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Income Distribution in Network Markets

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Abstract We enquiry about the effects of first and second order stochastic dominance shifts of the distribution of the consumers' willingness to pay, within the standard model of a market with network externalities and hump-shaped demand curve. This issue is analysed in the polar cases of perfect competition and monopoly. We find that, while under perfect competition both types of distributional changes result in higher output provided marginal costs are low enough, in the monopoly case the final outcome depends on the way income distribution and the network externality interact in determining market demand elasticity. *JEL Classification no.s*: D31, D40, L11.

Keywords Network externalities, income distribution, stochastic dominance.

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1 Introduction

Industries based on physical and non-physical networks have provided a significant boost to economic progress in recent decades. Among non-physical networks, information and telecommunication networks are characterized by positive consumption externalities, known as network externalities, which play an important role in defining the salient features of these markets. They occur when the benefit accruing to a consumer from a product depends positively on the number of the other users of the same product: the more consumers access a network, the more valuable the network becomes to each consumer.[1] These goods and services can confer either direct benefits associated to a larger installed base of compatible products or technologies (like the benefit from exchanging the good), or indirect benefits associated to a wider variety of the same product, a greater availability of complementary services, or a number of upgrades.

In this paper we examine the optimal output decisions of a network provider, drawing attention on the interactions between the firms' choices and the demand changes resulting from changes in the distribution of the consumers' willingness to pay. This is modelled by assuming that market demand derives from the cumulative distribution of consumers' incomes under the hypothesis of unit demand, and by introducing network externalities into consumer preferences. In this sense, we interpret the consumers' heterogeneity in terms of a non-degenerate income distribution – accordingly, we focus on the relationship between the equilibrium network size and the personal distribution of incomes.[2] Following Rohlfs (1974) and Economides and Himmelberg (1995), we model consumer utility in the presence of demand-side externalities as a function of the expected network size that equals, under fulfilled expectations, the actual network size[3] – which, as is well known, causes the demand curve to be hump-shaped. We examine the firms' reactions to distributive shocks in the two polar cases of a perfect competition and monopoly, and ask how changes in income distribution interact with

[1] Well known examples of network commodities include mobile phones, fax machines, game consoles, software industry, etc. General surveys are provided by Economides (1996a) and Shy (2011). A further source of network (known as 'tariff-mediated') externalities is related to the pricing of network goods (Laffont *et al*, 1998).

[2] As we model the distribution of the consumers' willingness to pay as grounded on income heterogeneity, our approach might be seen as complementary to other works focusing on preference heterogeneity (e.g., Katz and Shapiro, 1985; Lambertini, 2001).

[3] See Swann (2002) for an investigation of the functional form of network effects, and Kate and Niels (2006) for an analysis of the properties of fulfilled-expectations demand. A very general framework, tailored to assess existence problems in the context of oligopoly, is provided by Amir and Lazzati (2011).

the strength of network effects in determining the optimal firms' choices and hence market equilibria.[4] These changes in income distribution are modeled as first and second order stochastic dominance shifts of the distribution itself (about which we make no *a priori* assumption): as is well known, the former entails an increase in aggregate (mean) income, while the latter allows considering purely distributive changes amounting to inequality shocks in a well defined sense.

Overall, we find that in the perfect competition case both kinds of income changes exert a positive effect on equilibrium output levels, provided marginal costs are sufficiently low. On the other hand, in the monopoly case a crucial role is played by the interplay between the given cost level, and the way the network externality and income distribution contribute to determine the price elasticity of demand. Indeed, while in general a strong externality narrows the scope for monopoly power, one key insight we draw from our analysis is that the firm will react differently to a change in the same level of its demand elasticity, following the different ways in which income distribution and the externality may combine to deliver that elasticity level. As a result, if the distributional shock is of the first order type, a monopolist will expand equilibrium output when a sufficiently weak network externality is associated to sufficiently high marginal costs; whereas she may react to such a shock by restricting her output, when a strong network externality is associated to a low enough level of the marginal costs. By contrast, a purely distributive second order shock (decreasing dispersion for given average willingness to pay) leads to higher output for the monopolist, if the network externality is sufficiently weak and marginal costs lie within some interval which – depending on parameters – may include zero costs.

The paper is organized as follows. In the next Section we place our work in the context of the relevant literature. In Section 3 market demand is derived within a general discrete choice framework with network externalities, while in Section 4 we enquire about the way different (first or second order stochastic dominance) distributional shifts of the consumers' willingness to pay affect the hump-shaped demand curve. Sections 5 and 6 present our main results on how these distributional changes affect market equilibria, in the polar cases of perfect competition and monopoly, respectively. Finally, some concluding remarks are gathered in Section 7.

[4]The way income inequality can affect the firm's optimal choices and equilibrium is studied, amomg others, by Foellmi and Zweimüller in a monopolistic competition model with product diversity (2004), and in a growth model with innovators (2006).

2 Related literature

Beginning with the seminal work by Rohlfs (1974), network externalities have become crucial to a better understanding of many important issues, most notably technology adoption and standard compatibility, with relevant implications for competition policy. Indeed, it is well known that network externalities may affect the adoption of new technologies through their effect (*via* expectations) on consumers' choices: a minimum market size is typically called for in order to get a positive sale feedback, as people switch from an old product to the new one. This may lead to path-dependence, as network effects make previous outcomes matter in the relevant dynamics. In this scenario, there may arise a limit on the number of active firms, with one technology emerging as the sole standard (Arthur, 1989), and whether markets coordinate on the appropriate one is clearly a key question – the answer to which crucially depends on switching costs (Farrell and Klemperer, 2007): thus, e.g., the social and private incentives for technology adoption may diverge in the presence of compatibility benefits, with lock-in effects as consumers settle at an inferior standard (Farrell and Saloner, 1985).

The key role of compatibility was addressed in the seminal paper by Katz and Shapiro (1985). More generally, compatibility with existing products is just one way firms can take advantage of network externalities: exploiting them in a product's introductory stage may account for strategies such as penetration pricing (Katz and Shapiro, 1994, p. 107) which allow firms to attract an installed-base, product pre-announcements which affect technology adoption decisions by encouraging prospective customers' purchase (Farrell and Saloner, 1986) or deter competitors' entry (Haan, 2003), or strategies allowing the diffusion of trial versions (Haruvy and Prasad, 1998) or pirated software (Conner and Rumelt, 1991; Takeyama, 1994; Peitz and Waelbroeck, 2006) aimed to raise the users' base and charge higher prices on the major segments of complete or legal versions.

More generally, theoretical insights and empirical results provided by the literature on network externalities turn out quite often to pose challenging questions: indeed, economic predictions are often shown to fail in markets with network externalities, and a series of paradoxes are shown to revert standard results. In this vein, standard issues in Industrial Organization theory have been revisited assuming the presence of network externalities in both static and dynamic contexts, focusing on the one hand on the ways in which firms benefit from the presence of network externalities, and on the other investigating whether network externalities support market competition and improve welfare, or rather hurt market efficiency. From an empirical perspective, econometric evidence on network effects in many industries is discussed by Gandal (2008), while the theoretical works which revisit many standard issues in IO range from pricing strategies such as submarginal-cost pricing (Yano and Dei, 2006) and introductory prices (Cabral *et al.*, 1999), which increase over time due to the need of acquiring a large consumer base thus reverting the Coasian dynamics, to product choices which may overturn the Spence's conclusion about product quality provisions (Lambertini and Orsini, 2001) or show the existence of incentives for quality degradation (Haan, 2004), and R&D investments which lead to the adoption of a less risky technology than the socially optimal one (Choi, 1994).[5] The IO literature also shows a number of cases in which network externalities play a competitive effect - e.g. they provide incentives for inviting rivals to enter the market (Economides, 1996b) or for vertical integration (Dogan, 2009), and limit the detrimental effects of exclusive contracts under multi-homing (Doganoglu and Wright, 2010) - or rather they affect market share dynamics leading to market tipping (Cabral, 2011), with negative consequences on welfare.[6] In this general perspective, the theoretical question of how the distribution of the consumers' willingness to pay affect market equilibrium in the case of network externalities seems worth investigating.

3 Network externalities and market demand

In this section we set out our basic assumptions. Let $s(y,x)$, $s(y,0)=0$, be the (gross) surplus enjoyed by a consumer with income $y \in [y_m, y_M]$, $0 \leq y_m < y_M$, when she consumes one unit of the network commodity, where x denotes the network size. The standard indifference condition at price $p \geq 0$ identifies the marginal consumers by their income level y^* such that $s(y^*, x) = p$. Letting subscripts denote derivatives, by standard assumptions $y^* = y(p,x)$ satisfies $y_p(p,x) > 0$ and $y_x(p,x) < 0$.[7] For any given pair (p,x), the condition $y \geq y(p,x)$ identifies the set of buyers.

Let now $F(y,\theta)$ be the distribution of income, so that $F : [y_m, y_M] \times \Theta \rightarrow$

[5] The choice of excessively risky R&D strategies is the standard result in the latter models.

