proportion of the entrepreneurs to not transfer the control of the family firms to their heirs (Fig. 2b). At this point, the economy set in the stage described by Figure 4b when mobility out of the family businesses starts and continues until the steady state n^* is reached; in this equilibrium, the wage rate w^* is such to exactly compensate the incentives of entry and exit from the entrepreneurial sector.

Corollary 2. *If $g^* > \bar{\phi}$, it exists a unique wage rate w^* that satisfy the competitive equilibrium at the steady state $n_t = n^*$.*

Proof. See Appendix B. □

For a stationary rate of technological progress, the stability of the long-run equilibrium identified by the couple $\{n^*, w^*\}$ is guaranteed by the fact that the distribution of the abilities converges to stationarity since the occupational choices of the parents generation, and hence the thresholds a^{ϕ} and a^{a}, are unaffected by the changes in the *level* of the technological progress A_{t+1}. Indeed, as the wage rate is increasing, with a unitary elasticity (ε^A_w), in the level of the technological progress, it follows from (5) that also the profits per-efficiency units of managerial capital are increasing, at unitary rate, in A_{t+1}.

Lemma 2.

$$\frac{dw^*}{dA_{t+1}} > 0$$
$$\varepsilon^A_w = \frac{dw^*}{dA_{t+1}} \frac{A_{t+1}}{w^*} = 1$$

Proof. See Appendix B. □

Lemma 3. *For a stationary rate of technological progress $g_t = g^*$, the thresholds a^{ϕ} and a^{a} are independent of the level of the technological progress A_{t+1}*

$$\frac{\partial a^{\phi}}{\partial A_{t+1}} = 0$$
$$\frac{\partial a^{a}}{\partial A_{t+1}} = 0$$

Proof. See Appendix B. □

Consistently with the empirical evidence reported in Section 1, it is noticeable that, in the steady state equilibrium, there is no difference in the firm level performance of the family and non-family firms operated by entrepreneurs at the top of the ability distribution as both the heirs and the descendants of the workers accumulate the same level of entrepreneurial skills. Instead, at aggregate level, the lower average performance of the family firms, with respect to the non-family ones, is driven entirely by the distortions generated by the private benefits of the family connections that induce part of the entrepreneurs to transfer the control of the family businesses also to some of the low ability heirs without investing in their entrepreneurial skills. Pérez-González (2006), for instance, shows that, on average, firms with a family-descendant CEO have lower operating returns than firms which promote unrelated CEOs, but that such a disadvantage is limited to the case of descendants who have attended less selective college.

Comparative statics Although the equilibrium distribution of the abilities of the entrepreneurs is not responsive to variations in the level of the technological progress, it reacts to changes in the growth rate of the technology and in the institutional arrangement of the society since these affect the individuals occupational choice by influencing the productivity of the family connections. In particular, for a stationary rate of technological progress, $g_{t+1} = g^*$, changes in ϕ do affect the allocation of the abilities across the population through two channels. On a side, increases in ϕ augment the net productivity of the family connections such to make more profitable for the entrepreneurs to retain the control of the firm within the family by introducing their low ability heirs in the network of connections embedded in the family business. Correspondingly, the mass of family firms operated by the low ability heirs increases as for a larger set of them it becomes more rewarding to exploit the connections transferred by their parents rather than working for a wage. On the other side, the initial increment in the mass of family firms generates also an increase in the total proportion of firms, and hence in the total demand of human capital, that brings about, at aggregate level, an increase in the equilibrium wage rate. While this latter effect would lessen the profitability of the firms, the combined effect of the two channels implies that the mass of the lowest ability heirs who manage the family firms increases since the elasticity of the wage with respect to ϕ is lower than one (Figure 7).

Lemma 4.

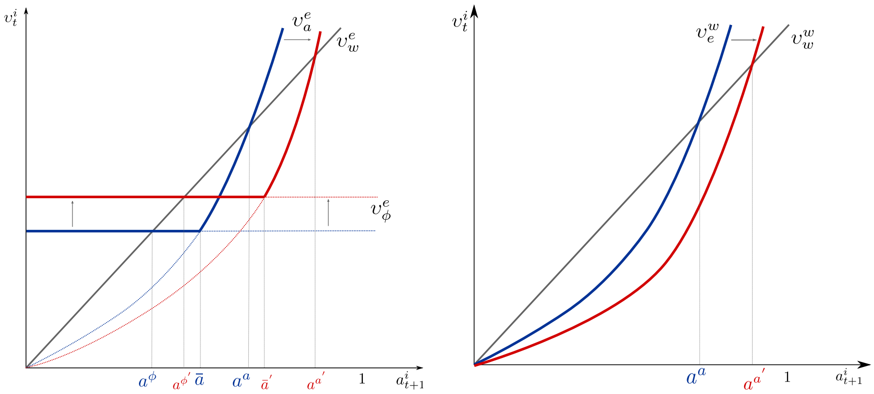
$$\frac{dw^*}{d\phi} > 0$$

$$\varepsilon_w^\phi = \frac{dw^*}{d\phi}\frac{\phi}{w^*} < 1$$

Proof. See Appendix B. □

Instead, increases in ϕ, through the general equilibrium channel, do reduce the proportion of the high ability entrepreneurs of the society. By decreasing the profit rate, via the

Figure 7: Occupational choice: changes in ϕ



(a) The entrepreneurs (b) The workers

increase in the wage rate, increases in ϕ, indeed, induces part of the highest ability heirs and descendants of the workers to not opt for the entrepreneurial sector, but to choose to work for a wage.

Proposition 4 (Comparative statics)**.** *The thresholds a^ϕ and a^a are increasing in the institutional setting* (ϕ)

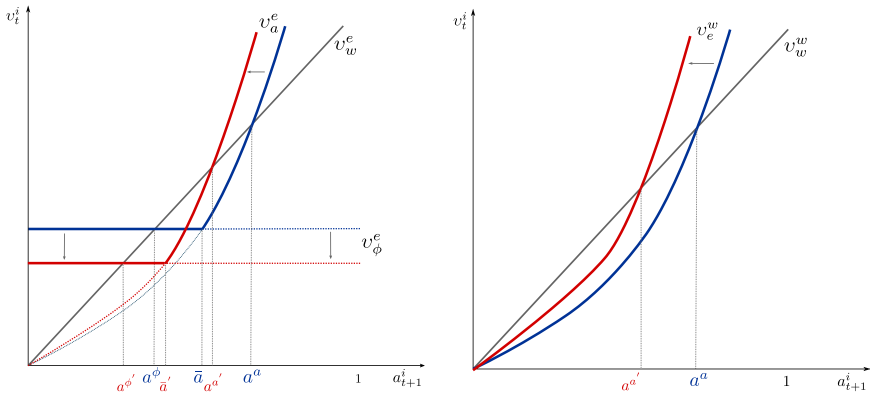
$$\frac{\partial a^\phi}{\partial \phi} > 0$$
$$\frac{\partial a^a}{\partial \phi} > 0$$

Proof. See Appendix B. □

Likewise, changes in the growth rate of the technology have both direct and compositional effects. Although accelerations in the rate of technological progress increase the level of the aggregate technology, these effects, as shown above, do not generate any changes in the individuals occupational choice and then leave the threshold unchanged. Notwithstanding, increases in the growth rate of the aggregate technology, through the erosion effect, cause a reduction in the net productivity of the family connections (i.e. $\bar{a}$), inducing some of the low ability heirs of the current generation of entrepreneurs to abandon the family business. The temporary reduction in the total mass of firms of the economy and, consequently in the total demand of human capital, generates a reduction in the equilibrium wage rate[16] that, in turn, stimulates some of the individuals in the upper tail of the ability distribution to start preferring the entrepreneurial sector as the unitary profits increase correspondingly (Figure 8).

[16]The total effect on the wage of an increase in the growth rate g^* is composed by a direct effect due to the increase in the level of the TFP and in a compositional effect, which affects the distribution of the individuals into the occupations. Although the combined effect of these two elements on the wage rate is ambiguous, it is not ambiguous the effect of the increase in the growth rate on the thresholds a^ϕ and a^a since the elasticity of the wage with respect to g is, in absolute value, lower than one.

Figure 8: Occupational choice: the erosion effect



(a) The entrepreneurs (b) The workers

Proposition 5 (Comparative statics)**.** *The thresholds a^{ϕ} and a^{a} are decreasing in the growth rate of aggregate technology (g^*)*

$$\frac{\partial a^{\phi}}{\partial g^*} < 0$$
$$\frac{\partial a^{a}}{\partial g^*} < 0$$

Proof. See Appendix B. □

The overall effect on the total number of firms depends on the relative responsiveness of the two thresholds to changes in the growth rate of the technology. Let's define the difference between the elasticities of the upper and lower thresholds with respect to changes in g^* as

$$\Delta\varepsilon_g \equiv \varepsilon^g_{a^a} - \varepsilon^g_{a^{\phi}} = g\left(\frac{a^{a'}(g)}{a^a} - \frac{a^{\phi'}(g)}{a^{\phi}}\right) \tag{31}$$

where $\varepsilon^g_{a^a}$ and $\varepsilon^g_{a^{\phi}}$ are, respectively, the elasticity of the upper and the lower threshold with respect to changes in the steady state growth rate, and $a^{a'}(g)$ and $a^{\phi'}(g)$ are the derivatives of the thresholds with respect to g.

