Equilibrium Market and Pricing Structures in Virtual Platform Duopoly: Coexistence on Competing Online Auction Sites revisited

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Abstract

We investigate the equilibrium market structure in virtual platform duopoly such as that of eBay and Yahoo! auctions. Building on the model of Ellison, Fudenberg, & Möbius (2004) we take full account of the complexity of network effects on such platforms. We extend the model by looking at the implication of exogenous and endogenous buyer and seller charges (i.e. vertical product differentiation) making use of the concept of insulating tariffs. This extension brings in line the theory with the empirical findings of Brown & Morgan (2006). Eventually we investigate welfare effects, look at the viability of duopoly with size differentials, and the implications for large markets and policy.

1 Introduction

Virtual market platforms such as auctions often reveal very different price strategies despite the fact that such intermediaries offer homogenous products. Competition between eBay vs. Yahoo! auctions are a case in point with Yahoo! having substantially lower fees and commissions than eBay both in the US and in Japan. Despite these similarities markets were eventually dominated by eBay in the US and by Yahoo! in Japan (see Yin (2004)). One explanation for this observation is the presence of network externalities.

Intermediation between heterogenous agents such as bargaining buyers and sellers generates direct, 'congestion' externalities (from agents of their own type) and indirect network externalities (from agents of the other type). This complex interaction of network externalities often remains unmodelled an exception being the work of Ellison, Fudenberg, and Möbius (EFM, 2004).

[1] I am grateful for comments by Matthias Blonski, Thomas Gall, Dominik Grafenhofer,

The analysis in EFM shows that stable equilibria in such duopoly markets exist and that they may be asymmetric. The consequences of asymmetry for optimal platform pricing strategies are however not pursued. Competitive pricing decisions are studied by Caillaud and Jullien (2001, 2003) albeit in a continuum model with homogenous agents on both sides of the platform focusing on indirect network externalities only.

Recently the implications of the EFM framework for platform competition have been investigated in Brown and Morgan (2009). In their extension of the EFM model they look at *exogenous* vertical platform differentiation. One of their findings (see Proposition 4 in their paper), is that given eBay is the dominant platform and provides an exogenous vertical differentiation advantage to sellers, 1) more buyers are attracted to a given Yahoo! auction than an eBay auction, and 2) prices for the traded goods are higher on Yahoo! than on eBay. The authors note that both predictions are *exactly contradicted* by their evidence from field experiments. As an alternative they offer a dynamic disequilibrium model with boundedly rational players that will eventually lead to 'tipping'.

In this paper we are offering a more parsimonious extension of EFM that is in accordance with equilibrium coexistence taking differences in seller charges into account. This extension is empirically warranted as eBay has almost always been the more expensive platform for sellers in practice, charging listing fees *and* commissions. However treating pricing/vertical differentiation as exogenous is clearly not fully satisfactory in the context of competing platforms either.

We thus investigate the effects of *endogenous* seller charges on the equilibrium market structure. Making use of the equilibrium concept of "insulating tariffs" for competition in *two-sides markets* put forward in Weyl (2010) for monopoly and White and Weyl (2011) for oligopoly we find optimal pricing decisions of platforms that target allocations. Such charges reflect the potentially *asymmetric* market shares of buyers and sellers on each platform and we show that our extensions are sufficient to explain the empirical evidence in Brown and Morgan (2009) and allows for long run equilibrium coexistence.

We are also able to tighten the set of equilibria compared to EFM and show that, contrary to the original model, endogenizing seller charges allows for coexistence of platforms with arbitrarily small numbers of participants.

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2 The Model

We model the duopolistic platform competition departing from a simple two-stage game presented in EFM (2004).

The timing of the game is as follows: In the first stage B risk-neutral buyers ($B \in \mathcal{N}_0$) with unit demand and S risk-neutral sellers ($S \in \mathcal{N}_0$) with a single unit of the good to sell simultaneously decide whether to attend platform 1 or platform 2. In the second stage they learn their valuations that are uniformly i.i.d. distributed and bargaining for the object takes place. We model this bargain as a uniform price (multiobject if $S > 1$) auction on each platform. By the revenue equivalence theorem this choice of the bargaining process is quite general. Each buyer only demands one homogeneous good. In order to guarantee strictly positive prices we make the 'non-triviality assumption' that

$$B > S + 1 \tag{1}$$

for both being positive integers. Risk neutral sellers have zero reservation value and their expected utility is given by the expected price on their chosen platform. A buyer's utility on a platform with B buyers and S sellers is given by his expected net utility conditional on winning the good i.e.

$$u_B = E\left\{v - v^{S+1,B} \middle| v \geq v^{S,B}\right\} \Pr\left\{v \geq v^{S,B}\right\} \tag{2}$$

where $v^{k,n}$ gives the k highest order statistic of a draw of n values and thus in this auction format the uniform price is simply the $S+1$ highest of the buyers valuations $v^{S+1,B}$ (i.e. the highest losing bid). This is the typical mathematical convention as long as we deal with a discrete model.