[6] All of which cannot but affecting the antitrust and the regulatory perspective relevant for network industries. In this connection Regibeau (2004) focuses on network externalities as sources of antitrust concern, while some regulatory implications of the presence of network externalities are discussed by Cabral (2011) and Yannelis (2002) with reference to access charges, and by Gruber and Verboven (2001) with regard to entry and standard regulation.

[7] Possible specifications are $y(p,x) = p/v(x)$, with $v(x)$ positive and increasing (Economides and Himmelberg, 1995), or $y(p,x) = p - kx$ with $k > 0$ (Katz and Shapiro, 1985).

$[0,1]$, where $\theta \in \Theta$ is a parameter to be made precise below.[8] If the whole population is normalized to unity, market demand for given network size x is given by

$$D(p, x; \theta) = 1 - F(y(p, x), \theta) \tag{1}$$

clearly decreasing in p and increasing in x. By the standard condition of fulfilled expectations, $x = D(p, x; \theta)$, any consistent price-quantity pair has to satisfy

$$x = 1 - F(y^*, \theta) \tag{2}$$

from which the (inverse) demand function $p(x, \theta)$ is derived.

It is well known that the slope of $p(x, \theta)$ can in general be positive or negative, depending on x: indeed, by (2) $p = 0$ is consistent with both $x = 0$ (demand price is zero when network size is zero), and $x = 1$ (all consumer buy for free, and network size is at maximum width).[9] Letting $f(y, \theta) = F_y(y, \theta)$ denote the positive income density, this slope can easily be obtained as

$$p_x(x, \theta) = -\frac{1 + f(y^*, \theta) y_x(p, x)}{f(y^*, \theta) y_p(p, x)} \tag{3}$$

which, recalling that $y_x(p, x) < 0$, and defining the (positive) elasticities $\varepsilon^x(p, x) = -(y_x x / y^*) > 0$ and $\varepsilon^p(p, x) = y_p p / y^*$, amounts to

$$p_x(x, \theta) = -\frac{1 - \eta(y^*, \theta)\varepsilon^x(p, x)}{\eta(y^*, \theta)\varepsilon^p(p, x)} \frac{p}{x} \tag{4}$$

where $\eta(y, \theta) = y f(y, \theta) / [1 - F(y, \theta)]$ is the (non negative) elasticity of $1 - F(y, \theta)$. This distribution elasticity can be used to characterize the income distribution, and will play a major role in what follows.

Easy manipulations allow one to note some interesting relationship between elasticities. On the one hand, the elasticity of demand from definition (1) is simply given by

$$H(p, x; \theta) = -\frac{\partial D(p, x; \theta)}{\partial p} \frac{p}{D(p, x; \theta)} = \eta(y^*, \theta)\varepsilon^p(p, x) \tag{5}$$

[8] In general it will be $\Theta \subseteq \mathbb{R}$, the boundaries of Θ depending on the specific model at hand.

[9] Actually, one can have $p = p_m > 0$ at $x = 1$, whenever $y(p_m, 1) = y_m$ solves for a positive price, that is, whenever the poorest consumer can afford to buy at p_m (see examples 1 and 2 below). Most of the literature implicitly assumes $y_m = 0$, which implies $p_m = 0$.

which can be taken as the no-externality demand elasticity;[10] on the other hand, the price elasticity from equation (4) obviously satisfies

$$\Gamma(p, x; \theta) = -\frac{dx}{dp}\frac{p}{x} = \frac{\eta(y^*, \theta)\varepsilon^p(p, x)}{1 - \eta(y^*, \theta)\varepsilon^x(p, x)} \tag{6}$$

such that, with no network effect (i.e., $y_x = 0 = \varepsilon^x(p, x)$), one would have $\Gamma = H$. Equations (5) and (6) may be specialized to well known formulation of y^*. If $y^* = p - kx$, we have, $H = \eta(y^*, \theta) + kf(y^*, \theta)$ and (6) reduces to

$$\Gamma_1(p, x; \theta) = \frac{\eta(y^*, \theta) + kf(y^*, \theta)}{1 - kf(y^*, \theta)} \tag{6'}$$

obviously yielding $\Gamma_1 = \eta$ whenever $k = 0$. On the other hand, if $y^* = p/v(x)$, we have $\varepsilon^p(p, x) = 1$ and $\varepsilon^x(p, x) = e(x)$, with $e(x)$ the (positive) elasticity of $v(\cdot)$. Hence, $H = \eta(y^*, \theta)$ and (6) boils down to

$$\Gamma_2(p, x; \theta) = \frac{\eta(y^*, \theta)}{1 - \eta(y^*, \theta)e(x)} \tag{6''}$$

which again gives $\Gamma_2 = \eta$ for $e(x) = 0$.

We are now in the position to address the issue of the effects of distributional shocks on the demand curve.

4 Income distribution, network externalities, and the demand curve

In this section we enquire about the way shifts in the income distribution, modeled by changes in the parameter θ, affect the demand curve. To this end, we note preliminarily that, given (1), the following must hold

$$p_\theta(x, \theta) = -\frac{F_\theta(y^*, \theta)}{f(y^*, \theta)y_p(p, x)} \tag{7}$$

which of course has different meanings according to the kind of shock one is studying. Before going into the details of the latter, we note that in general the demand curve shifts upwards or downwards according as F_θ is negative or positive, that is, according as a change in θ shifts the distribution F down or up around income y^*.

[10] The demand function would then be $1 - F(y^*, \theta)$, which can be thought of as the 'base' demand, whose price elasticity amounts to the distribution elasticity weighted by the p-elasticity of y^*. With no network effects, the latter would be one for $s(y^*, \cdot) = y^*$ independently of x (e.g., Benassi *et al.*, 2002).

4.1 Increasing aggregate income: First Order Stochastic Dominance

Let now θ be a parameter measuring first order stochastic dominance (FSD). This means that

$$F_\theta(y, \theta) \leq 0 \tag{8}$$

for all $y \in [y_m, y_M]$ (the inequality holding strictly somewhere). As is well known, if θ is a FSD parameter, an increase thereof makes all income classes richer, in the sense that for any given y the probability decreases of having income below y: accordingly, aggregate (mean) income μ is increasing in θ.

Unsurprisingly, (7) implies that this distributional shock shifts up the demand curve, as $p_\theta(x, \theta) > 0$ for all $x \in (0, 1)$, which in turn means that for a given p the critical mass is now lower: a richer market makes for the network externality to have a stronger effect. As an example, Figure 1 reports the case where $y(p, x) = p/\sqrt{x}$, and $f(y, \theta) = 1/\theta$, so that an increase in θ 'stretches' rightwards a uniform distribution with support $[0, \theta]$: higher θ means higher demand at all prices, and hence lower critical mass for a given constant marginal cost $c > 0$.

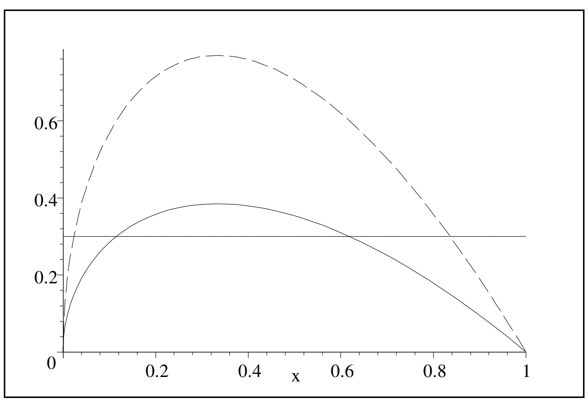

Fig. 1: Demand shift with FSD shock.

Rather obviously, if $c = p$ is the supply curve of a perfectly competitive industry working at constant marginal costs, higher θ (dashed curve) means also higher equilibrium quantity: in a richer market, the network settles at a larger size.

4.2 Increasing concentration: Second Order Stochastic Dominance

A standard way to formalize income concentration is *via* the notion of Second Order stochastic dominance (SSD). The parameter θ shifts the distribution is the SSD sense if

$$\int_{y_m}^{y} F_\theta(z,\theta)dz \leq 0 \tag{9}$$

for all $y \in [y_m, y_M]$. It is well known that, under this definition, higher θ means a less unequal distribution – an inequality averse social planner would prefer it to a lower θ distribution (e.g., Lambert, 2001 ch.3); if the further restriction is added that $\int_{y_m}^{y_M} F_\theta(z,\theta)dz = 0$ (i.e., mean income is not altered by changes in θ), θ ranks equal mean distributions by their Lorenz curve. This we shall assume to be the case, which allows us to focus on purely distributive effects.