Lemma 5. *For $g^* > \bar{\phi}$, $\Delta\varepsilon_g > 0$; namely, the upper threshold is more elastic than the lower threshold to changes in the steady state growth rate of the technology.*

Proof. See Appendix B. □

Corollary 3. *For $g^* > \bar{\phi}$, an increase in the steady state growth rate generates an increase in the steady state total number of firms:*

$$\frac{\partial n^*}{\partial g^*} > 0 \tag{32}$$

Proof. See Appendix B. □

4.1.2 Crony society

If $g^* < \bar{\phi}$, the dynamical system governing the evolution of the size and the composition of the entrepreneurial sector is the following:

$$n_{t+1} = \begin{cases} 1 - a^a_{t+1}\left(1 - n_t\right) \equiv n^C_1\left(n_t\right) & \text{if } n_t \leq \underset{\sim}{n}_t \\ n_t \equiv n^C_2\left(n_t\right) & \text{if } n_t \in \left(\underset{\sim}{n}_t, \hat{\hat{n}}\right) \\ n_t a^{\phi}_{t+1} \equiv n^C_3\left(n_t\right) & \text{if } n_t \geq \hat{\hat{n}} \end{cases} \tag{33}$$

If the connections established over the generations of entrepreneurs strongly affect the profitability of the family businesses, the entrepreneurs have never the incentive to invest in their heir entrepreneurial ability. Furthermore, the stronger is the productivity of the family connections, the less attractive is the outside option of working as employee since the potential profits generated by the system of relations inherited by the parents are more rewarding than the wage possibly earned on the labor market. Hence, in crony societies where the parental effect is strong, the heirs of the entrepreneurs would never leave the family businesses unless the equilibrium wage rate is high enough (see Fig. 5). Hence, using also Corollary 1, if $n_t < \hat{\hat{n}}$, the total number of firms at each time $t+1$, n_{t+1}, is composed by all the heirs who continue the family business by exploiting the system of relations inherited by their parent entrepreneurs. In addition, if $n_t < \underset{\sim}{n}_t$ also holds, even part of the highest ability descendants of the workers choose to set up new firms; hence, as represented by the law of motion $n^C_1\left(n_t\right)$, the total number of firms at time $t+1$ would be composed by the number of pre-existing firms, n_t, as well as by the proportion $\left(1 - n_t\right)\left(1 - a^a_{t+1}\right)$ of new firms installed by individuals, whose parents were workers at period t. Otherwise, if $n_t > \underset{\sim}{n}_t$, the descendants of the workers would never set up new firms such that, at each time $t+1$, all the firms of the economy are of the family-type one. Depending on the equilibrium wage rate, either all firms of each generation t, n_t, are continued within the family at time $t+1$ such that $n_{t+1} = n^C_2\left(n_t\right)$; or, if the wage rate is high enough (i.e. $n_t > \hat{\hat{n}}$), only the proportion $n_t a^{\phi}_{t+1}$ of the lowest ability heirs continue the family business since for the proportion $n_t(1 - a^{\phi}_{t+1})$ of the highest ability heirs is more rewarding to leave the family business to join the employment sector, as represented by the law of motion $n^C_3\left(n_t\right)$.

Lemma 6 (Properties of the dynamical system)**.** *If $g^* < \bar{\phi}$, the dynamical system in (33) is characterized by the following properties:*

1. $n^C_1\left(0\right) > 0$*;* $n^C_1\left(\underset{\sim}{n}_t\right) = \underset{\sim}{n}_t = n^C_2\left(\underset{\sim}{n}_t\right)$*;* $n^{C'}_1\left(n_t\right) < 0$*;* $n^{C''}_1\left(n_t\right) > 0$.
2. $n^C_3\left(\hat{\hat{n}}\right) = \hat{\hat{n}} = n^C_2\left(\hat{\hat{n}}\right)$*;* $n^C_3\left(1\right) < 1$*;* $n^{C'}_3\left(n_t\right) > 0$*;* $n^{C''}_3\left(n_t\right) < 0$.

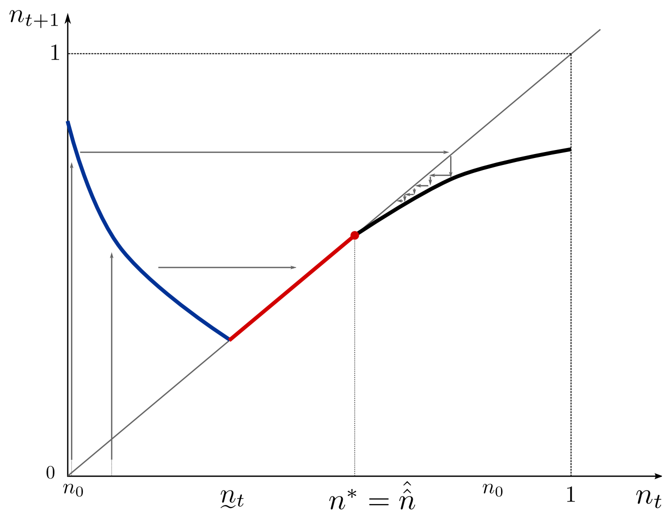
Proof. See Appendix B. □

Proposition 6. *If $g^* < \bar{\phi}$, for any initial number of firms n_0 the economy converges into a fully immobile society. Regardless of the innate individuals ability, the firms heirs operate all the firms of the economy (n^*), while the descendants of the workers, $(1 - n^*)$, remain in the employment sector.*

Proof. See Appendix B. □

Proposition 6 shows that if the institutional arrangement of the society is such to allow the family connections to be highly productive, the economy converges to a steady state with no exit of the heirs from the family businesses as well as with no entry of the descendants of the workers in the entrepreneurial sector. Although the steady state number of firms is indeterminate as it depends on its initial level, the allocation of the individuals across the sectors converges, independently of its initial state, to the distribution represented in Figure 5b and corresponding to either one of the points on the red segment of the transitional dynamics in Figure 9. Only if the initial number of firms n_0 is either very small or very large, there exists a unique path leading to the determinate steady state number of firms $n^* = \hat{\hat{n}} \equiv \theta^2 \theta_4^{-1}$.

Figure 9: Dynamics: crony society



Since the network of relations of the entrepreneurs strongly determines, and more than the entrepreneurial skills, the profitability of the firms, the entrepreneurs always choose to transfer the control of the firms within the family to their heirs, regardless of the heir innate ability. The indirect effect of the persistence, over time, of the family businesses is that also the entry of potential high ability descendants of the workers is blocked; indeed, due to the large presence of the firms heirs, the total demand of human capital is high enough to sustain a correspondingly high equilibrium wage rate that causes the unitary profits to be low enough to discourage the entry of all the descendants of the workers.

4.2 Endogeneous growth

TBA

5 Conclusion

TBA

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Appendix A Characterization of the equilibrium

A.1 Entrepreneurial society

If $g_{t+1} > \bar{\phi}$, the occupational choices the parents make for their descendants (Section 2.4 and Fig. 4) imply that the aggregate supply of human capital $H^{S,E}_{t+1}$ is given by

$$H^{S,E}_{t+1} = \begin{cases} (1-n_t) \int_0^{a^a_{t+1}} a^i_{t+1} \tau^h da^i_{t+1} & \text{if } w_{t+1} \leq \hat{w} \\ n_t \int_{a^\phi_{t+1}}^{a^a_{t+1}} a^i_{t+1} \tau^h da_{i,t+1} + (1-n_t) \int_0^{a^a_{t+1}} a^i_{t+1} \tau^h da^i_{t+1} & \text{if } w_{t+1} \in (\hat{w}, \tilde{w}) \\ n_t \int_{a^\phi_{t+1}}^1 a^i_{t+1} \tau^h da^i_{t+1} + (1-n_t) \int_0^1 a^i_{t+1} \tau^h da^i_{t+1} & \text{if } w_{t+1} \geq \tilde{w} \end{cases} \tag{A.1}$$

and the aggregate demand of human capital $H^{D,E}_{t+1}$ by

$$H^{D,E}_{t+1} = \begin{cases} n_t \left[\int_0^{\bar{a}_{t+1}} H^{i,\phi}_{t+1} da^i_{t+1} + \int_{\bar{a}_{t+1}}^1 H^{i,a}_{t+1} da^i_{t+1} \right] + (1-n_t) \int_{a^a_{t+1}}^1 H^{i,a}_{t+1} da^i_{t+1} & \text{if } w_{t+1} \leq \hat{w} \\ n_t \int_0^{a^\phi_{t+1}} H^{i,\phi}_{t+1} da^i_{t+1} + \int_{a^a_{t+1}}^1 H^{i,a}_{t+1} da^i_{t+1} & \text{if } w_{t+1} \in (\hat{w}, \tilde{w}) \\ n_t \int_0^{a^\phi_{t+1}} H^{i,\phi}_{t+1} da^i_{t+1} & \text{if } w_{t+1} \geq \tilde{w} \end{cases} \tag{A.2}$$

where $H^{i,\phi}_{t+1}$ and $H^{i,a}_{t+1}$ are, respectively, the firm level optimal demands of human capital of the entrepreneurs (3), obtained substituting in (6) the optimal choices of the parents; formally, $H^{i,\phi}_{t+1} = \left[(1-\alpha) \tau^e A_{t+1} w^{-1}_{t+1} \phi (1-g_{t+1}) \right]^{1/\alpha}$ and $H^{i,a}_{t+1} = \left[(1-\alpha) \tau^e A_{t+1} w^{-1}_{t+1} a^i_{t+1} \right]^{1/\alpha}$, with $\tau^e \equiv \tau^a = \tau^\phi$.

Solving the integrals, we can rewrite $H^{S,E}_{t+1}$ and $H^{D,E}_{t+1}$ as:

$$H^{S,E}_{t+1} = \begin{cases} \frac{\tau^h}{2} (1-n_t) \left(a^a_{t+1}\right)^2 & \text{if } w_{t+1} \leq \hat{w} \\ \frac{\tau^h}{2} \left[\left(a^a_{t+1}\right)^2 - n_t \left(a^\phi_{t+1}\right)^2 \right] & \text{if } w_{t+1} \in (\hat{w}, \tilde{w}) \\ \frac{\tau^h}{2} \left[1 - n_t \left(a^\phi_{t+1}\right)^2 \right] & \text{if } w_{t+1} \geq \tilde{w} \end{cases} \tag{A.3}$$

and

$$H^{D,E}_{t+1} = \begin{cases} \frac{\mathcal{A}}{1+\alpha} \left\{ n_t \bar{a}^{\frac{1+\alpha}{\alpha}}_{t+1} + \alpha \left[1 - (1-n_t) \left(a^a_{t+1}\right)^{\frac{1+\alpha}{\alpha}} \right] \right\} & \text{if } w_{t+1} \leq \hat{w} \\ \mathcal{A} \left\{ n_t a^\phi_{t+1} \bar{a}^{\frac{1}{\alpha}}_{t+1} + \frac{\alpha}{1+\alpha} \left[1 - \left(a^a_{t+1}\right)^{\frac{1+\alpha}{\alpha}} \right] \right\} & \text{if } w_{t+1} \in (\hat{w}, \tilde{w}) \\ \mathcal{A} n_t a^\phi_{t+1} \bar{a}^{\frac{1}{\alpha}}_{t+1} & \text{if } w_{t+1} \geq \tilde{w} \end{cases} \tag{A.4}$$

where $\mathcal{A} \equiv \left((1-\alpha) \tau^e A_{t+1} w^{-1}_{t+1} \right)^{\frac{1}{\alpha}}$. Substituting (22) and (23) in (A.3) and (A.4), the equilibrium wage schedule results as

$$w^E_{t+1} = \begin{cases} A_{t+1}\left[\frac{\alpha + n_t \bar{a}_{t+1}^{\frac{1+\alpha}{\alpha}}}{(1-n_t)\theta_5}\right]^{\frac{\alpha(1-\alpha)}{1+\alpha}} \equiv \omega^E_1(n_t) & \text{if } n_t \leq \hat{n}_t \\ \omega^E_2(n_t) & \text{if } n_t \in (\hat{n}_t, \tilde{n}_t) \\ \theta_4^{\frac{\alpha}{2}} n_t^{\frac{\alpha}{2}} A_{t+1}\bar{a}_{t+1} \equiv \omega^E_3(n_t) & \text{if } n_t \geq \tilde{n}_t \end{cases} \tag{A.5}$$

where $\omega^E_2(n_t)$ is the wage w_{t+1} implicitly defined by the function:

$$\Omega(w) = \theta_1 \left(\frac{w_{t+1}}{A_{t+1}}\right)^{\frac{1+\alpha}{\alpha(1-\alpha)}} - \theta_2 n_t \bar{a}_{t+1}^{\frac{2}{\alpha}} \left(\frac{A_{t+1}}{w_{t+1}}\right)^{\frac{1}{\alpha}} - \theta_3 = 0 \tag{A.6}$$

where $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 > 0$ are collections of parameters.