Larger markets are more efficient than smaller ones as they come closer to the ex-post efficient outcome to allocate a good to a buyer iff his valuation is high. The ex-post efficient outcome implies that the buyers with the S highest values obtain the good, so that the expectation of the maximum total ex-ante surplus (welfare) is

$$\begin{aligned} &B \Pr\left\{v \geq v^{S,B}\right\} E\left\{v \middle| v \geq v^{S,B}\right\} = SE\left\{v \middle| v \geq v^{S,B}\right\} = \\ &SE\left\{v \middle| v > v^{S+1,B}\right\} = S\int_0^1 \left(\int_x^1 vf(v\,|v > x)dv\right) f^{S+1,B}(x)dx \end{aligned} \tag{3}$$

where $f^{S+1,B}$ is the density function of $v^{S+1,B}$, the $S+1$ highest order statistic of a draw of B values under the uniform distribution.

Lemma 0 (EFM):

Under the uniform distribution total welfare on one platform can be written as the sum of buyer and seller utilities

$$w(B,S) = S(1 - \frac{1}{2}\frac{1+S}{B+1}) = S\left(\frac{B-S}{B+1}\right) + B\left(\frac{S(1+S)}{2B(B+1)}\right)$$

Proof:
See Appendix.■

The result is intuitive: The total value of a sale is $E\left\{v|\, v > v^{S+1,B}\right\}$, i.e. expected value of v *given* $v > p$. Under the uniform distribution this is $1 - \frac{1}{2}\frac{1+S}{B+1} = p + \frac{1-p}{2}$. Clearly the second term is the value for one buyer $E\left\{v - v^{S+1,B}\middle|\, v > v^{S+1,B}\right\} = \frac{1-p}{2}$ with the remaining p (as calculated above) going to the seller and to obtain total welfare we multiply with the number of sales.

Note that

$$\frac{\partial w(1, \frac{S}{B} = \bar{x} < 1)}{\partial B} = \frac{1}{2}\bar{x}\frac{(2-\bar{x})(B+2)B+1}{(B+1)^2} > 0 \tag{4}$$

showing that for constant shares of sellers to buyers larger markets are more efficient than smaller ones. The efficiency deficit makes it more difficult for small markets to survive but the sequential structure of the game allows for equilibria with two active platforms whenever the impact of switching of buyer and/or seller on his expected surplus more than outweighs the efficiency advantage.

The game is solved by backward induction and the solution concept is Subgame Perfect Nash equilibrium (SPNE). The transaction of the good in stage two yields ex-ante utility in stage one for a seller of

$$u_S(B,S) = p = \frac{B-S}{B+1} \tag{5}$$

and for a potential buyer of

$$u_B(B,S) = \frac{1-p}{2}\frac{S}{B} = \frac{S(1+S)}{2B(1+B)}. \tag{6}$$

Note that holding S/B (the relative advantages of buyers and sellers) constant, sellers prefer larger, *more liquid* markets (where the expected equilibrium price is higher) and buyers prefer small, less efficient markets as

$$\frac{\partial u_S(1, \frac{S}{B} = \bar{x} < 1)}{\partial B} = \frac{\partial p(1, \frac{S}{B} = \bar{x} < 1)}{\partial B} > 0 \tag{7}$$

and

$$\frac{\partial u_B(1, \frac{S}{B} = \bar{x} < 1)}{\partial B} < 0 \tag{8}$$

Extending the setting of EFM we assume that platforms can charge buyers and/or sellers some fee for participating. Without loss of generality we assume that such a fee takes a non-negative value.

As buyers and sellers simultaneously decide which platform to join in stage one, we can set up the relevant constraints that determine the set of *all* possible SPNE of the game subject to the qualification that the integer constraint holds. Otherwise we will speak of a *quasi-equilibrium*. This restriction is investigated in detail in Anderson, Ellison and Fudenberg (2010). The constraints to keep buyers in place in stage one given buyer charge difference $p_{2B} - p_{1B} \equiv \Delta_B \geq 0$ are (B1)

$$u_B(B_1, S_1) \geq u_B(B_2 + 1, S_2) - \Delta_B \tag{9}$$

and (B2)

$$u_B(B_2, S_2) - \Delta_B \geq u_B(B_1 + 1, S_1) \tag{10}$$

In words: A buyer on platform 1 needs to have an expected utility from the bargaining stage correcting for charges paid to the platform owner such that a change to the other platform and the implied effect on the equilibrium bargaining outcome there deters him from doing so.