Now notice that the condition that different values of θ leave mean income unchanged, together with (9), implies that the function $F_\theta(y,\theta)$ crosses zero at least once, being negative around $F_\theta(y_m,\theta) = 0$. This immediately implies, from (7), that $p_\theta = 0$ for all points (p,x) on the demand curve, such that the income of the consumers who are marginal at (p,x), $y^* = y(p,x)$, solves $F_\theta(y^*,\theta) = 0$: following a change in θ the before-change and the after-change demand curves cross at least once. We can formalize this intuition in the following Proposition 1:

Proposition 1 *Let θ be an equal-mean* SSD *shift variable of the income distribution $F(y,\theta)$. Then, following a marginal change in θ the (inverse) demand functions cross at least once, the rightmost crossing being such that the lower concentration curve cuts from above the higher-concentration curve.*

Proof. See Appendix ■

A noteworthy case may be that of 'single crossing', in which case $F_\theta(\cdot,\theta) = 0$ is verified in the interior of $[y_m, y_M]$ only once: Proposition 1 and equation (2) then imply that the demand curves cross only once, as can be seen in Figure 2.[11]

[11] We use a Cauchy distribution $f(y,\theta) = 2\theta/\{[4\theta^2 + (2u-1)^2]\arctan((2\theta)^{-1})\}$, normalized over $[0,1]$, unimodal and symmetric around its mean $\mu = 1/2$; in this example $\theta > 0$ is an inverse single crossing ssd parameter, with higher θ associated to lower concentration. Using $y^* = p/\sqrt{x}$, we have $p(x,\theta) = \sqrt{x}\{\theta\tan[(1-2x)\arctan((2\theta)^{-1})] + 1/2\}$. Figure 2 compares $p(x,1/2)$ (solid curve) with $p(x,1/4)$ (dashed curve): since $f(y,\theta)$ is symmetric, crossing occurs at $x = 1/2$.

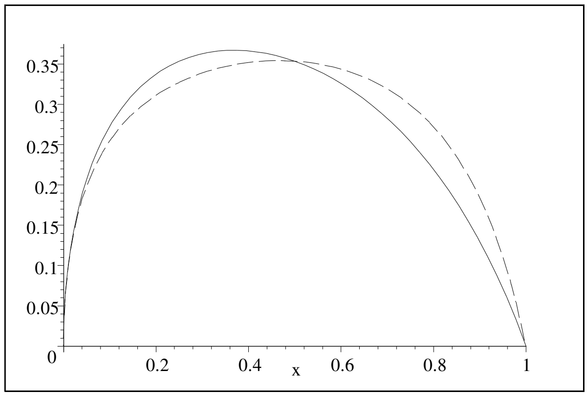

Fig. 2: Demand shift with single crossing SSD shock.

We are now ready to use our framework to enquire about the way market equilibrium is affected by a change in the distribution of income across consumers, which we model as a FSD (aggregate income) or a SSD (concentration) change in the parameter θ of $F(\cdot, \theta)$. We focus on the polar cases of perfect competition and monopoly, and assume throughout constant marginal costs, $c \geq 0$.

5 Income distribution and market equilibrium: perfect competition

Given a flat supply curve, the perfect competition case is in general quite simple, particularly so as far as a FSD shock is concerned: as already seen in Section 3.1, such a shock shifts upwards the demand curve: hence it lowers the critical mass, and (assuming equilibrium to be at the stable demand-supply intersection) it raises equilibrium output (see again Figure 1).

As to the SSD case, any given increase in income concentration as measured by a change in θ can in principle have different effects, depending on the position of the intersection of the (flat) supply curve with the after-shock demand curve: indeed, from Proposition 1 we know that multiple or single crossing of the demand curves may take place, according as the SSD shift of the distribution entails multiple or single crossing of the latter. However, if the marginal costs are sufficiently low, greater concentration always yields higher output: from the same Proposition 1, the demand curve shifts out for sufficiently high quantity levels. We can formalize this in the following:

Proposition 2 *Let θ be an equal-mean* SSD *shift variable of the distribution $F(y, \theta)$: then there exists a level $\overline{c} > 0$ of the marginal cost, such that an* *increase in income concentration leads to higher output under perfect competition for all $c < \overline{c}$.*

Proof. Let $p(x, \theta) = c$ identify the competitive (stable) equilibrium for some given marginal cost $c \geq 0$. By implicit differentiation, equilibrium output increases following an increase in θ if $dx/d\theta = -(dp/d\theta)/(dp/dx) > 0$. Since $dp/dx < 0$ at equilibrium, $dx/d\theta > 0$ holds if $dp/d\theta > 0$ which, by (7), occurs whenever $F_\theta(y(c, x), \theta) < 0$. Let now $\overline{y} \in (y_m, y_M)$ be the lowest income value satisfying $F_\theta(\cdot, \theta) = 0$: then by Proposition 1 it must be $F_\theta(y, \theta) < 0$ for all $y < \overline{y}$. Also, let $\overline{c} > 0$ be defined by $y(\overline{c}, x) = \overline{y}$: such $\overline{c}$ certainly exists and is unique, since (a) $\lim_{p \to 0} y(p, x) = y_m$ and $\lim_{p \to \infty} y(p, x) = y_M$ (all consumers buy at nil price and none does for a sufficiently high price); (b) $y_p(p, x) > 0$ for any given positive $x \in (0, 1)$. Then for any $c < \overline{c}$ we have $y(c, x) < \overline{y}$, and hence $F_\theta(y(c, x), \theta) < 0 = F_\theta(y(\overline{c}, x), \theta)$, so that indeed $dp/d\theta > 0$. ■

Two final remarks can be made about the perfect competition case. First, the result of Proposition 2 is quite general: in particular, it is independent of there being single or multiple crossing of the distribution. Secondly, the output expansion effect of greater income concentration is conditional on the level of marginal costs only, in the sense that $\overline{c}$ is strictly positive: as c tends to zero, equilibrium output is bound to increase independently of the strength of the externality effect as measured by the elasticity ε^x. As we shall see, this is not so in the monopoly case, to which we now turn.

6 Income distribution and market equilibrium: monopoly

In the standard monopoly model, the usual elasticity condition must hold at the profit-maximizing price-quantity pair (p^*, x^*) such that $p^* = p(x^*, \theta)$:

$$p(x^*, \theta)\left(1 - \frac{1}{\Gamma(p^*, x^*; \theta)}\right) = c \tag{10}$$

about which two general considerations stand out.

First, one can re-write (10) as

$$p(x^*, \theta)\left(1 - \frac{1}{H(p^*, x^*; \theta)} + \sigma(p^*, x^*)\right) = c \tag{10'}$$

where $0 \leq \sigma(p, x) = \varepsilon^x(p, x)/\varepsilon^p(p, x)$, and $H(p^*, x^*; \theta) = \eta(y^*, \theta)\varepsilon^p(p^*, x^*)$ is the no-externality price-elasticity of demand: indeed, the weaker the network externality effect measured by $\varepsilon^x(p,x)$, the closer is the monopoly solution to the standard case. Notice that (10') implies a Lerner index (and hence a price-cost margin) which is lower, the stronger the externality: in this sense the latter narrows the scope for monopoly power.

Secondly, the second-order conditions for a maximum (given constant marginal costs) impose that the left-hand side of (10) (the marginal revenue function) be downward sloping, while the profit-maximizing price must exceed marginal cost. All of which implies that the pair (p^*,x^*) has to lie on the downward sloping region of the demand curve. Letting y^* be the income of the marginal consumer, this means that we can use (4) to obtain

$$\eta(y^*,\theta) \leq \frac{1}{\varepsilon^x(p^*,x^*)} \tag{11}$$

with equality holding where the demand curve has a maximum. On the other hand, equation (10) implies that the demand elasticity be not smaller than one, which gives

$$\eta(y^*,\theta) \geq \frac{1}{\varepsilon^x(p^*,x^*)+\varepsilon^p(p^*,x^*)} \tag{12}$$

where we use equations (6) and (11), and equality holds when marginal revenue is zero.

Equations (11) and (12) characterize the profit maximizing equilibrium by upper and lower bounds on the position of the marginal consumer along the income scale, which depend on the strength of the externality and price effects *at the individual level.* In order to parametrize the model with respect to the externality, from now on we assume ε^x to be constant; if one also sets $\varepsilon^p = 1$, we have $\varepsilon^x(p^*,x^*) = \varepsilon^x = \sigma$, to be treated as a parameter.[12] Under these assumptions, marginal revenue $M(x,\theta;\sigma)$ is

$$M(x,\theta;\sigma) = p(x,\theta)\left(1 - \frac{1}{\eta(y(p(x,\theta),x),\theta)} + \sigma\right) \tag{13}$$

where we substituted back into (10) for the definition (5) of H. Given constant marginal costs $c \geq 0$, the issue whether equilibrium output reacts positively to a (FSD or SSD) change in θ boils down to studying the sign of

[12] This is often implicit in many utility functions used to model network externalities (e.g., Economides and Himmelberg, 1995). Actually, for our purposes σ could be treated as a parameter so long as the ratio $\varepsilon^x/\varepsilon^p$ was kept constant; on the other hand, ε^p has just a scale effect on the way a change in price affects the income of the marginal consumer: with ε^p constant, the *relative* change of the latter is independent of aggregate demand x (as well as of income level); in this sense, $\varepsilon^p = 1$ can be looked at as a normalization.

the derivative $M_\theta(\cdot, \theta; \sigma)$ around $M(x^*, \theta; \sigma) = c$, crucial information in this respect coming from the income of the marginal consumer having to satisfy $1/(1+\sigma) \leq \eta(y^*, \theta) \leq 1/\sigma$.