Proofs of Proposition 2 and Corollary 1. The existence and the uniqueness of a wage w_{t+1} such that labor market clears follows directly by noticing that $\omega^E_1(n_t)$ and $\omega^E_3(n_t)$ are single-valued functions. Further, the existence and the uniqueness of the wage $\omega^E_2(n_t)$ follows from rewriting the expression in (A.6) as

$$\Omega_l(w) \equiv \theta_1 \left(\frac{w_{t+1}}{A_{t+1}}\right)^{\frac{1+\alpha}{\alpha(1-\alpha)}} = \theta_2 n_t \bar{a}_{t+1}^{\frac{2}{\alpha}} \left(\frac{A_{t+1}}{w_{t+1}}\right)^{\frac{1}{\alpha}} + \theta_3 \equiv \Omega_r(w) \tag{A.7}$$

and firstly noticing that $\Omega'_l(w) > 0$, while $\Omega'_r(w) < 0$. Consistently with the boundary conditions imposed by the equilibrium choices (Section 2.4 and Corollary 1), that $w \in (\hat{w}, \tilde{w})$, it is suffice to note that $\Omega_l(\hat{w}) = \theta_1 \theta^{\frac{1+\alpha}{1-\alpha}} \bar{a}_{t+1}^{\frac{1+\alpha}{\alpha}} < \theta_3 + \theta_2 \theta^{-1} n_t \bar{a}_{t+1}^{\frac{1+\alpha}{\alpha}} = \Omega_r(\hat{w})$ for any

$$n_t > \hat{n}_t \equiv \frac{\theta_6 \bar{a}_{t+1}^{\frac{1+\alpha}{\alpha}} - \alpha}{\bar{a}_{t+1}^{\frac{1+\alpha}{\alpha}}(1+\theta_6)} \tag{A.8}$$

and $\Omega_l(\tilde{w}) = \theta_1 \theta^{\frac{1+\alpha}{1-\alpha}} > \theta_3 + \theta_2 \theta^{-1} n_t \bar{a}_{t+1}^{\frac{2}{\alpha}} = \Omega_r(\tilde{w})$ for any

$$n_t < \tilde{n}_t \equiv \frac{\theta^2}{\theta_4 \bar{a}^{\frac{2}{\alpha}}} \tag{A.9}$$

where θ_6 is a collection of parameters. It is also verified that $\omega^E_1(n_t) : w_{t+1} < \hat{w}$ as long as $n_t < \hat{n}_t$, while $\omega^E_3(n_t) : w_{t+1} > \tilde{w}$ as long as $n_t > \tilde{n}_t$. □

Dynamical system Using (22) and (23) and substituting the equilibrium wage in (A.5), the dynamical system in (30) can be written as:

$$n_{t+1}=\begin{cases}1-\left(\frac{1-n_t}{\theta_6^{\alpha}}\right)^{\frac{1}{1+\alpha}}\left(\alpha+n_t\bar{a}^{\frac{1+\alpha}{\alpha}}\right)^{\frac{\alpha}{1+\alpha}}\equiv n_1^E(n_t) & \text{if } n_t\leq\hat{n}_t\\ n_t\theta\left[\frac{\bar{a}A_{t+1}}{\omega_2^E(n_t)}\right]^{\frac{1}{\alpha}}+1-\left(\frac{\omega_2^E(n_t)}{\theta^{\alpha}A_{t+1}}\right)^{\frac{1}{1-\alpha}}\equiv n_2^E(n_t) & \text{if } n_t\in(\hat{n}_t,\tilde{n}_t)\\ \theta\left(\frac{n_t}{\theta_4}\right)^{\frac{1}{2}}\equiv n_3^E(n_t) & \text{if } n_t\geq\tilde{n}_t\end{cases} \tag{A.10}$$

As follows from Proposition 3 and using (22) and (23), the unique globally steady state equilibrium is given by:

$$n^*=\frac{1-a^a}{1-a^{\phi}}=\frac{\left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}}-w_{t+1}^{\frac{1}{1-\alpha}}\right)w_{t+1}^{\frac{1}{\alpha}}}{\left(w_{t+1}^{\frac{1}{\alpha}}-\theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)\theta^{\frac{\alpha}{1-\alpha}}A_{t+1}^{\frac{1}{1-\alpha}}} \tag{A.11}$$

where $w_{t+1}\equiv\omega_2^E(n_t)$ is the wage implicitly defined in (A.6).

A.2 Crony society

If $g_{t+1}<\bar{\phi}$, the aggregate supply of human capital $H_{t+1}^{S,C}$ is given by

$$H_{t+1}^{S,C}=\begin{cases}(1-n_t)\int_0^{a_{t+1}^a}a_{t+1}^i\tau^h da_{t+1}^i & \text{if } w_{t+1}\leq\tilde{w}\\ (1-n_t)\int_0^1 a_{t+1}^i\tau^h da_{t+1}^i & \text{if } w_{t+1}\in(\hat{w},\tilde{w})\\ n_t\int_{a_{t+1}^{\phi}}^1 a_{t+1}^i\tau^h da_{t+1}^i+(1-n_t)\int_0^1 a_{t+1}^i\tau^h da_{t+1}^i & \text{if } w_{t+1}\geq\tilde{w}\end{cases} \tag{A.12}$$

and the aggregate demand of human capital $H_{t+1}^{D,C}$ by

$$H_{t+1}^{D,C}=\begin{cases}n_t\int_0^1 H_{t+1}^{i,\phi}da_{t+1}^i+(1-n_t)\int_{a_{t+1}^a}^1 H_{t+1}^{i,a}da_{t+1}^i & \text{if } w_{t+1}\leq\tilde{w}\\ n_t\int_0^1 H_{t+1}^{i,\phi}da_{t+1}^i & \text{if } w_{t+1}\in\left(\tilde{w},\hat{\hat{w}}\right)\\ n_t\int_0^{a_{t+1}^{\phi}} H_{t+1}^{i,\phi}da_{t+1}^i & \text{if } w_{t+1}\geq\hat{\hat{w}}\end{cases} \tag{A.13}$$

Solving the integrals, $H_{t+1}^{S,C}$ and $H_{t+1}^{D,C}$ are given by the followings

$$H_{t+1}^{S,C}=\begin{cases}\frac{\tau^h}{2}(1-n_t)\left(a_{t+1}^a\right)^2 & \text{if } w_{t+1}\leq\tilde{w}\\ \frac{\tau^h}{2}(1-n_t) & \text{if } w_{t+1}\in\left(\tilde{w},\hat{\hat{w}}\right)\\ \frac{\tau^h}{2}\left[1-n_t\left(a_{t+1}^{\phi}\right)^2\right] & \text{if } w_{t+1}\geq\hat{\hat{w}}\end{cases} \tag{A.14}$$

$$H^D_{C,t+1} = \begin{cases} \mathcal{A}\left\{n_t \bar{a}_{t+1}^{\frac{1}{\alpha}} + \frac{\alpha(1-n_t)}{1+\alpha}\left[1-\left(a^a_{t+1}\right)^{\frac{1+\alpha}{\alpha}}\right]\right\} & \text{if } w_{t+1} \leq \tilde{w} \\ \mathcal{A} n_t \bar{a}_{t+1}^{\frac{1}{\alpha}} & \text{if } w_{t+1} \in \left(\tilde{w}, \hat{\hat{w}}\right) \\ \mathcal{A} n_t a^{\phi}_{t+1} \bar{a}_{t+1}^{\frac{1}{\alpha}} & \text{if } w_{t+1} \geq \hat{\hat{w}} \end{cases} \tag{A.15}$$

and the equilibrium wage schedule by

$$w_{t+1} = \begin{cases} A_{t+1}\left[\frac{n_t\left(\bar{a}_{t+1}^{\frac{1}{\alpha}}(1+\alpha)+\alpha\right)-\alpha}{(1-n_t)\theta_5}\right]^{\frac{\alpha(1-\alpha)}{1+\alpha}} \equiv \omega^C_1(n_t) & \text{if } n_t \leq \underset{\sim}{n}_t \\ \theta_7 A_{t+1} \bar{a}_{t+1}\left(\frac{n_t}{1-n_t}\right)^{\alpha} \equiv \omega^C_2(n_t) & \text{if } n_t \in \left(\underset{\sim}{n}_t, \hat{\hat{n}}\right) \\ \theta_4^{\frac{\alpha}{2}} n_t^{\frac{\alpha}{2}} A_{t+1}\bar{a}_{t+1} \equiv \omega^C_3(n_t) & \text{if } n_t \geq \hat{\hat{n}} \end{cases} \tag{A.16}$$

where θ_7 is a collection of parameters. Finally, consistently with the boundary conditions imposed by the equilibrium choices (Section 2.4 and Corollary 1), it is verified that $\omega^C_1(n_t) : w_{t+1} < \tilde{w}$ as long as $n_t < \underset{\sim}{n}_t$, $\omega^C_2(n_t) : w_{t+1} \in \left(\tilde{w}, \hat{\hat{w}}\right)$ as long as $n_t \in \left(\underset{\sim}{n}_t, \hat{\hat{n}}\right)$, and $\omega^C_3(n_t) : w_{t+1} > \hat{\hat{w}}$ as long as $n_t > \hat{\hat{n}}_t$, with

$$\underset{\sim}{n}_t \equiv \frac{\theta_6 - \alpha}{\theta_6 - \alpha + \bar{a}_{t+1}^{\frac{1}{\alpha}}(1+\alpha)} \tag{A.17}$$

and

$$\hat{\hat{n}} \equiv \frac{\theta^2}{\theta_4} \tag{A.18}$$

Dynamical system Using (22) and (23) and substituting the equilibrium wage in (A.16), the dynamical system in (33) can be written as:

$$n_{t+1} = \begin{cases} 1 - \left(\frac{1-n_t}{\theta_6^{\alpha}}\right)^{\frac{1}{1+\alpha}}\left[n_t\left((1+\alpha)\bar{a}^{\frac{1}{\alpha}} - \alpha\right) + \alpha\right]^{\frac{\alpha}{1+\alpha}} \equiv n^C_1(n_t) & \text{if } n_t \leq \underset{\sim}{n}_t \\ n_t \equiv n^C_2(n_t) & \text{if } n_t \in \left(\underset{\sim}{n}_t, \hat{\hat{n}}\right) \\ \theta\left(\frac{n_t}{\theta_4}\right)^{\frac{1}{2}} \equiv n^C_3(n_t) & \text{if } n_t \geq \hat{\hat{n}} \end{cases} \tag{A.19}$$

Appendix B Proofs

Lemma 1. From (A.10) it follows:

1. Properties of $n_1^E(n_t)$:

   (a) $n_1^E(0) = 1 - \left(\frac{\alpha}{\theta_6}\right)^{\frac{\alpha}{1+\alpha}} \in (0,1)$.