To keep sellers in place in stage one given seller charge difference $\Delta_S \geq 0$ we need (S1)

$$u_S(B_1, S_1) \geq u_S(B_2, S_2 + 1) - \Delta_S \tag{11}$$

and (S2)

$$u_S(B_2, S_2) - \Delta_S \geq u_B(B_1, S_1 + 1) \tag{12}$$

to hold. The motivation for the constraints is analogous. Clearly these constraints matter only for interior equilibria.

3 Exogenous buyer charges

We now look explicitly at the form of the constraints and thus at the set of possible SPNE with some exogenous charge differences $\Delta_B > 0$ to (winning) buyers in auction two. Note that this does not imply that charges are made only by one of the platforms but only that it is the difference between such charges that influence location incentives.

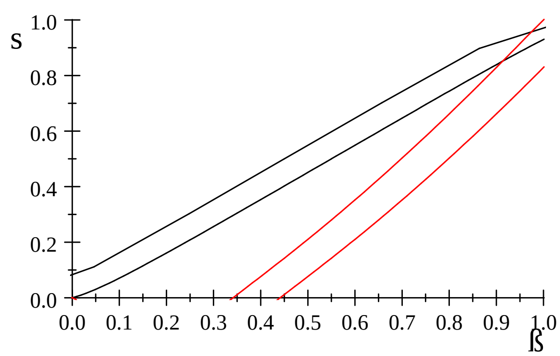
Denoting s as the *share* of sellers on platform one and β as the *share* of buyers at platform one the buyer constraint (9) becomes

$$\frac{sS(1+sS)}{2\beta B(1+\beta B)} \geq \frac{(1-s)S(1+(1-s)S)}{2((1-\beta)B+1)(1+(1-\beta)B+1)} - \Delta_B \tag{13}$$

and (B2) is

$$\frac{(1-s)S(1+(1-s)S)}{2(1-\beta)B(1+(1-\beta)B)} - \Delta_B \geq \frac{sS(1+sS)}{2(\beta B+1)(1+\beta B+1)} \tag{14}$$

A numerical example (with $B = 10, S = 5$) may make clear how the buyer constraints change. The two buyer constraints with $\Delta_B = 0$ and $\Delta_B = 0.3$ are:



where the share of sellers on platform one (s) is on the ordinate and the share of buyers on platform one (β) is on the abscissa.

The interpretation of this finding is as follows: The lower (B1) constraint gives the condition that buyers stay on platform one if the fraction of sellers s is large enough or, alternatively if β is low enough. The higher, (B2) constraint gives the condition under which buyers stay on platform 2, i.e. if s is small (and thus $(1-s)$ the fraction of seller on his own platform is large enough). Between the two curves is the candidate set of SPNE (we still need to check if the seller constraints hold).

Now with a charge of $\Delta_B > 0$ to buyers on the second platform both the (B1) and the (B2) constraint shift downwards, i.e. the set of SPNE allows for equilibria with a lower share of sellers on platform one for a given share of buyers. The (B2) constraint also shifts downwards. i.e. buyers move from the second platform at higher levels of s already, (and thus for a lower fraction of seller $(1-s)$ on his own platform) than before given the new charge.

4 Exogenous seller charges

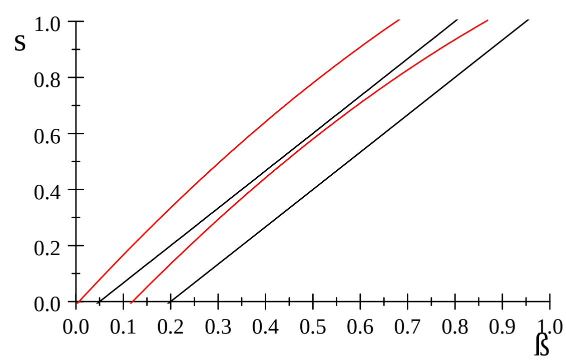
We now introduce an exogenous charge difference Δ_S for sellers of platform 2. Seller constraints are (S1)

$$\frac{\beta B - sS}{\beta B + 1} \geq \frac{(1-\beta)B - ((1-s)S + 1)}{(1-\beta)B + 1} - \Delta_S \tag{15}$$

and (S2)

$$\frac{(1-\beta)B - (1-s)S}{(1-\beta)B + 1} - \Delta_S \geq \frac{\beta B - (sS + 1)}{\beta B + 1} \tag{16}$$

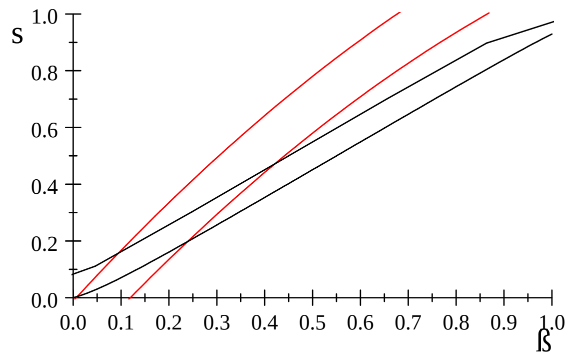
With $\Delta_S = 0.3$ we find the picture with the seller constraints becomes:



The interpretation of this finding is as follows: For the upper linear (S1) constraint, a seller stays on platform 1 if s is not too high for a given share of β, otherwise he will go to platform 2. For the lower linear (S2) constraint, a seller stays at platform 2 if s is high (i.e. his own seller share $1-s$ is low) otherwise he will go to platform one. Between the two curves is the candidate set of SPNE (we need to check if the buyer constraint holds simultaneously).