We now take up in turn the first order and the second order stochastic dominance cases.

6.1 Increasing aggregate income: network externalities and monopoly equilibrium

In this section we enquire about the reaction of the monopolist firm to a demand increase driven by a FSD increase in mean income. Two preliminary remarks are in order.

First, we shall focus on a (very wide) subset of FSD shocks to F, which can be identified by means of the income share elasticity (Esteban, 1986). The latter is a function $\pi(y, \theta)$ which obeys the following definition

$$\pi(y, \theta) = \lim_{h \to 0} \frac{d \log \left(\frac{1}{\mu} \int_y^{y+h} x f(x, \theta) dx \right)}{d \log y} = 1 + \frac{y f_y(y, \theta)}{f(y, \theta)}$$

where μ is average income. This elasticity measures the relative marginal change in the share of income accruing to class y, brought about by a marginal increase in y. Esteban (1986) shows that the conventional density representation of the size distribution of income can be mapped one-to-one with π, and identifies restrictions on π which seem empirically well supported.[13] Using this definition, a noteworthy class of FSD shock to the income distribution is identified by the condition that an increase in θ shifts up the π function for all values of $y \in [y_m, y_M]$, that is

$$\pi_\theta(y, \theta) \geq 0 \tag{15}$$

This condition covers most applications, and obtains when the function $\lambda(y, \theta) = f_\theta(y, \theta)/f(y, \theta)$ is increasing in y – a case which is known in the contract theoretic literature as "monotone likelihood property".[14]

[13] These restrictions are the weak-weak Pareto Law (i.e., π approaches some negative constant as y tends to infinity), the existence of at least one mode (i.e., $\pi(y, \theta) = 1$ has at least one solution), and that π declines at a constant rate: a generalized Gamma distribution satisfies all of these. For an empirical application of Esteban's model, see Chakravarty and Majumder (1990). In several contexts the π this formulation can be analytically convenient, as "the Pareto, Gamma and Normal density functions correspond to constant, linear and quadratic elasticities, respectively" (Esteban, 1986, p.442).

[14] For a general assessment on the relationship between the behaviour of π and stochastic dominance, see Benassi and Chirco (2006).

Secondly we shall assume throughout that the distribution elasticity $\eta(y,\theta)$ is increasing in y, which is equivalent to the condition

$$\eta(y,\theta)+\pi(y,\theta)>0 \qquad (16)$$

This restriction is satisfied by most currently used distributions,[15] and implies that the marginal revenue function is downward sloping when so is the demand function (which in turn is a necessary condition in equilibrium for maximum profit when marginal costs are constant).[16]

Given these assumptions, from the discussion of Section 3 we know that a FSD shift of the distribution raises the demand price at any quantity $x \in (0,1)$, as consumers are richer on average. This is in principle consistent with both increasing and decreasing equilibrium quantity, as the before-change and after-change marginal revenue curves might cross – in which case the level of the marginal costs, *vis à vis* the level of marginal revenue at which the crossing occurs, is of course crucial. Indeed, a FSD increase in aggregate income and market demand may also deliver a decrease in output:

Proposition 3 *Assume a change in θ is a* FSD *shock to F such that (15) applies, and let F satisfy (16). Then (a) there exist levels $\overline{\sigma}>0$ of σ and $\overline{c}>0$ of c such that, for all positive σ lower than $\overline{\sigma}$, equilibrium output increases following an increase in θ for all $c \in (\overline{c}, p_{\max}]$; (b) if the income density satisfies $y_m f(y_m,\theta)>0$ and $\pi(y_m,\theta)$ is finite, there exist levels $\widetilde{\sigma}\geq 0$ of σ and $\widetilde{c}>0$ of c such that, for all σ greater than $\widetilde{\sigma}$, equilibrium output decreases following an increase in θ for all $c\in[0,\widetilde{c})$.*

Proof. See Appendix ■

Proposition 3 sets up sufficient conditions on costs and the strength of the network externality, for output to react (positively or negatively) to a demand increase driven by a FSD to income. The boundary assumptions relevant for case (b) amount to having a positive non-vertical density at the minimum income level.

As a premise to commenting on this result, notice first a general feature of the framework we are using: since by (6) overall elasticity Γ (as distinct from the core no-externality elasticity η) obeys $\Gamma=\eta/\left(1-\eta\sigma\right)$, the same level of the price elasticity of demand is consistent with different combinations of η

[15] E.g., this property is shared by the Gamma, Lognormal, Exponential and Uniform distributions. The limit case is the standard Pareto distribution, for which $\eta=-\pi$ is constant, so that $\eta+\pi=0$ for all x.

[16] Using (13), we have $sign\{M_x(x,\theta)\} = -sign\{(1-\eta\sigma)(1-1/\eta+\sigma)+(\pi+\eta)/\eta\}$, where constraints (11) and (12) ensure that $(1-\eta\sigma)(1-1/\eta+\sigma)>0$.

and σ – however, the firm's reaction to a demand shock for given Γ will be sensitive to the specific combination at hand. Indeed, given Γ, high η (low σ) implies the marginal consumer is "rich" (i.e. only high-income consumers buy), while low η (high σ) implies the marginal consumer is "poor" (i.e. both poor and rich consumer are served by the firm). Secondly, recall that in a standard monopoly model, equilibrium output will increase following an increase in demand, whenever the firm is led to exploit its extensive margin: i.e., to keep a "low" price and take advantage of larger demand, which of course requires a high enough demand price elasticity.

By Proposition 3(a), a low externality makes for output expansion, if costs are sufficiently high: for in that case the price-quantity pair lies near the turning point of the (inverse U-shaped) demand curve, where price-elasticity is high and $1/\eta$ is near σ – which, for given σ, constrains the value of η and hence the income of the marginal consumer along the lines sketched above. Indeed, for marginal revenue to shift out, one requires that the change in income distribution be such that the marginal consumer be endowed with (relatively) "high" income, as high income density is the component of demand raised by an FSD shock. Since η is increasing in income, this happens when σ is sufficiently low: for in that case, the increase in elasticity required to bring price near marginal cost is obtained *via* a high distribution elasticity, and hence a high-income marginal consumer: as the network externality works as a substitute for the distribution elasticity (which, absent the externality, would coincide with demand elasticity), in the neighbourhood of the turning point of demand if the former is low, the latter is bound to be high .

By contrast, the firm's reaction to higher demand will be to decrease its output whenever it is convenient to exploit the intensive margin, which in general is the case when the demand increase is associated to a strong enough decrease in its price elasticity.[17] By Proposition 3(b), on the one hand this happens when costs are low enough, as they have to lie beneath the level at which marginal revenue cross; on the other hand, this crossing does actually take place when the externality effect is strong enough: while it is true that higher σ means *ceteris paribus* higher Γ, it is also true that marginal revenue is zero when elasticity Γ is one, a lower bound to which equilibrium elasticity necessarily tends as costs tend to zero – and which yields marginal revenue crossing only if elasticity is high enough to begin with.[18] More precisely,

[17] That this is quite a possibility was already pointed out by J. Robinson (1969, p.70), who argued that in the monopoly case an increase in the consumers' income may decrease price elasticity so much as to deliver a lower equilibrium output (Benassi and Chirco, 2004).

[18] It should be noticed that if $\sigma = 0$ (no network effect), for marginal revenues to cross following a marginal increase in θ a strong enough decreasing effect on η (and hence a strong

since η is increasing in income, a high value of σ in the neighbourhood of the point where $\eta = 1/(1+\sigma)$ means a low income of the consumer who is marginal at equilibrium: "poor" consumers (whose density is lowered by an FSD shock) weigh relatively high on demand and marginal revenue shifts down, which makes it more convenient to exploit the intensive margin.

Thus overall the effect on output of a FSD shock to income will be positive or negative, depending on the trade-off between costs and externalities: high externality and low costs point to decreasing output, low externality and high cost point to increasing output. Here we report a simple example.