   (b) $n_1^E(\hat{n}_t) = \frac{(1+\theta_6)\bar{a}^{1/\alpha} - \left(\alpha + \bar{a}^{\frac{1+\alpha}{\alpha}}\right)}{(1+\theta_6)\bar{a}^{1/\alpha}} > \hat{n}_t \Rightarrow (1+\theta_6)\,\bar{a}^{\frac{1+\alpha}{\alpha}} - \left(\alpha + \bar{a}^{\frac{1+\alpha}{\alpha}}\right) > \theta_6 \bar{a}^{\frac{1+\alpha}{\alpha}} - \alpha \Rightarrow (1-\bar{a})\left(\alpha + \bar{a}^{\frac{1+\alpha}{\alpha}}\right) > 0$.

   (c) Differentiating and rearranging,

$$\frac{\partial n_1^E(n_t)}{\partial n_t} = \frac{1}{(1+\alpha)\,\theta_6^{\frac{\alpha}{1+\alpha}}}\left[\left(\frac{\alpha + n_t \bar{a}^{\frac{1+\alpha}{\alpha}}}{1-n_t}\right)^{\frac{\alpha}{1+\alpha}} - \alpha \bar{a}^{\frac{1+\alpha}{\alpha}}\left(\frac{1-n_t}{\alpha + n_t \bar{a}^{\frac{1+\alpha}{\alpha}}}\right)^{\frac{1}{1+\alpha}}\right] \tag{B.1}$$

   Hence, $\frac{\partial n_1^E(n_t)}{\partial n_t} > 0$ if $\alpha + n_t \bar{a}^{\frac{1+\alpha}{\alpha}} > \alpha \bar{a}^{\frac{1+\alpha}{\alpha}}(1-n_t) \Rightarrow \alpha\left(1 - \bar{a}^{\frac{1+\alpha}{\alpha}}\right) + n_t \bar{a}^{\frac{1+\alpha}{\alpha}}(1+\alpha) > 0$.

   (d) From point (c),

$$\frac{\partial^2 n_1^E(n_t)}{\partial n_t^2} = \frac{1}{(1+\alpha)\,\theta_6^{\frac{\alpha}{1+\alpha}}}\left[\frac{\alpha}{1+\alpha}\left(\frac{\alpha + n_t \bar{a}^{\frac{1+\alpha}{\alpha}}}{(1-n_t)}\right)^{\frac{\alpha}{1+\alpha}-1}\left(\frac{\bar{a}^{\frac{1+\alpha}{\alpha}} + \alpha}{(1-n_t)^2}\right) + \frac{\alpha \bar{a}^{\frac{1+\alpha}{\alpha}}}{1+\alpha}\left(\frac{1-n_t}{\alpha + n_t \bar{a}^{\frac{1+\alpha}{\alpha}}}\right)^{\frac{1}{1+\alpha}-1}\left(\frac{\left(\alpha + n_t \bar{a}^{\frac{1+\alpha}{\alpha}}\right) + (1-n_t)\,\bar{a}^{\frac{1+\alpha}{\alpha}}}{\left(\alpha + n_t \bar{a}^{\frac{1+\alpha}{\alpha}}\right)^2}\right)\right] > 0 \tag{B.2}$$

2. Properties of $n_2^E(n_t)$:

   (a) From Corollary 1 and (A.5), for $n_t = \hat{n}_t \Leftrightarrow w_{t+1}^E = \omega_1^E(n_t) = \omega_2^E(n_t)$, such that from (22) and (23) it results that $a_{t+1}^{\phi}(\hat{n}_t) = \bar{a} = a_{t+1}^{a}(\hat{n}_t)$. From (30), $n_1^E(\hat{n}_t) = 1 - \bar{a}(1-\hat{n}_t) = n_2^E(\hat{n}_t)$. From point 1.b, we have also that $n_1^E(\hat{n}_t) = n_2^E(\hat{n}_t) > \hat{n}_t$.

   (b) From Corollary 1 and (A.5), $n_t = \tilde{n}_t \Leftrightarrow w_{t+1}^E = \omega_2^E(n_t) = \omega_3^E(n_t)$, such that from (22) and (23) it results that $a_{t+1}^{\phi}(\tilde{n}_t) = \bar{a}^{\frac{1}{\alpha}}$ and $a_{t+1}^{a}(\tilde{n}_t) = 1$; using (30), $n_2^E(\tilde{n}_t) = \tilde{n}_t \bar{a}^{\frac{1}{\alpha}} = n_3^E(\tilde{n}_t) < \tilde{n}_t$ since $\bar{a} \in (0,1)$.

   (c) Differentiating $n_2^E(n_t)$ w.r.t. n_t and rearranging,

$$\frac{\partial n_2^E(n_t)}{\partial n_t} = a_{t+1}^{\phi}\left(1 - \frac{n_t}{\alpha w_{t+1}}\frac{dw_{t+1}}{dn_t}\right) - \frac{a_{t+1}^a}{(1-\alpha)\, w_{t+1}}\frac{dw_{t+1}}{dn_t} \tag{B.3}$$

which implies that $\frac{\partial n_2^E(n_t)}{\partial n_t} > 0$ if $1 - \frac{n_t}{\alpha w_{t+1}}\frac{dw_{t+1}}{dn_t} > \frac{a_{t+1}^a}{(1-\alpha)w_{t+1}a_{t+1}^{\phi}}\frac{dw_{t+1}}{dn_t}$, namely if

$$n_t < \alpha\left[w_{t+1}\left(\frac{dw_{t+1}}{dn_t}\right)^{-1} - \frac{a_{t+1}^a}{(1-\alpha)\, a_{t+1}^{\phi}}\right] \equiv \mathring{n}_t \tag{B.4}$$

where $w_{t+1} = \omega_2^E(n_t)$; hence, applying the implicit function theorem to (A.6),

$$\frac{dw_{t+1}}{dn_t} = -\frac{\partial\Omega(w)/\partial n_t}{\partial\Omega(w)/\partial w_{t+1}} =$$

$$= \frac{\theta_2 \bar{a}^{\frac{2}{\alpha}}\left(A_{t+1}w_{t+1}^{-1}\right)^{\frac{1}{\alpha}}}{\theta_1\left(\frac{1+\alpha}{\alpha(1-\alpha)}\right)\left(\frac{w_{t+1}^{1+\alpha^2}}{A_{t+1}^{1+\alpha}}\right)^{\frac{1}{\alpha(1-\alpha)}} + \theta_2\frac{n_t\bar{a}^{\frac{2}{\alpha}}}{\alpha w_{t+1}}\left(\frac{A_{t+1}}{w_{t+1}}\right)^{\frac{1}{\alpha}}} \tag{B.5}$$

Substituting (22), (23) and (B.5) in (B.4), we have

$$n_t < \alpha\left[\frac{n_t}{\alpha} + \left(\frac{1+\alpha}{\alpha(1-\alpha)}\right)\frac{\theta_1}{\theta_2\bar{a}^{\frac{2}{\alpha}}}\left(\frac{w_{t+1}}{A_{t+1}}\right)^{\frac{1+\alpha}{\alpha(1-\alpha)}}\left(\frac{w_{t+1}}{A_{t+1}}\right)^{\frac{1}{\alpha}} - \frac{1}{\theta(1-\alpha)}\left(\frac{w_{t+1}}{\bar{a}A_{t+1}}\right)^{\frac{1}{\alpha}}\left(\frac{w_{t+1}}{\theta^{\alpha}A_{t+1}}\right)^{\frac{1}{1-\alpha}}\right] \Rightarrow$$

$$\Rightarrow w_{t+1} > \mathring{w} \equiv \left(\frac{\alpha\theta_2}{(1+\alpha)\,\theta_1}\right)^{\alpha(1-\alpha)}\frac{A_{t+1}\bar{a}^{1-\alpha}}{\theta^{\alpha}} \tag{B.6}$$

Summing up, we have that

$$\frac{\partial n_2^E(n_t)}{\partial n_t}\begin{cases} < 0 & \text{if } w_{t+1} < \mathring{w} \\ > 0 & \text{if } w_{t+1} > \mathring{w}\end{cases} \tag{B.7}$$

From (A.10), $n_2^E(n_t) \in (\hat{n}_t, \tilde{n}_t)$ is defined only for values of $n_t > \hat{n}_t \Leftrightarrow w_{t+1} > \hat{w}$. Comparing $\mathring{w}$ and $\hat{w}$, it results that $\mathring{w} \equiv \left(\frac{\alpha\theta_2}{(1+\alpha)\theta_1}\right)^{\alpha(1-\alpha)}\frac{A_{t+1}\bar{a}^{1-\alpha}}{\theta^{\alpha}} < \theta^{\alpha}A_{t+1}\bar{a}^{1-\alpha} \equiv \hat{w} \Rightarrow \left(\frac{\alpha\theta_2}{(1+\alpha)\theta_1}\right)^{\alpha(1-\alpha)}\frac{1}{\theta^{2\alpha}} < 1$ always holds since substituting the parameters defined in Appendix C and after some algebra it derives that $\left(\frac{\alpha^{\frac{1}{1-\gamma}}+2(1-\alpha)(1-\gamma(1-\alpha))^{\frac{\gamma}{1-\gamma}}}{(1+\alpha)\alpha^{\frac{\gamma}{1-\gamma}}+2(1-\alpha)(1-\gamma(1-\alpha))^{\frac{\gamma}{1-\gamma}}}\right)^{\alpha(1-\alpha)} < 1 \Rightarrow \alpha < (1+\alpha)$. Finally, since

$w_{t+1}^{'}(n_t) > 0$, $\mathring{w} < \hat{w}$ also implies that $\mathring{n}_t < \hat{n}_t$; hence, for any $n_t > \hat{n}_t$ for which $n_2^E(n_t)$ is active, it finally derives that $n_2^{E'}(n_t) > 0$.