Example 1 Suppose $y(p,x) = px^{-\sigma}$ with $\sigma > 0$, and let the income distribution be exponential over the support $[1,\infty)$: hence, $f(y,\theta) = e^{(1-y)/\theta}/\theta$ and $\theta > 0$ is a FSD parameter – indeed, higher θ means higher mean as $\mu = 1+\theta$, while $F(y,\theta) = 1 - e^{(1-y)/\theta}$ is clearly decreasing in θ for $y \in (1,\infty)$.[19] It is easily verified that the (inverse) demand curve obeys

$$p(x,\theta;\sigma) = (1 - \theta \ln x)\, x^{\sigma}$$

which has a turning point at $x_{\max} = e^{(\sigma-\theta)/\sigma\theta}$.[20] The corresponding marginal revenue function is

$$M(x,\theta;\sigma) = \{1 + \sigma - \theta\,[1 + (1+\sigma)\ln x]\}\, x^{\sigma}$$

Following a marginal increase in θ, $M_\theta(x,\theta;\sigma)$ is positive (negative) as x is lower (higher) than $e^{-1/(1+\sigma)} = x'$: this places restrictions on σ, since the viable portion of the marginal revenue curve corresponds to the downward sloping region of the demand curve, so that output expansion requires $x' > x_{\max}$. As a consequence, $M_\theta(x,\theta;\sigma) > 0$ requires $\sigma < \frac{1}{2}\left(\sqrt{1+4\theta} - 1\right) = \overline{\sigma}$, strictly positive for all $\theta > 0$; on the other hand $M_\theta(\cdot,\theta;\sigma)$ is nil at x', where a marginal change in θ generates a crossing of the marginal revenue; hence, output increases for $c > M(x',\theta;\sigma) = (1+\sigma)\, e^{-\sigma/(1+\sigma)} = \overline{c}$, provided of course that $c < p_{\max} = M(x_{\max},\theta;\sigma) = \theta e^{(\sigma-\theta)/\theta}/\sigma$. By the same token, marginal revenue shifts down following a marginal change in θ for all $x \in (x', 1]$: this means that $x_{1+\sigma}$, such that $M(x_{1+\sigma},\theta;\sigma) = 0$, has to satisfy $x' < x_{1+\sigma}$, i.e. $e^{-1/(1+\sigma)} < e^{-1/(1+\sigma)+1/\theta}$, which is satisfied for all positive

effect on the income distribution) would be called for – something which Proposition 3 does not require.

[19] The Esteban elasticity is $\pi(y,\theta) = 1 - y/\theta$, clearly obeying (15).

[20] For the problem to make sense, we must have $x_{\max} \in (0,1)$, which imposes $\theta > \sigma$. The income of the marginal consumer at $(x_{\max}, p_{\max})$ is $y_\sigma = \theta/\sigma > 1$, such that $\eta(y_\sigma,\theta) = y_\sigma/\theta = 1/\sigma$. Notice that $y(p,1) = y_m = 1$ solves for $p = p_m = 1 > 0$: see f.note 9.

values of σ:[21] in this simple example we have $\widetilde{\sigma} = 0$ and, clearly, $\widetilde{c} = \overline{c}$, as marginal revenues cross only once.□

In the next section we concentrate on income dispersion.

6.2 Monopoly equilibrium, network externalities, and income dispersion

Here we interpret the parameter θ as an index of SSD: an increase in θ is an equal-mean distributional shift which lowers inequality. Our result is as follows.

Proposition 4 *Let θ be an equal-mean* SSD *shift variable of the income distribution $F(y, \theta)$, and let F satisfy (16). If the income density satisfies $y_m f(y_m, \theta) > 0$ and $\pi(y_m, \theta)$ is finite, then there exist a level $\widetilde{\sigma} > 0$ of σ and a positive interval $(\widetilde{c}, \overline{c})$, such that, for all positive σ lower than $\widetilde{\sigma}$, equilibrium output increases following an increase in θ for all $c \in (\widetilde{c}, \overline{c})$. If σ is lower than $\widetilde{\sigma} - 1 > 0$, then $\widetilde{c} = 0$ and output increases for any nonnegative cost level lower than $\overline{c}$.*

Proof. See Appendix ■

We recall that (16) is required to hold in equilibrium by the second order conditions for maximum profits. Also, as was the case for Proposition 3(b), the boundary assumptions rule out that the density be vertical at the minimum income level.

The result of Proposition 4 points to the fact that (contrary to the FSD case) a low enough externality delivers an expansive effect on output for sufficiently low costs. Generally speaking, an increase in income concentration makes for both a higher density of middle income consumers and a lower density in the tails of the distribution, while the firm expands its output to match increasing demand whenever marginal revenue shifts out. This will be unambiguously the case, if costs fall within some range – too high a cost would lead to price middle income consumers out of the market and leave the firm with only high income consumers, whose density is falling; and, by the same token, too low a cost would bring in "too many" low income consumers, whose decreasing demand would add to that of high-income consumers to counterbalance marginal revenue from increasing middle-income demand.

[21] Marginal revenue will cross the zero axis within $(0, 1)$ if $\sigma < \theta - 1$. This is equivalent to the income $y_{\sigma+1}$ of the marginal consumer at $x_{1+\sigma}$ (zero marginal revenue), such that $\eta(y_{\sigma+1}, \theta) = y_{1+\sigma}/\theta = 1/(1+\sigma)$, obeying $y_{1+\sigma} > y_m = 1$.

All of which will happen if σ is not too large, as we know that the same price-cost margin is associated to a different distribution of demand by income classes, according to the strength of the externality effect measured by σ: too strong an externality would be inconsistent with pricing out low-income consumers.[22] Indeed, if the externality is very low, zero costs are consistent with the income of the marginal consumer being high enough, for the firm to capture the increase in the weight of middle income while pricing out low income consumers. Here follows an example.

Example 2 Suppose again $y(p,x) = px^{-\sigma}$ with $\sigma > 0$, and let income be distributed as a simple uniform distribution over $\left[\frac{1}{2} - \frac{1}{\theta}, \frac{1}{2} + \frac{1}{\theta}\right] \subset [0,1]$, so that the mean is $\mu = 1/2$: we then have $F(y,\theta) = \frac{1}{2}\left(1 + \left(y - \frac{1}{2}\right)\theta\right)$, where $\theta \geq 2$ is a SSD parameter.[23] It is easily verified that the (inverse) demand curve obeys

$$p(x,\theta;\sigma) = \left(\frac{1}{2} + \frac{1}{\theta}(1-2x)\right)x^{\sigma}$$

having a turning point at $x_{\max} = \frac{\sigma}{4}\frac{2+\theta}{1+\sigma}$;[24] the corresponding marginal revenue function is

$$M(x,\theta;\sigma) = \tfrac{1}{2\theta}\left[(2+\theta)(1+\sigma) - (\sigma+2)\,4x\right]x^{\sigma}$$

which cuts the x-axis at $x_{\sigma+1} = \frac{2+\theta}{4}\frac{1+\sigma}{2+\sigma}$.[25] Following a marginal increase in θ, demand curves cross at $x = 1/2$ as the distribution is symmetric, while

[22] Higher externality means lower maximum willingness to pay: at the turning point where $p_x(x,\theta) = 0$, one has $\eta(y^*,\theta) = 1/\sigma$; since $\eta_y(y,\theta) > 0$, higher σ means lower y^*, with $y^* = y(p,x)$ and $y_p > 0$, which for given x yields lower p. Indeed, at $p_x(x,\theta) = 0$ the externality driven marginal increase in the willingness to pay is offset by the ordinary negative effect of quantity: if the externality is strong, so must be at that point the marginal negative effect of quantity on the consumers' willingness to pay. In elasticity terms, the latter is simply given by (the inverse of) the elasticity of demand, i.e. $1/\eta$ as the no-externality (direct) demand function is simply $1 - F(y^*,\theta)$: a strong externality is offset by a strong (inverse) elasticity, hence at a low-income marginal consumer and a low price.

[23] With $\theta = 2$ one ricovers the ordinary uniform distribution on $[0,1]$. The corresponding density is $f(y,\theta) = \theta/2$; its Esteban elasticity is $\pi(y,\theta) = 1$, while $\eta(y,\theta) = 2\theta y / \left[2 + \theta(1-2y)\right]$, so that $\pi(y,\theta) + \eta(y,\theta) = (2+\theta) / \left[2 + \theta(1-2y)\right] > 0$ for all $y < y_M$.

[24] As the lowest income is $y_m = \frac{1}{2} - \frac{1}{\theta}$, the market is fully covered for x such that $y_m = px^{-\sigma}$, that is $\frac{1}{2} - \frac{1}{\theta} = \frac{1}{2} + \frac{1}{\theta}(1-2x)$; hence, $p(1,\theta;\sigma) = \frac{1}{2} - \frac{1}{\theta} > 0$ for $\theta > 2$, as the poorest consumer is able to buy at a positive price (see again f.note 9); also, we require $\sigma < \frac{4}{\theta-2}$ for $x_{\max} \in (0,1)$. The income of the marginal consumer at the pair $(x_{\max}, p_{\max})$ is $y_\sigma = \frac{1}{2}\frac{\theta+2}{\theta(1+\sigma)}$, obeying $\eta(y_\sigma,\theta) = 1/\sigma$.

[25] At which point the income of the marginal consumer is $y_{\sigma+1} = \frac{1}{2}\frac{2+\theta}{(2+\sigma)\theta}$, obeying $\eta(y_{\sigma+1},\theta) = 1/(\sigma+1)$.

$M_\theta(x, \theta; \sigma)$ is positive (negative) as x is higher (lower) than $\frac{1}{2}\frac{1+\sigma}{2+\sigma} = \overline{x}$. Since $M(\overline{x}, \theta; \sigma) = \frac{1+\sigma}{2}\left(\frac{1}{2}\frac{1+\sigma}{2+\sigma}\right)^\sigma > 0$ and $\overline{x} < 1/2$ for all $\sigma > 0$, output increases for any low enough cost level: i.e., $\widetilde{\sigma} = \infty$, $\sigma < \widetilde{\sigma} - 1$ is satisfied for all σ, and $\widetilde{c} = 0$. The upper bound on cost is then $\overline{c} = \min\{M(\overline{x}, \theta; \sigma), M(x_{\max}, \theta; \sigma)\} = p(x_{\max}, \theta; \sigma) = \frac{1}{2\theta}\left(\frac{\sigma}{4}\right)^\sigma \left(\frac{2+\theta}{1+\sigma}\right)^{\sigma+1}$.□

7 Concluding remarks

Markets with network externalities are likely to exhibit specific features, taking which into account led to revisiting many results in Industrial Organization theory. As is well known, one way to model the relevant externality boils down to market demand taking a hump-shaped pattern. In this basic framework, in this paper we enquired about the way demand is affected by changes in the distribution of the consumers' willingness to pay, and the related effects on market equilibria in the two polar cases of perfect competition and monopoly. Overall, while in the former case both a (first order stochastic dominance) increase in aggregate income and a (second order stochastic dominance) increase in concentration deliver higher equilibrium output (provided costs are sufficiently low), in the latter case a key role is played by the way income distribution and the externality effect combine themselves to give equilibrium price elasticity.

Although our framework is obviously constrained by its simplifying assumption (in particular with respect to constant marginal costs and unit individual demand), it would seem to support for the monopoly case two broad conclusions, which may be relevant also in a regulatory perspective. On the one hand, different industries may react differently to similar demand shocks, depending on the relative weight of the externality effect and the cost levels: thus, e.g., 'high costs' are consistent with a positive output reaction to a (first order stochastic dominance) demand expansion, while 'low costs' may support an output contraction, depending on the strength of the externality effect. On the other hand, the same industry may react differently to a demand shock, according as the latter is due to a first order or a (purely distributive) second order shock on the distribution of the consumers' willingness to pay.

References

[1] AMIR, R. and LAZZATI, N. (2011). Network Effects, Market Structure and Industry Performance. *Journal of Economic theory*, **146**, 2389–2419.

[2] ARTHUR, B. (1989). Competing Technologies, Increasing Returns and Lock-in by Historical Events. *Economic Journal*, **99**, 106–131.

[3] BENASSI, C., CHIRCO, A. and SCRIMITORE, M. (2002). Income Concentration and Market Demand. *Oxford Economic Papers*, **54**, 584–96.

[4] BENASSI, C. and CHIRCO, A. (2004). Income Distribution, Price Elasticity and the 'Robinson Effect'. *Manchester School*, **72**, 591–600.

[5] BENASSI, C. and CHIRCO, A. (2006). Income Share Elasticity and Stochastic Dominance. *Social Choice and Welfare*, **26**, 511–25.

[6] CABRAL, L. (2011). Dynamic Price Competition with Network Effects. *Review of Economic Studies,* **78**, 83–111.

[7] CABRAL, L., SALANT, D. J. and WOROCH, G. A. (1999). Monopoly Pricing with Network Externalities. *International Journal of Industrial Organization,* **17**, 199–214.

[8] CHAKRAVARTY, S.R. and MAJUMDER, A. (1990). Distribution of Personal Income: Development of a New Model and its Application to U.S. Income Data. *Journal of Applied Econometrics*, **5**, 189–196.

[9] CHOI, J. P. (1994). Irreversible Choice of Uncertain Technologies with Network Externalities. *RAND Journal of Economics,* **25**, 382–401.

[10] CONNER, K.R. and RUMELT, R.P. (1991). Software Piracy: An Analysis of Protection Strategies. *Management Science,* **37**, 25–139.

[11] DOGAN, P. (2009). Vertical Networks, Integration, and Connectivity. *Journal of Economics and Management Strategy,* **18**, 347–392.

[12] DOGANOGLU, T. and WRIGHT, J. (2010). Exclusive Dealing with Network Effects. *International Journal of Industrial Organization,* **28**, 145–154.

[13] ECONOMIDES, N. (1996a). The Economics of Networks. *International Journal of Industrial Organization*, **16**, 415–44.

[14] ECONOMIDES, N. (1996b). Network Externalities, Complementarities, and Invitations to Enter. *European Journal of Political Economy*, **12**, 211–233.

[15] ECONOMIDES, N. and HIMMELBERG, C. (1995). *Critical Mass and Network Size with Application to the US Fax Market*. Discussion Paper EC-95-11, Stern School of Business, New York University.

[16] ESTEBAN, J. (1986). Income Share elasticity and the Size Distribution of Income. *International Economic Review*, **27**, 439–44.

[17] FARRELL, J., and KLEMPERER, P. (2007). Coordination and Lock-in: Competition with Switching Costs and Network Effects. In M. Armstrong and R. Porter (eds.), *Handbook of Industrial Organization*, Vol. 3. New York: Elsevier.

[18] FARRELL, J. and SALONER, G. (1985). Standardization, Compatibility, and Innovation. *Rand Journal of Economics*,**16**, 70–83.

[19] FARRELL, J. and SALONER, G. (1986). Installed Base and Compatibility: Innovation, Product Preannouncements, and Predation. *American Economic Review*, **76**, 940–955.

[20] FOELLMI, R. and ZWEIMÜLLER, J. (2004). Inequality, Market Power, and Product Diversity. *Economics Letters*, **82**, 139–145.

[21] FOELLMI, R. and ZWEIMÜLLER, J. (2006). Income Distribution and Demand-Induced Innovations. *Review of Economic Studies*, **73**, 941–960.

[22] GANDAL, N. (2008). Network Effects - Empirical Studies. In L. Blume and S. Durlauf (eds.), *The New Palgrave Dictionary of Economics*. New York: Palgrave Macmillan.

[23] GRUBER, H. and VERBOVEN, F. (2001). The Evolution of Markets under Entry and Standards Regulation - the Case of Global Mobile Telecommunications. *International Journal of Industrial Organization*, **19**, 1189–212.

[24] HAAN, A. M. (2003). Vaporware as a Means of Entry Deterrence. *Journal of Industrial Economics*, **51**, 345–358.

[25] HAAN, A. M. (2004). The Welfare Effect of Quality Degradation in the Presence of Network Externalities. *Information Economics and Policy*, **16**, 535–552.

[26] HARUVY, E. and A.PRASAD (1998). Optimal Product Strategies in the Presence of Network Externalities. *Information Economics and Policy*, **10**, 489–99.

[27] KATE, A.T. and NIELS, G. (2006). Fulfilled-Expectations Equilibria in Network Demand. *Review of Network Economics*, **5**, 320–36.

[28] KATZ, M. L. and SHAPIRO, C. (1985). Network Externalities, Competition, and Compatibility. *The American Economic Review*, **75**: 424–40.

[29] KATZ, M. L. and SHAPIRO, C. (1994). Systems Competition and Network Effects. *The Journal of Economic Perspectives*, **8**, 93–115.

[30] LAFFONT, J.-J., REY, P. and TIROLE, J. (1997). Network Competition: II. Price Discrimination. *RAND Journal of Economics,* **29**, 1–37.

[31] LAMBERT, P.J. (2001). *The Distribution and Redistribution of Income.* Manchester and New York: Manchester University Press.

[32] LAMBERTINI, L. and ORSINI, R. (2001). Network Externalities and the Overprovision of Quality by a Monopolist. *Southern Economic Journal*, **67**, 969–82.

[33] PEITZ, M. and WAELBROECK, p. (2006). Piracy of Digital Products: a Critical Review of the Theoretical Literature. *Information Economics and Policy*, **18**, 449–76.

[34] REGIBEAU, P. (2004). Network Externalities and Competition Policy. Comments on Koski and Kretschmer. *Journal of Industry, Competition and Trade*, **4**, 33–3.

[35] ROBINSON, J. (1969). *The Economics of Imperfect Competition*, 2^{nd} edn. London and New York: MacMillan.

[36] ROHLFS, J.H. (1974). A Theory of Interdependent Demand for a Communication Service. *Bell Journal of Economics and Management Science*, **5**, 16–37.

[37] SHY, O. (2011). A Short Survey of Network Economics. *Review of Industrial Organization*, **38**, 119–49.

[38] SWANN, G.M.P. (2002). The Functional Form of Network Effects. *Information Economics and Policy*, **14**, 417–29.

[39] TAKEYAMA, L.N. (1994). The Welfare Implications of Unauthorized Reproduction of Intellectual Property in the Presence of Demand Network Externalities. *Journal of Industrial Economics*, **17**, 155–66.

[40] YANO, M. and DEI, F. (2006). Network Externalities, Discrete Demand Shifts, and Submarginal-cost Pricing. *Canadian Journal of Economics,* **39**, 455–476.