3. The properties of $n_3^E(n_t)$ easily follows from (A.10):

   (a) $\frac{\partial n_3^E(n_t)}{\partial n_t} = \frac{\theta}{2}(n_t\theta_4)^{-\frac{1}{2}} > 0$

   (b) $\frac{\partial^2 n_3^E(n_t)}{\partial n_t^2} = -\frac{\theta}{4}\left(n_t^3\theta_4\right)^{-\frac{1}{2}} < 0$

   (c) $n_3^E(1) = \theta\theta_4^{-\frac{1}{2}} = \frac{\alpha^{\frac{1}{1-\gamma}}}{\alpha^{\frac{1}{1-\gamma}}+2(1-\alpha)(1-\gamma(1-\alpha))^{\frac{\gamma}{1-\gamma}}} < 1.$

□

Proposition 3. Lemma 1 ensures that the unique admissible steady state number of firms can lie only within the interval $(\hat{n}_t, \tilde{n}_t)$. To see this, notice that the law of motion $n_1^E(n_t)$ is increasing and convex and such that $n_1^E(\hat{n}_t) > \hat{n}_t$. Further, $n_1^E(n_t)$ admits two steady states in the whole interval $n_t \in (0,1)$. One steady state is $n = 1$ such that $n_1^E(1) = 1$. The other possible value of n such that $n_1^E(n) = n$ is defined as $n = \frac{\theta_6-\alpha}{\theta_6+\bar{a}^{\frac{1+\alpha}{\alpha}}} > \frac{\theta_6\bar{a}^{\frac{1+\alpha}{\alpha}}-\alpha}{\bar{a}^{\frac{1+\alpha}{\alpha}}(1+\theta_6)} \equiv \hat{n}_t$ since, after rearranging, $(\theta_6-\alpha)(1+\theta_6)\bar{a}^{\frac{1+\alpha}{\alpha}} > \left(\theta_6\bar{a}^{\frac{1+\alpha}{\alpha}}-\alpha\right)\left(\bar{a}^{\frac{1+\alpha}{\alpha}}+\theta_6\right) \Rightarrow \bar{a}^{\frac{1+\alpha}{\alpha}} + \alpha\left(1-\bar{a}^{\frac{1+\alpha}{\alpha}}\right) > \bar{a}^{\frac{2(1+\alpha)}{\alpha}}$ as $\bar{a} \in (0,1)$. Hence, both these values of the number of firms are not admissible steady states. Instead, as follows from Lemma 1, in the interval $n_t \in (\hat{n}_t, \tilde{n}_t)$, $n_2^E(\hat{n}_t) > \hat{n}_t$, $n_2^E(\tilde{n}_t) < \tilde{n}_t$ and $n_2^E(n_t)$ is monotonically increasing in n_t; hence, it must intersect the 45° degree line from above, with a slope less than one, at most once. Similarly, it can be proved that there not exists any admissible steady state in the interval $n_t \in (\tilde{n}_t, 1)$. □

Corollary 2. Substituting (A.11) in (A.6) and rearranging, the wage rate in steady state w^* is defined by the value of w_{t+1} that solves

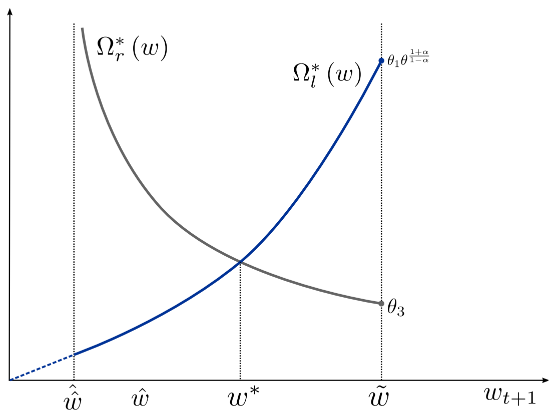
$$\Omega^*(w) = \theta_1\left(\frac{w_{t+1}}{A_{t+1}}\right)^{\frac{1+\alpha}{\alpha(1-\alpha)}} - \frac{\theta_2\bar{a}^{\frac{2}{\alpha}}A_{t+1}^{\frac{1-2\alpha}{\alpha(1-\alpha)}}}{\theta^{\frac{\alpha}{1-\alpha}}}\frac{\left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}}-w_{t+1}^{\frac{1}{1-\alpha}}\right)}{\left(w_{t+1}^{\frac{1}{\alpha}}-\theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)} - \theta_3 = 0 \qquad \text{(B.8)}$$

that can also be written as

$$\Omega_l^*(w) \equiv \theta_1\left(\frac{w_{t+1}}{A_{t+1}}\right)^{\frac{1+\alpha}{\alpha(1-\alpha)}} = \frac{\theta_2\bar{a}^{\frac{2}{\alpha}}A_{t+1}^{\frac{1-2\alpha}{\alpha(1-\alpha)}}}{\theta^{\frac{\alpha}{1-\alpha}}}\frac{\left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}}-w_{t+1}^{\frac{1}{1-\alpha}}\right)}{\left(w_{t+1}^{\frac{1}{\alpha}}-\theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)} + \theta_3 \equiv \Omega_r^*(w) \qquad \text{(B.9)}$$

For any $w \in (0, \infty)$, $\Omega_l^* (w)$ is increasing $\left(\Omega_l^{*'} (w) > 0\right)$ and convex $\left(\Omega_l^{*''} (w) > 0\right)$, with $\lim_{w \to \infty} \Omega_l^* (w) = \infty$. For any $w \in \left(\hat{\hat{w}}, \tilde{w}\right)$ $\lim_{w \to \hat{\hat{w}}} \Omega_r^* (w) = \infty$ and $\Omega_r^{*'} (w) < 0$. Furthermore, $\Omega_l^* (\tilde{w}) = \theta_1 \theta^{\frac{1+\alpha}{1-\alpha}} > \theta_3 = \Omega_r^* (\tilde{w})$. Hence, there exists a unique wage rate w^* that satisfies the condition in (B.8). Finally, consistently with the boundary conditions that govern the existence of the thresholds a^ϕ and a^a, $\Omega_r^* (\hat{w}) > \Omega_l^* (\hat{w})$ ensures that this equilibrium wage rate is higher than $\hat{w}$ and lower than $\tilde{w}$; namely, $w^* \in (\hat{w}, \tilde{w})$. Figure B.1 offers a graphical representation of the equilibrium

Figure B.1: Steady state equilibrium



□

Lemma 2. Using (B.8),

$$\frac{dw^*}{dA_{t+1}} = -\frac{\partial \Omega^* / \partial A_{t+1}}{\partial \Omega^* / \partial w^*} > 0$$

since

$$\frac{\partial \Omega^*}{\partial w^*} = \theta_1 \left(\frac{1+\alpha}{\alpha (1-\alpha)} \right) \left(\frac{w_{t+1}^{1+\alpha^2}}{A_{t+1}^{1+\alpha}} \right)^{\frac{1}{\alpha(1-\alpha)}} +$$
$$- \frac{\theta_2 \bar{a}^{\frac{2}{\alpha}} A_{t+1}^{\frac{1-2\alpha}{\alpha(1-\alpha)}}}{\theta^{\frac{\alpha}{1-\alpha}}} \left[\frac{-\left(\frac{1}{1-\alpha}\right) w_{t+1}^{\frac{\alpha}{1-\alpha}} \left(w_{t+1}^{\frac{1}{\alpha}} - \theta (\bar{a} A_{t+1})^{\frac{1}{\alpha}} \right) - \left(\frac{1}{\alpha}\right) w_{t+1}^{\frac{1-\alpha}{\alpha}} \left(A_{t+1}^{\frac{1}{1-\alpha}} \theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}} \right)}{\left(w_{t+1}^{\frac{1}{\alpha}} - \theta (\bar{a} A_{t+1})^{\frac{1}{\alpha}} \right)^2} \right] > 0 \tag{B.10}$$

and

$$\frac{\partial \Omega^*}{\partial A_{t+1}} = -\theta_1 \left(\frac{1+\alpha}{\alpha(1-\alpha)}\right)\left(\frac{w_{t+1}}{A_{t+1}}\right)^{\frac{1+\alpha}{\alpha(1-\alpha)}} \frac{1}{A_{t+1}} +$$

$$-\frac{\theta_2 \bar{a}^{\frac{2}{\alpha}}}{\theta^{\frac{\alpha}{1-\alpha}}} \left\{ \frac{\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)\left[\left(\frac{1-2\alpha}{\alpha(1-\alpha)}\right) A_{t+1}^{\frac{1-3\alpha+\alpha^2}{\alpha(1-\alpha)}} \left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right) + \left(\frac{\theta^{\frac{\alpha}{1-\alpha}}}{1-\alpha}\right) A_{t+1}^{\frac{1-2\alpha+\alpha^2}{\alpha(1-\alpha)}}\right]}{\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)^2} + \right.$$

$$\left. + \frac{A_{t+1}^{\frac{1-2\alpha}{\alpha(1-\alpha)}} \left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right)\left(\frac{\theta \bar{a}^{\frac{1}{\alpha}}}{\alpha}\right) A_{t+1}^{\frac{1-\alpha}{\alpha}}}{\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)^2} \right\} < 0$$