[41] YANNELIS, D. (2002). On Access Pricing with Network Externalities. *Atlantic Economic Journal*, **30**, 186–190.

Appendix

Proof of Proposition 1

Let $p(x,\theta)$ be the implicit demand curve derived from (2). Proving the proposition amounts to proving that (a) there exists at least one value of x, $\overline{x}$ say, such that $\frac{\partial p(\overline{x},\theta)}{\partial\theta} = p_\theta(\overline{x},\theta) = 0$; and (b) that $\frac{\partial^2 p(\overline{x},\theta)}{\partial x\partial\theta} = p_{\theta x}(\overline{x},\theta) > 0$ if $\overline{x}$ is the highest value of x where a crossing takes place.

(a) Observe that $\int_{y_m}^{y_M} F_\theta(y,\theta)dy = 0$ implies that $F_\theta(y,\theta)$ takes on both negative and positive values, while $\int_{y_m}^{y} F_\theta(z,\theta)dz \leq 0$ for all y implies the existence of some $\overline{y} \in (y_m, y_M)$ such that $F_\theta(y,\theta) < 0$ for all $y < \overline{y}$, $F_\theta(y,\theta) > 0$ for any y greater than and close enough to $\overline{y}$, and $F_\theta(\overline{y},\theta) = 0$. Take now any quantity $x \geq 0$ and let $y(p(x,\theta),x) = \widehat{y}(x,\theta)$ be the income of the corresponding marginal consumers: using (2) and (3), it is straightforward that $\partial\widehat{y}(x,\theta)/\partial x = -1/f(\widehat{y}(x,\theta),\theta) < 0$, $\lim_{x\to 0}\widehat{y}(x,\theta) = y_M$, and $\lim_{x\to 1}\widehat{y}(x,\theta) = y_m$. Since this amounts to monotonicity, one can uniquely associate to $\overline{y}$ the quantity $\overline{x}$ such that $\widehat{y}(\overline{x},\theta) = \overline{y}$: there follows that, using (7), $p_\theta(\overline{x},\theta) = 0$.

(b) Let $x' > \overline{x}$: the corresponding income value satisfies $\widehat{y}(x',\theta) = y' < \overline{y}$ so that the same argument as above can be invoked to yield $F_\theta(y',\theta) < 0$ and hence, by (7), $p_\theta(x',\theta) > 0$; on the other hand, the argument can be reversed for any $x'' < \overline{x}$ close enough to $\overline{x}$, to the effect that for $y'' > \overline{y}$ close enough to $\overline{y}$, $F_\theta(y'',\theta) > 0$ and hence $p_\theta(x'',\theta) < 0$. Since by construction $x'' < \overline{x} < x'$, while $p_\theta(x'',\theta) < 0 = p_\theta(\overline{x},\theta) < p_\theta(x',\theta)$, $p_\theta(x,\theta)$ is increasing in x around $\overline{x}$, and hence $p_{\theta x}(\overline{x},\theta) > 0$. Finally, since $\overline{y}$ is the lowest value of y such that $F_\theta(\cdot,\theta) = 0$, and $\widehat{y}(x,\theta)$ is monotonically decreasing in x, $\overline{x}$ is the highest (i.e. rightmost) value of x such that $p_\theta(\cdot,\theta) = 0$.■

Proof of Proposition 3

Consider the marginal revenue defined in (13). Letting subscripts denote derivatives, differentiation wrt θ yields

$$\begin{aligned} M_\theta(x,\theta;\sigma) &= p_\theta(x,\theta)\left(1 - \frac{1}{\eta(\widehat{y},\theta)} + \sigma\right) \\ &\quad + \frac{p(x,\theta)}{\eta^2(\widehat{y},\theta)}\left[\eta_y(\widehat{y},\theta)y_p(p(x,\theta),x)p_\theta(x,\theta) + \eta_\theta(\widehat{y},\theta)\right] \end{aligned}$$

where $\widehat{y} = \widehat{y}(x,\theta) = y(p(x,\theta),x)$. To ease notation, let $\lambda(y,\theta) = f_\theta(y,\theta)/f(y,\theta)$: we have $\eta_\theta(y,\theta) = \eta(y,\theta)\left(\lambda(y,\theta) + \frac{F_\theta(y,\theta)}{1-F(y,\theta)}\right)$, and using (7) it is immediate to verify that $sign\left\{M_\theta(x,\theta;\sigma)\right\} = sign\left\{L(x,\theta)\right\}$,with

$$L(x,\theta) = -\frac{F_\theta(\widehat{y},\theta)}{1-F(\widehat{y},\theta)}\left(1 - \frac{1}{\eta(\widehat{y},\theta)} + \sigma + \frac{\pi(\widehat{y},\theta)}{\eta(\widehat{y},\theta)}\right) + \lambda(\widehat{y},\theta)$$

where $\pi(y,\theta)$ is Esteban's income share elasticity. By differentiation it is easily seen that $\pi_\theta(y,\theta)>0$ is equivalent to $y\lambda_y(y,\theta)>0$, so that the class of FSD shocks identified by (15) delivers a monotonically increasing $\lambda(y,\theta)$.

(a) Let $p_{\max}=p(x_{\max},\theta)$ be such that $p_x(x_{\max},\theta)=0$. Consider the income level y_σ, defined from (11) by the condition $\eta(y_\sigma,\theta)=1/\sigma$, and the associated price-quantity pair (p_σ,x_σ), such that $p_\sigma=p(x_\sigma,\theta)$ and $y_\sigma=\widehat{y}(x_\sigma,\theta)$. Clearly, $p_\sigma=p_{\max}$ and

$$L(x_\sigma,\theta)=-\frac{F_\theta(y_\sigma,\theta)}{1-F(y_\sigma,\theta)}\left(1+\frac{\pi(y_\sigma,\theta)}{\eta(y_\sigma,\theta)}\right)+\lambda(y_\sigma,\theta)$$

Now notice that

(a) $F_\theta(y,\theta)\leq 0$ for all y, with at most equality at the endpoints of the support, by the definition of FSD;

(b) given (16), $1+\pi(y,\theta)/\eta(y,\theta)\geq 0$ for all y;

(c) given (15), $\lambda(y,\theta)$ is monotonically increasing: hence, $\lim_{y\to y_m}\lambda(y,\theta)<0$ and $\lim_{y\to y_M}\lambda(y,\theta)>0$, since by definition

$$F_\theta(y_M,\theta)=\int_{y_m}^{y_M}f(y,\theta)\lambda(y,\theta)dy=0$$

and $f(y,\theta)>0$ for $y\in(y_m,y_M)$. So there is a unique value $y_\theta\in(y_m,y_M)$ such that $\lambda(y_\theta,\theta)=0$, with $\lambda(y,\theta)\lesseqgtr 0$ as $y\lesseqgtr y_\theta$.

From (a), (b) and (c) there follows that $L(x_\sigma,\theta)>0$ if $y_\sigma>y_\theta$, in which case there exists some $y'\in[y_\theta,y_\sigma)$ such that $L(x,\theta)>0$ for any x such that $y^*(p(x,\theta),x)$ lies in $[y',y_\sigma]$. Since y_θ depends only on the income distribution, while clearly y_σ is decreasing in σ and such that $\lim_{\sigma\to\infty}y_\sigma=y_m$ and $\lim_{\sigma\to 0}y_\sigma=y_M$, there exists a positive threshold value $\overline{\sigma}$ of σ (for which $y_{\overline{\sigma}}=y_\theta$), such that for all $\sigma\in(0,\overline{\sigma})$ it is indeed true that $y_\sigma>y_\theta$: hence, there is a corresponding value y' and an associated quantity $x'=1-F(y',\theta)$, such that $L(x,\theta)>0$ for $x\in[x_\sigma,x']$. There follows that $M_\theta(x,\theta)>0$ for x in that interval, and hence output increases for all cost levels c such that $c>\overline{c}=M(x',\theta)$ (obviously, it must be $c<M(x_\sigma,\theta)=p_{\max}$).

(b) Let $y_{1+\sigma}$ be defined by condition (12), i.e. $\eta(y_{1+\sigma},\theta)=1/(1+\sigma)$: letting the associated price-quantity pair be $(p_{1+\sigma},x_{1+\sigma})$, such that $y(p_{1+\sigma},x_{1+\sigma})=y_{1+\sigma}$ and $p_{1+\sigma}=p(x_{1+\sigma},\theta)$, it must be $M(x_{1+\sigma},\theta;\sigma)=0$, to which there corresponds the income $y_{1+\sigma}$ of the marginal consumer, such that $y_{1+\sigma}=\widehat{y}(x_{1+\sigma},\theta)$, $\widehat{y}(x,\theta)$ monotonically decreasing in x. As before, $sign\{M_\theta(x,\theta;\sigma)\}=sign\{L(x,\theta)\}$, while at $x=x_{1+\sigma}$ we have

$$L(x_{1+\sigma},\theta)=-\frac{F_\theta(y_{1+\sigma},\theta)}{1-F(y_{1+\sigma},\theta)}\frac{\pi(y_{1+\sigma},\theta)}{\eta(y_{1+\sigma},\theta)}+\lambda(y_{1+\sigma},\theta)$$

To ease notation, let the RHS of the above be $\Lambda(y_{1+\sigma}, \theta)$, and observe that

$$\lim_{y \to y_m} \Lambda(y, \theta) \leq 0$$

since by assumption $\eta(y_m, \theta) = y_m f(y_m, \theta) > 0$ and by definition $F_\theta(y_m, \theta) = F(y_m, \theta) = 0$, while $\pi(y_m, \theta)$ is finite by assumption and $\lim_{y \to y_m} \Lambda(y, \theta) = \lim_{y \to y_m} \lambda(y, \theta) < 0$.

Now notice that, using Hopital's rule,

$$\lim_{y \to y_M} \Lambda(y, \theta) = \lim_{y \to y_M} \lambda(y, \theta) \left(1 + \frac{\pi(y, \theta)}{\eta(y, \theta)}\right) > 0$$

as by (16) $\pi(y, \theta) + \eta(y, \theta) > 0$ for all $y \in [y_m, y_M]$, and $\lim_{y \to y_M} \lambda(y, \theta) > 0$. Hence, by continuity there exists some $\widetilde{y} \in (y_m, y_M)$ such that $\Lambda(\widetilde{y}, \theta) = 0$ and $\Lambda(y, \theta) < 0$ for all $y \in [y_m, \widetilde{y})$. Let now $\widetilde{\sigma} \geq 0$ be uniquely defined by $\eta(\widetilde{y}, \theta) = \min\{1, 1/(1 + \widetilde{\sigma})\} = \eta(y_{1+\widetilde{\sigma}}, \theta)$: since $\eta(y, \theta)$ is monotonically increasing in y, we have $y_{1+\sigma} < \widetilde{y}$ for any $\sigma > \widetilde{\sigma}$ and, by the monotonicity of $\widehat{y}(x, \theta)$, we can associate to $\widetilde{y}$ the quantity $\widetilde{x} = x_{1+\widetilde{\sigma}}$ such that $\widetilde{y} = \widehat{y}(\widetilde{x}, \theta)$: so we have that $L(x_{1+\sigma}, \theta) < 0$, and hence $M_\theta(x_{1+\sigma}, \theta; \sigma) < 0$, for any $\sigma > \widetilde{\sigma}$; there follows that, for any such σ, there exists an $x' < x_{1+\sigma}$, such that $M_\theta(x, \theta; \sigma) < 0$ for all $x \in (x', x_{1+\sigma}]$: output then decreases for all cost levels c such that $M(x_{1+\sigma}, \theta; \sigma) = 0 \leq c < \widetilde{c} = M(x', \theta; \sigma)$.■

Proof of Proposition 4

We first need the following Lemma:

Lemma *Assume θ is a* SSD *parameter of the distribution F, and let $\lambda(y, \theta) = f_\theta(y, \theta)/f(y, \theta)$. Then $\lim_{y \to y_m} \lambda(y, \theta) \leq 0$.*

Proof. By definition (9) of SSD, we have that $\int_{y_m}^{y} F_\theta(x, \theta)dx \leq 0$ for all $y \in [y_m, y_M]$. Since $F_\theta(y_m, \theta) = 0$, $F_\theta(\cdot, \theta)$ cannot be positive around y_m, which also holds for its derivative $f_\theta(\cdot, \theta)$ as $F_\theta(\cdot, \theta)$ cannot point upwards there. Recall now that $f(y, \theta) > 0$ for all $y \in (y_m, y_M)$: the result then follows trivially, as $sign\{\lambda(y, \theta)\} = sign\{f_\theta(y, \theta)\}$. ■

Assume now that $\sigma > 0$, and recall from the proof of Proposition 3 above that $sign\{M_\theta(x, \theta; \sigma)\} = sign\{L(x, \theta)\}$,with

$$L(x, \theta) = -\frac{F_\theta(\widehat{y}, \theta)}{1 - F(\widehat{y}, \theta)} \left(1 - \frac{1}{\eta(\widehat{y}, \theta)} + \sigma + \frac{\pi(\widehat{y}, \theta)}{\eta(\widehat{y}, \theta)}\right) + \lambda(\widehat{y}, \theta)$$

where as before $\widehat{y} = \widehat{y}(x, \theta)$, monotonically decreasing in x.

Let now the RHS be denoted as $\mathcal{L}(\widehat{y},\theta)$. Clearly, $\lim_{x\to 1} L(x,\theta) = \lim_{y\to y_m} \mathcal{L}(y,\theta) = \lim_{y\to y_m} \lambda(y,\theta) \leq 0$ by the Lemma above, since $F_\theta(y_m,\theta) = 0$, and the term in brackets tends to a finite value as y tends to y_m, which follows from $y_m f(y_m,\theta) > 0$ and $\pi(y_m,\theta)$ being finite.

Moreover, by the definition of SSD, there exists an income level $\overline{y}$ such that $F_\theta(y,\theta) < 0$ for all $y \in (y_m,\overline{y})$ and $F_\theta(\overline{y},\theta) = 0$. Indeed, from Proposition 2 we know that θ being SSD implies the existence of a quantity $\overline{x} \in (0,1)$ such that $p_\theta(\overline{x},\theta) = 0$ and $p_{\theta x}(\overline{x},\theta) > 0$: this is the rightmost point where the demand curves cross following a marginal increase in θ, and it satisfies $\overline{y} = \widehat{y}(\overline{x},\theta)$. Using (7), it is then immediate to verify that it must be $f_\theta(\overline{y},\theta) > 0$ and hence $\mathcal{L}(\overline{y},\theta) = \lambda(\overline{y},\theta) = f_\theta(\overline{y},\theta)/f(\overline{y},\theta) > 0$.

All of which implies the existence of some $\widetilde{y} \in [y_m,\overline{y})$, such that $\mathcal{L}(\widetilde{y},\theta) = 0$ and $\mathcal{L}(y,\theta) > 0$ for all $y \in (\widetilde{y},\overline{y}]$, and hence $M_\theta(x,\theta;\sigma) > 0$ for all $x \in [\overline{x},\widetilde{x})$, where $\widetilde{x}$ is uniquely identified by the condition $\widetilde{y} = \widehat{y}(\widetilde{x},\theta)$.

Let now $\widetilde{\sigma} > 0$ be identified by the condition $\widetilde{\sigma} = 1/\eta(\widetilde{y},\theta)$. Clearly, this implies that for any σ such that $0 < \sigma < \widetilde{\sigma}$ we have $y_\sigma > \widetilde{y}$ and hence $\widetilde{x} > x_{\max}$, as the latter satisfies $\sigma = 1/\eta(\widehat{y}(x_{\max},\theta),\theta)$. For any such σ, this means that $M_\theta(x,\theta;\sigma) > 0$ in the downward sloping region of the demand curve, the only viable for the monopolist problem. Hence, output increases for all costs levels $c \in (\widetilde{c},\overline{c})$, where $\widetilde{c} = \max\{M(\widetilde{x},\theta;\sigma),0\}$ and $\overline{c} = \min\{M(x_{\max},\theta;\sigma), M(\overline{x},\theta;\sigma)\}$.

Take now some positive $\sigma < \widetilde{\sigma}$, and consider the income level $y_{1+\sigma}$, defined by $\eta(y_{1+\sigma},\theta) = 1/(1+\sigma)$, so that at the corresponding quantity $x_{1+\sigma}$ marginal revenue is zero. We then have two possibilities: either $y_{1+\sigma} \geq \widetilde{y}$, i.e. $x_{1+\sigma} \leq \widetilde{x}$, in which case $M_\theta(x,\theta;\sigma) > 0$ for all $x \in [\overline{x},x_{1+\sigma}]$, $\max\{M(\widetilde{x},\theta;\sigma),0\} = 0$, and output increases for all $c \in [0,\overline{c}]$; or $y_{1+\sigma} < \widetilde{y}$, i.e. $x_{1+\sigma} > \widetilde{x}$, in which case $\max\{M(\widetilde{x},\theta;\sigma),0\} = M(\widetilde{x},\theta;\sigma)$, and output increases for all $c \in [\widetilde{c},\overline{c}]$, $\widetilde{c} = M(\widetilde{x},\theta;\sigma) > 0$. The latter cannot apply if $\sigma < \widetilde{\sigma} - 1$, since in that case $\overline{\sigma} = \widetilde{\sigma} - 1$ satisfies $\eta(\widetilde{y},\theta) = 1/(1+\overline{\sigma})$ and we always have $[\overline{x},x_{1+\sigma}] \subset [\overline{x},\widetilde{x})$: hence, output increases for any cost level low enough, if $\widetilde{\sigma} > 1$ and $\sigma \in (0,\widetilde{\sigma}-1)$. ■