Further, using the above derivative and rearranging

$$\varepsilon_w^A = \frac{dw^*}{dA_{t+1}} \frac{A_{t+1}}{w^*} = 1 \Rightarrow -\frac{\partial \Omega^*}{\partial A_{t+1}} A_{t+1} = \frac{\partial \Omega^*}{\partial w^*} w^*$$

since

$$-\frac{\partial \Omega^*}{\partial A_{t+1}} A_{t+1} = \theta_1 \left(\frac{1+\alpha}{\alpha(1-\alpha)}\right)\left(\frac{w_{t+1}}{A_{t+1}}\right)^{\frac{1+\alpha}{\alpha(1-\alpha)}} + \frac{\theta_2 \bar{a}^{\frac{2}{\alpha}}}{\theta^{\frac{\alpha}{1-\alpha}}} \left\{ \frac{(1-2\alpha) A_{t+1}^{\frac{1-2\alpha}{\alpha(1-\alpha)}} \left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right)}{\alpha(1-\alpha)\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)} + \right.$$

$$\left. + \frac{\theta^{\frac{\alpha}{1-\alpha}} A_{t+1}^{\frac{1}{\alpha}}}{(1-\alpha)\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)} + \frac{A_{t+1}^{\frac{2-3\alpha}{\alpha(1-\alpha)}} \left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right) \theta \bar{a}^{\frac{1}{\alpha}}}{\alpha\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)^2} \right\} = \theta_1 \left(\frac{1+\alpha}{\alpha(1-\alpha)}\right)\left(\frac{w_{t+1}}{A_{t+1}}\right)^{\frac{1+\alpha}{\alpha(1-\alpha)}} +$$

$$+ \frac{\theta_2 \bar{a}^{\frac{2}{\alpha}} A_{t+1}^{\frac{1-2\alpha}{\alpha(1-\alpha)}}}{\theta^{\frac{\alpha}{1-\alpha}}} \left[\frac{w_{t+1}^{\frac{1}{1-\alpha}}}{(1-\alpha)\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)} + \frac{w_{t+1}^{\frac{1}{\alpha}} \left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right)}{\alpha\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)^2} \right] = \frac{\partial \Omega^*}{\partial w^*} w^* \Rightarrow$$

$$\Rightarrow \frac{(1-2\alpha)\left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right)}{\alpha(1-\alpha)\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)} + \frac{\theta^{\frac{\alpha}{1-\alpha}} A_{t+1}^{\frac{1}{1-\alpha}}}{(1-\alpha)\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)} + \frac{\theta \bar{a}^{\frac{1}{\alpha}} A_{t+1}^{\frac{1}{\alpha}} \left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right)}{\alpha\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)^2} =$$

$$= \frac{w_{t+1}^{\frac{1}{1-\alpha}}}{(1-\alpha)\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)} + \frac{w_{t+1}^{\frac{1}{\alpha}} \left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right)}{\alpha\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)^2} \Rightarrow$$

$$\Rightarrow \frac{\left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right)}{\alpha\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)} = \frac{\left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right)}{\alpha\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)}$$

□

Lemma 3. Differentiating (22) and (23) w.r.t. A_{t+1}, and using Lemma 2

$$\frac{\partial a^{\phi}}{\partial A_{t+1}} = \frac{\theta\bar{a}^{\frac{1}{\alpha}}}{\alpha}\left(\frac{A_{t+1}}{w^*}\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{w^* - A_{t+1}\left(\frac{dw^*}{dA_{t+1}}\right)}{{w^*}^2}\right) = \frac{a^{\phi}\left(1-\varepsilon_w^A\right)}{\alpha A_{t+1}} = 0$$

$$\frac{\partial a^{a}}{\partial A_{t+1}} = \frac{1}{(1-\alpha)}\left(\frac{w^*}{\theta A_{t+1}}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{A_{t+1}\left(\frac{dw^*}{dA_{t+1}}\right) - w^*}{A_{t+1}^2}\right) = \frac{a^{a}\left(\varepsilon_w^A - 1\right)}{(1-\alpha)A_{t+1}} = 0$$

□

Lemma 4. Using (B.8),

$$\frac{dw^*}{d\phi} = -\frac{\partial\Omega^*/\partial\phi}{\partial\Omega^*/\partial w^*} > 0$$

since

$$\frac{\partial\Omega^*}{\partial\phi} = -\frac{\theta_2\left(1-g\right)^{\frac{2}{\alpha}}A_{t+1}^{\frac{1-2\alpha}{\alpha(1-\alpha)}}\left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right)}{\theta^{\frac{\alpha}{1-\alpha}}}\times$$
$$\times\left[\frac{\left(\frac{2}{\alpha}\right)\phi^{\frac{2-\alpha}{\alpha}}\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right) + \left(\frac{1}{\alpha}\right)\theta\phi^{\frac{2}{\alpha}}\phi^{\frac{1-\alpha}{\alpha}}\left(\left(1-g\right)A_{t+1}\right)^{\frac{1}{\alpha}}}{\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)^2}\right] < 0$$

and, from above, $\partial\Omega^*/\partial w^* > 0$.
Further, using the above results and rearranging

$$\varepsilon_w^{\phi} = \frac{dw^*}{d\phi}\frac{\phi}{w^*} < 1 \Rightarrow -\frac{\partial\Omega^*}{\partial\phi}\phi < \frac{\partial\Omega^*}{\partial w^*}w^*$$

since

$$-\frac{\partial \Omega^*}{\partial \phi}\phi = \frac{\theta_2 \bar{a}^{\frac{2}{\alpha}} A_{t+1}^{\frac{1-2\alpha}{\alpha(1-\alpha)}} \left(A_{t+1}^{\frac{1}{1-\alpha}} \theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}} \right)}{\theta^{\frac{\alpha}{1-\alpha}} \left(w_{t+1}^{\frac{1}{\alpha}} - \theta \left(\bar{a} A_{t+1} \right)^{\frac{1}{\alpha}} \right)^2} \left[\left(\frac{2}{\alpha} \right) \left(w_{t+1}^{\frac{1}{\alpha}} - \theta \left(\bar{a} A_{t+1} \right)^{\frac{1}{\alpha}} \right) + \left(\frac{1}{\alpha} \right) \theta \left(\bar{a} A_{t+1} \right)^{\frac{1}{\alpha}} \right] <$$

$$\theta_1 \left(\frac{1+\alpha}{\alpha(1-\alpha)} \right) \left(\frac{w_{t+1}}{A_{t+1}} \right)^{\frac{1+\alpha}{\alpha(1-\alpha)}} + \frac{\theta_2 \bar{a}^{\frac{2}{\alpha}} A_{t+1}^{\frac{1-2\alpha}{\alpha(1-\alpha)}}}{\theta^{\frac{\alpha}{1-\alpha}}} \left[\frac{w_{t+1}^{\frac{1}{1-\alpha}}}{(1-\alpha) \left(w_{t+1}^{\frac{1}{\alpha}} - \theta \left(\bar{a} A_{t+1} \right)^{\frac{1}{\alpha}} \right)} + \right.$$

$$\left. + \frac{w_{t+1}^{\frac{1}{\alpha}} \left(A_{t+1}^{\frac{1}{1-\alpha}} \theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}} \right)}{\alpha \left(w_{t+1}^{\frac{1}{\alpha}} - \theta \left(\bar{a} A_{t+1} \right)^{\frac{1}{\alpha}} \right)^2} \right] = \frac{\partial \Omega^*}{\partial w^*} w^* \Rightarrow$$

$$\Rightarrow \frac{\theta_2 \bar{a}^{\frac{2}{\alpha}} A_{t+1}^{\frac{1-2\alpha}{\alpha(1-\alpha)}} \left(A_{t+1}^{\frac{1}{1-\alpha}} \theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}} \right)}{\alpha \theta^{\frac{\alpha}{1-\alpha}} \left(w_{t+1}^{\frac{1}{\alpha}} - \theta \left(\bar{a} A_{t+1} \right)^{\frac{1}{\alpha}} \right)} <$$

$$\theta_1 \left(\frac{1+\alpha}{\alpha(1-\alpha)} \right) \left(\frac{w_{t+1}}{A_{t+1}} \right)^{\frac{1+\alpha}{\alpha(1-\alpha)}} + \frac{\theta_2 \bar{a}^{\frac{2}{\alpha}} A_{t+1}^{\frac{1-2\alpha}{\alpha(1-\alpha)}} w_{t+1}^{\frac{1}{1-\alpha}}}{(1-\alpha) \theta^{\frac{\alpha}{1-\alpha}} \left(w_{t+1}^{\frac{1}{\alpha}} - \theta \left(\bar{a} A_{t+1} \right)^{\frac{1}{\alpha}} \right)} \Rightarrow$$

$$\Rightarrow \frac{\theta_2 \bar{a}^{\frac{2}{\alpha}} A_{t+1}^{\frac{1-2\alpha}{\alpha(1-\alpha)}} \left((1-\alpha) A_{t+1}^{\frac{1}{1-\alpha}} \theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}} \right)}{\alpha (1-\alpha) \theta^{\frac{\alpha}{1-\alpha}} \left(w_{t+1}^{\frac{1}{\alpha}} - \theta \left(\bar{a} A_{t+1} \right)^{\frac{1}{\alpha}} \right)} < \theta_1 \left(\frac{1+\alpha}{\alpha(1-\alpha)} \right) \left(\frac{w_{t+1}}{A_{t+1}} \right)^{\frac{1+\alpha}{\alpha(1-\alpha)}} \Rightarrow$$

$$\Rightarrow -\alpha \left[\frac{\theta_2 \bar{a}^{\frac{2}{\alpha}} A_{t+1}^{\frac{1}{\alpha}}}{\left(w_{t+1}^{\frac{1}{\alpha}} - \theta \left(\bar{a} A_{t+1} \right)^{\frac{1}{\alpha}} \right)} + \theta_1 \left(\frac{w_{t+1}}{A_{t+1}} \right)^{\frac{1+\alpha}{\alpha(1-\alpha)}} \right] < \theta_3$$

□

Proposition 4.

$$\frac{\partial a^\phi}{\partial \phi} = \frac{\theta \left((1-g^*) A_{t+1} \right)^{\frac{1}{\alpha}}}{\alpha} \left(\frac{\phi}{w^*} \right)^{\frac{1-\alpha}{\alpha}} \left(\frac{w^* - \phi \left(\frac{dw^*}{d\phi} \right)}{w^{*2}} \right) = \frac{a^\phi \left(1 - \varepsilon_w^\phi \right)}{\alpha \phi} > 0$$

$$\frac{\partial a^a}{\partial \phi} = \frac{w^{*\frac{\alpha}{1-\alpha}}}{(1-\alpha) \theta^{\frac{\alpha}{1-\alpha}} A_{t+1}^{\frac{1}{1-\alpha}}} \frac{dw^*}{d\phi} = \frac{a^a \varepsilon_w^\phi}{(1-\alpha) \phi} > 0$$

□

Proposition 5. Using $A_{t+1} = (1+g)\,A_t$ in (22), (23) and (B.8),

1. $\partial a^{\phi}/\partial g$:

$$\frac{\partial a^{\phi}}{\partial g} = \frac{\theta\,(\phi A_t)^{\frac{1}{\alpha}}}{\alpha}\left(\frac{1-g^2}{w^*}\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{-2gw^* - (1-g^2)\left(\frac{dw^*}{dg}\right)}{{w^*}^2}\right) < 0$$

if

$$-2gw^* - (1-g^2)\left(\frac{dw^*}{dg}\right) < 0 \Leftrightarrow \frac{dw^*}{dg} > \frac{-2gw^*}{1-g^2}$$

Noticing that

$$\frac{dw^*}{dg} = -\frac{\partial\Omega^*/\partial g}{\partial\Omega^*/\partial w^*}$$

it results that $\frac{\partial a^{\phi}}{\partial g^*} < 0$ if

$$-\frac{\partial\Omega^*}{\partial g} > -\frac{\partial\Omega^*}{\partial w^*}\frac{2gw^*}{1-g^2} \Rightarrow \frac{\partial\Omega^*}{\partial w^*}\frac{2gw^*}{1-g^2} > \frac{\partial\Omega^*}{\partial g} \tag{B.11}$$

Using the implicit function theorem, first, from (B.8) we derive

$$\frac{\partial\Omega^*}{\partial g} = -\theta_1\left(\frac{1+\alpha}{\alpha\,(1-\alpha)}\right)\left(\frac{w_{t+1}}{(1+g)\,A_t}\right)^{\frac{1+\alpha}{\alpha(1-\alpha)}}\frac{1}{(1+g)} - \frac{\theta_2\phi^{\frac{2}{\alpha}}A_t^{\frac{1-2\alpha}{\alpha(1-\alpha)}}}{\theta^{\frac{\alpha}{1-\alpha}}}\times$$
$$\times\left\{\frac{\partial}{\partial g}\left[\frac{(1-g^2)^{\frac{2}{\alpha}}\left((1+g)^{\frac{1}{1-\alpha}}A_t^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right)}{(1+g)^{\frac{1}{\alpha(1-\alpha)}}\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\phi\,(1-g^2)\,A_t\right)^{\frac{1}{\alpha}}\right)}\right]\right\}$$

Substituting for the derivative in the braces on the second line and rearranging,

$$\frac{\partial\Omega^*}{\partial g} = -\theta_1 \left(\frac{1+\alpha}{\alpha(1-\alpha)}\right)\left(\frac{w_{t+1}}{(1+g)A_t}\right)^{\frac{1+\alpha}{\alpha(1-\alpha)}} \frac{1}{(1+g)} - \frac{\theta_2 \phi^{\frac{2}{\alpha}} A_t^{\frac{1-2\alpha}{\alpha(1-\alpha)}}}{\theta^{\frac{\alpha}{1-\alpha}}} \times$$

$$\times \left\{ -\frac{\left(\frac{2}{\alpha}\right) 2g\left(1-g^2\right)^{\frac{2-\alpha}{\alpha}} \left((1+g)^{\frac{1}{1-\alpha}} A_t^{\frac{1}{1-\alpha}} \theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right)}{(1+g)^{\frac{1}{\alpha(1-\alpha)}} \left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\phi\left(1-g^2\right)A_t\right)^{\frac{1}{\alpha}}\right)} + \right.$$

$$+ \frac{\left(\frac{1}{1-\alpha}\right)\left(1-g^2\right)^{\frac{2}{\alpha}} (1+g)^{\frac{\alpha}{1-\alpha}} A_t^{\frac{1}{1-\alpha}} \theta^{\frac{\alpha}{1-\alpha}}}{(1+g)^{\frac{1}{\alpha(1-\alpha)}} \left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\phi\left(1-g^2\right)A_t\right)^{\frac{1}{\alpha}}\right)} +$$

$$- \frac{\left(\frac{1}{\alpha(1-\alpha)}\right)\left(1-g^2\right)^{\frac{2}{\alpha}} (1+g)^{-1} \left((1+g)^{\frac{1}{1-\alpha}} A_t^{\frac{1}{1-\alpha}} \theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right)}{(1+g)^{\frac{1}{\alpha(1-\alpha)}} \left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\phi\left(1-g^2\right)A_t\right)^{\frac{1}{\alpha}}\right)} +$$

$$\left. - \frac{\left(\frac{1}{\alpha}\right) 2g\left(1-g^2\right)^{\frac{1-\alpha}{\alpha}} (1+g)^{\frac{1}{\alpha(1-\alpha)}} \theta\left(\phi A_t\right)^{\frac{1}{\alpha}} \left(1-g^2\right)^{\frac{2}{\alpha}} \left((1+g)^{\frac{1}{1-\alpha}} A_t^{\frac{1}{1-\alpha}} \theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right)}{\left[(1+g)^{\frac{1}{\alpha(1-\alpha)}} \left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\phi\left(1-g^2\right)A_t\right)^{\frac{1}{\alpha}}\right)\right]^2} \right\}$$

Finally, using this with (B.10) and simplifying, it results that condition (B.11) holds since

$$\theta_3 (1+g) + \alpha (1+g) \theta_1 \left(\frac{w_{t+1}}{A_{t+1}}\right)^{\frac{1+\alpha}{\alpha(1-\alpha)}} + \frac{\theta_2 \bar{a}^{\frac{2}{\alpha}} A_{t+1}^{\frac{1-2\alpha}{\alpha(1-\alpha)}}}{\theta^{\frac{\alpha}{1-\alpha}} \left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a} A_{t+1}\right)^{\frac{1}{\alpha}}\right)} \times$$

$$\times \left[\alpha g \left(A_{t+1}^{\frac{1}{1-\alpha}} \theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right) + \alpha A_{t+1}^{\frac{1}{1-\alpha}} \theta^{\frac{\alpha}{1-\alpha}} + \alpha g w_{t+1}^{\frac{1}{1-\alpha}}\right] > 0$$

2. $\partial a^a / \partial g$:

$$\frac{\partial a^a}{\partial g} = \frac{1}{(1-\alpha)\theta^{\frac{\alpha}{1-\alpha}} A_t^{\frac{1}{1-\alpha}}} \left(\frac{w^*}{1+g}\right)^{\frac{\alpha}{1-\alpha}} \left(\frac{\left(\frac{dw^*}{dg}\right)(1+g) - w^*}{(1+g)^2}\right) < 0$$

if

$$\frac{dw^*}{dg} < \frac{w^*}{1+g}$$

Using

$$\frac{dw^*}{dg} = -\frac{\partial\Omega^*/\partial g}{\partial\Omega^*/\partial w^*}$$

it results that $\frac{\partial a^a}{\partial g^*} < 0$ if

$$-\frac{\partial\Omega^*}{dg} < \frac{\partial\Omega^*}{\partial w^*}\frac{w^*}{1+g} \tag{B.12}$$

Substituting the relevant values of $(\partial\Omega^*/\partial g)$ and $(\partial\Omega^*/\partial w^*)$ and rearranging, condition (B.12) results always verified since the followings holds

$$\frac{(1-\alpha)\,\theta_2\bar{a}^{\frac{2}{\alpha}}A_{t+1}^{\frac{1-2\alpha}{\alpha(1-\alpha)}}\left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}} - w_{t+1}^{\frac{1}{1-\alpha}}\right)}{(1-g)\,\theta^{\frac{\alpha}{1-\alpha}}\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)}\left[\frac{w_{t+1}^{\frac{1}{\alpha}} + g\theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}}{\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)} + 2g + 1\right] > 0$$

□

Lemma 5. Using $\partial a^{\phi}/\partial g$ and $\partial a^a/\partial g$ from Proposition 5, it derives that

$$\varepsilon^g_{a^\phi} \equiv \frac{\partial a^\phi}{\partial g}\frac{g}{a^\phi} = \frac{g}{\alpha\left(1-g^2\right)w^*}\left(-2gw^* - \left(1-g^2\right)\frac{dw^*}{dg}\right)$$

and

$$\varepsilon^g_{a^a} \equiv \frac{\partial a^a}{\partial g}\frac{g}{a^a} = \frac{g}{(1-\alpha)\,w^*}\left(\frac{\left(\frac{dw^*}{dg}\right)(1+g) - w^*}{(1+g)}\right)$$

Hence,

$$\Delta\varepsilon_g \equiv \varepsilon^g_{a^a} - \varepsilon^g_{a^\phi} = \frac{dw^*}{dg}\frac{g}{\alpha\,(1-\alpha)\,w^*} - \frac{g}{1+g}\left(\frac{\alpha\,(1+g) - 2g}{\alpha\,(1-\alpha)\,(1-g)}\right)$$

which implies that $\Delta\varepsilon_g > 0$ if

$$\frac{dw^*}{dg} > \frac{w^*}{1-g^2}\left(\alpha\,(1+g) - 2g\right) \Rightarrow$$

$$\Rightarrow -\frac{\partial\Omega^*}{\partial g} > \frac{\partial\Omega^*}{\partial w^*}\left[\frac{w^*}{1-g^2}\left(\alpha\,(1+g) - 2g\right)\right]$$

Substituting for $(\partial\Omega^*/\partial g)$ and $(\partial\Omega^*/\partial w^*)$ from above, it results that $\Delta\varepsilon_g > 0$ since the following holds

$$(1-\alpha)\left\{\theta_3(1+\alpha) - \frac{\alpha\theta_2\bar{a}^{\frac{2}{\alpha}}A_{t+1}^{\frac{1-2\alpha}{\alpha(1-\alpha)}}}{\theta^{\frac{\alpha}{1-\alpha}}\left(w_{t+1}^{\frac{1}{\alpha}} - \theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right)^2} \times \right.$$
$$\left. \times \left[\left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}}\right)\left(\theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right) - w_{t+1}^{\frac{1}{1-\alpha}}w_{t+1}^{\frac{1}{\alpha}}\right]\right\} > 0$$

by noticing that the term in square brackets is negative

$$\left(A_{t+1}^{\frac{1}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}}\right)\left(\theta\left(\bar{a}A_{t+1}\right)^{\frac{1}{\alpha}}\right) < w_{t+1}^{\frac{1}{1-\alpha}}w_{t+1}^{\frac{1}{\alpha}} \Rightarrow w_{t+1}^{\frac{1}{\alpha(1-\alpha)}} > A_{t+1}^{\frac{1}{\alpha(1-\alpha)}}\theta^{\frac{1}{1-\alpha}}\bar{a}^{\frac{1}{\alpha}} \Rightarrow$$

$$\Rightarrow w_{t+1} > A_{t+1}\theta^{\alpha}\bar{a}^{1-\alpha} \equiv \hat{w}$$

which is verified in steady state since $w^* \in (\hat{w}, \tilde{w})$. □

Corollary 3. From Proposition 3, $n^{*'}(g^*) > 0$ *iff* $(1-a^a)\,a^{\phi'}(g^*) > \left(1-a^{\phi}\right)a^{a'}(g^*)$; namely *iff*

$$a^{\phi'} - a^{a'} > a^{\phi'}a^a - a^{a'}a^{\phi} \tag{B.13}$$

From Lemma 5, it derives that $a^{a'}a^{\phi} - a^{\phi'}a^a > 0$ which implies also that $\frac{a^{a'}}{a^{\phi'}} > \frac{a^a}{a_{\phi}}$. Given that $\frac{a^a}{a_{\phi}} > 1$, it results that $\frac{a^{a'}}{a^{\phi'}} > 1$ as well. Since, from Proposition 5, $a^{a'}(g) < 0$ and $a^{\phi'}(g) < 0$, it must also be the case that $\mid a^{a'} \mid > \mid a^{\phi'} \mid$. Hence condition (B.13) is always verified, since the right-hand side is always negative, while the left-hand side is always positive. □

Lemma 6 and Proposition 6. Using (A.19), it follows

1. Properties of $n_1^C(n_t)$:

   (a) $n_1^C(0) = 1 - \left(\frac{\alpha}{\theta_6}\right)^{\frac{\alpha}{1+\alpha}} \in (0,1)$.

   (b) Notice that for $n_t = \underset{\sim}{n}_t \Leftrightarrow w_{t+1} = \omega_1^C(n_t)$, $a_{t+1}^a = 1$, which implies that $n_1^C(\underset{\sim}{n}_t) = \underset{\sim}{n}_t = n_2^C(\underset{\sim}{n}_t)$. Formally, $n_1^C(\underset{\sim}{n}_t) = 1 - \frac{\bar{a}^{\frac{1}{\alpha}}(1+\alpha)}{\theta_6 - \alpha + \bar{a}^{\frac{1}{\alpha}}(1+\alpha)} = \frac{\theta_6 - \alpha}{\theta_6 - \alpha + \bar{a}^{\frac{1}{\alpha}}(1+\alpha)} = \underset{\sim}{n}_t$.

   (c) Differentiating and rearranging,

$$\frac{\partial n_1^C(n_t)}{\partial n_t} = -\frac{1}{\theta_6^{\frac{\alpha}{1+\alpha}}(1+\alpha)}\left\{-(1-n_t)^{\frac{1}{1+\alpha}-1}\left[n_t\left(\bar{a}^{\frac{1}{\alpha}}(1+\alpha)-\alpha\right)+\alpha\right]^{\frac{\alpha}{1+\alpha}} + \right.$$
$$\left. + \alpha(1-n_t)^{\frac{1}{1+\alpha}}\left[n_t\left(\bar{a}^{\frac{1}{\alpha}}(1+\alpha)-\alpha\right)+\alpha\right]^{\frac{\alpha}{1+\alpha}-1}\left(\bar{a}^{\frac{1}{\alpha}}(1+\alpha)-\alpha\right)\right\} \quad \text{(B.14)}$$

Hence, $\frac{\partial n_1^C(n_t)}{\partial n_t} < (>)\, 0$ if $\alpha(1-n_t)\left(\bar{a}^{\frac{1}{\alpha}}(1+\alpha)-\alpha\right) > (<)\, n_t\left(\bar{a}^{\frac{1}{\alpha}}(1+\alpha)-\alpha\right)+$ α, which implies

$$\frac{\partial n_1^C(n_t)}{\partial n_t}\begin{cases} < 0 & \text{if } n_t < \underset{\sim}{n} \\ > 0 & \text{otherwise} \end{cases}$$

with

$$\underset{\sim}{n} \equiv \frac{\alpha\left(\bar{a}^{\frac{1}{\alpha}}-1\right)}{\bar{a}^{\frac{1}{\alpha}}(1+\alpha)-\alpha} \quad \text{(B.15)}$$

Further, notice that $\underset{\sim}{n} > \underset{\approx}{n} \Rightarrow \alpha(\bar{a}^{\frac{1}{\alpha}}-1)(\bar{a}^{\frac{1}{\alpha}}(1+\alpha)+\theta_6-\alpha) > (\theta_6-\alpha)(\bar{a}^{\frac{1}{\alpha}}(1+\alpha)-\alpha) \Rightarrow \alpha(1+\alpha)\,\bar{a}^{\frac{1}{\alpha}} - \theta_6 - \alpha^2 > 0$, which is always verified for any $\bar{a}$ sufficiently greater than one. Notice, finally, that if, otherwise, $\underset{\sim}{n} < \underset{\approx}{n}$ nothing would change for the dynamics and the results. Hence, for simplicity we can assume that $\bar{a}$ is sufficiently high that $n_1^{C'}(n_t) < 0$ for any $n_t < \underset{\approx}{n}_t$.

(d) From point (c), it results

$$\frac{\partial^2 n_1^C(n_t)}{\partial n_t^2} = -\frac{1}{\theta_6^{\frac{\alpha}{1+\alpha}}(1+\alpha)}\left\{-\frac{\alpha}{1+\alpha}(1-n_t)^{\left(\frac{1}{1+\alpha}-1\right)-1}\left[n_t\left(\bar{a}^{\frac{1}{\alpha}}(1+\alpha)-\alpha\right)+\alpha\right]^{\frac{\alpha}{1+\alpha}} + \right.$$
$$+\frac{\alpha}{1+\alpha}(1-n_t)^{\left(\frac{1}{1+\alpha}-1\right)}\left[n_t\left(\bar{a}^{\frac{1}{\alpha}}(1+\alpha)-\alpha\right)+\alpha\right]^{\frac{\alpha}{1+\alpha}-1}\left(\bar{a}^{\frac{1}{\alpha}}(1+\alpha)-\alpha\right)+$$
$$-\alpha\left(\bar{a}^{\frac{1}{\alpha}}(1+\alpha)-\alpha\right)\left[\left(\frac{1}{1+\alpha}\right)(1-n_t)^{\left(\frac{1}{1+\alpha}-1\right)}\left(n_t\left(\bar{a}^{\frac{1}{\alpha}}(1+\alpha)-\alpha\right)+\alpha\right)^{\frac{\alpha}{1+\alpha}-1}+\right.$$
$$\left.\left.+\left(\frac{1}{1+\alpha}\right)(1-n_t)^{\frac{1}{1+\alpha}}\left(n_t\left(\bar{a}^{\frac{1}{\alpha}}(1+\alpha)-\alpha\right)+\alpha\right)^{\left(\frac{\alpha}{1+\alpha}-1\right)-1}\left(\bar{a}^{\frac{1}{\alpha}}(1+\alpha)-\alpha\right)\right]\right\} > 0$$

since the term in the braces is negative.

2. Properties of $n_3^C(n_t)$:

(a) $n_3^C\left(\hat{n}\right) = \theta^2\theta_4^{-1} = \hat{n}$; $n_3^C(1) = \theta\theta_4^{-\frac{1}{2}} < 1$.

(b) $\frac{\partial n_3^C(n_t)}{\partial n_t} = \frac{\theta}{2}(n_t\theta_4)^{-\frac{1}{2}} > 0$, from which also follows that $\frac{\partial^2 n_3^C(n_t)}{\partial n_t^2} < 0$.

Finally, the proof of Proposition 6 follows directly from the properties of this Lemma. □

Appendix C Parameters

- $\theta \equiv \beta \exp\left[(\delta-\eta)(1-\gamma)^{-1}\right] = \dfrac{\alpha^{\frac{1}{1-\gamma}}\left[(1-\alpha)(1-\gamma)\right]^{\frac{1-\alpha}{\alpha}}}{(1-\gamma(1-\alpha))^{\frac{1-\gamma(1-\alpha)}{\alpha(1-\gamma)}}} \in (0,1)$

- $\theta_1 \equiv \dfrac{\tau^h}{2\theta^{\frac{2\alpha}{1-\alpha}}} + \dfrac{\alpha\left((1-\alpha)\tau^e\right)^{\frac{1}{\alpha}}}{(1+\alpha)\theta^{\frac{1+\alpha}{1-\alpha}}} = \dfrac{(1-\gamma)\left[(1+\alpha)\alpha^{\frac{\gamma}{1-\gamma}} + 2(1-\alpha)(1-\gamma(1-\alpha))^{\frac{\gamma}{1-\gamma}}\right]}{2(1+\alpha)\alpha^{\frac{\gamma}{1-\gamma}}\theta^{\frac{2\alpha}{1-\alpha}}}$

- $\theta_2 \equiv \theta\left((1-\alpha)\tau^e\right)^{\frac{1}{\alpha}} + \theta^2\left(\dfrac{\tau^h}{2}\right) = \dfrac{\theta^2(1-\gamma)}{2\alpha^{\frac{1}{1-\gamma}}}\left[\alpha^{\frac{1}{1-\gamma}} + 2(1-\alpha)(1-\gamma(1-\alpha))^{\frac{\gamma}{1-\gamma}}\right]$

- $\theta_3 \equiv \left(\dfrac{\alpha}{1+\alpha}\right)\left((1-\alpha)\tau^e\right)^{\frac{1}{\alpha}} = \dfrac{\theta(1-\gamma)(1-\alpha)(1-\gamma(1-\alpha))^{\frac{\gamma}{1-\gamma}}}{\alpha^{\frac{\gamma}{1-\gamma}}(1+\alpha)}$

- $\theta_4 \equiv \left[\dfrac{2\theta\left(\tau^e(1-\alpha)\right)^{1/\alpha}}{\tau^h} + \theta^2\right] = \theta^2\left[\dfrac{2(1-\alpha)(1-\gamma(1-\alpha))^{\frac{\gamma}{1-\gamma}}}{\alpha^{\frac{1}{1-\gamma}}} + 1\right]$

- $\theta_5 \equiv \left[\dfrac{(1+\alpha)\tau^h}{2\left(\tau^e(1-\alpha)\right)^{1/\alpha}\theta^{\frac{2\alpha}{1-\alpha}}} + \dfrac{\alpha}{\theta^{\frac{1+\alpha}{1-\alpha}}}\right] = \dfrac{1}{\theta^{\frac{1+\alpha}{1-\alpha}}}\left[\dfrac{(1+\alpha)\alpha^{\frac{1}{1-\gamma}}}{2(1-\alpha)(1-\gamma(1-\alpha))^{\frac{\gamma}{1-\gamma}}} + \alpha\right]$

- $\theta_6 \equiv \theta_5\theta^{\frac{1+\alpha}{1-\alpha}} = \left[\dfrac{(1+\alpha)\alpha^{\frac{1}{1-\gamma}}}{2(1-\alpha)(1-\gamma(1-\alpha))^{\frac{\gamma}{1-\gamma}}} + \alpha\right]$

- $\theta_7 \equiv \dfrac{2^\alpha(1-\alpha)\tau^e}{\tau^{h^\alpha}} = \left[\dfrac{(1+\alpha)\alpha^{\frac{1}{1-\gamma}}}{2(1-\alpha)(1-\gamma(1-\alpha))^{\frac{\gamma}{1-\gamma}}} + \alpha\right]$

- $\theta_8 \equiv \left((1-\alpha)^{(1-\alpha)}\tau^e\right)^{\frac{1}{\alpha}} = \dfrac{\theta_3(1+\alpha)}{\alpha(1-\alpha)} = \dfrac{\theta(1-\gamma)(1-\gamma(1-\alpha))^{\frac{\gamma}{1-\gamma}}}{\alpha^{\frac{1}{1-\gamma}}}$