Now that there is a charge of $\Delta_S > 0$ to the sellers on the second platform, the (S1) constraint in no longer linear and shifts upwards: Seller stay at platform 1 even if s is much higher than before for given β. Similarly the (S2) constraint is no longer linear and also shifts upwards: Seller will move from platform 2 even if s is much higher (hence their own seller share $1 - s$ much lower) than before.

The numerical example with $\Delta_S = 0.3$ yields both seller and buyer constraints as



Only $\beta = 0.2, s = 0.2$ is a viable equilibrium here and the previous candidate $\beta = 0.4, s = 0.4$ is no longer viable.

The result reveals that charging sellers on platform 2 allows for higher s tolerance for given β on platform 1. Also, equally sized platforms are no longer viable. As sellers like larger, more liquid platforms where the uncertainty about the resulting final price is lower we find that a positive and exogenous relative seller charge difference of platform 2 can only be an equilibrium if platform 2 also has the larger share of sellers.

5 Equilibrium properties

We now propose some general results that allow us to characterize the set of SPNE more tightly than in EFM:

Definition 1 *We call a platform duopoly equilibrium "proportional" if the fraction of buyers and sellers on each platform is identical, i.e.* $\beta = s$, $\forall S, B \in \mathcal{N}_0$.

Absent charges the set of subgame perfect quasi-equilibria as defined by the incentive constraints is disconnected from the cornered market outcomes. The cornered market outcomes ($\beta = 0$,$s = 0$ and $\beta = 1, s = 1$), i.e. the case where the market is "tipping" is always a true equilibrium and hence by definition part of the quasi-equilibrium set.

Lemma 2 *Given* $\Delta_S = \Delta_B = 0$ *the set of SPNE is not connected.*

Proof:

From looking at the numerical example for $\Delta_S = 0$ the seller constraint (S1) prevents corner outcomes (0,0) just as symmetrically (S2) prevents corner outcomes (1,1). The first constraint (15) reveals the intercept with the abscissa as

$$\beta = \frac{B - S - 1}{3B + BS} > 0 \tag{17}$$

which always holds from non-triviality.■

The practical implication of this finding is that there may always be an outcome with only one active platform (as emphasized in the work of Brown and Morgan (2009)). However given that there are two platforms operating there exists a *critical mass of buyers* necessary to render this second platform operational. (See EFM (2004), Proposition 4) which has implications for judging the competitive architecture of our setting that allows for complex but realistic and relevant network effects.

Most importantly we find that the set of SPNE can be characterized further than undertaken in EFM and we may often focus on the particular sub-class of proportional equilibria.

Proposition 3 *Given* $\Delta_B = 0$ *the set of SPNE of the game contains proportional equilibria only even if* $\Delta_S > 0$.

Proof:

See Appendix.■

The fact that the buyer constraint is the 'stricter' one relative to the seller constraint around the proportional quasi-equilibrium set can be seen in the numerical example. That this result holds for any B, S is quite intriguing and can be rationalized by the fact that by the non-triviality assumption there are strictly more buyers than sellers and hence they are more averse to inequalities with regard to the buyer seller ratio (and hence their ex-ante probability to obtain the good in the auction) than sellers. It is conjectured that the result also holds for other distributions of valuations.

6 Welfare

When thinking about welfare in this setting one should first note that due to the presence of direct and indirect network effects preferences of buyers and sellers are largely opposed so that the welfare benefits of consolidation as not as in more general models that neglect the former effect. Total welfare, given as the sum of welfare on each platform can be written as

$$W(\beta, s, B, S) = \frac{1}{2} S \frac{\begin{array}{c}(2S + SB) s^2 + (B - 2S - 2SB\beta - 2B\beta) s \\ -2B + B\beta + S + SB\beta - 2B^2\beta - 1 + 2B^2\beta^2\end{array}}{(B\beta + 1)(-B + B\beta - 1)}$$

subject to the constraints that $0 \leq s, \beta \leq 1$ for any $\Delta_B, \Delta_S \geq 0$ as those charges are only redistributed between buyers, sellers and the proprietors of the platform.

We then find that

Proposition 4 *Constraint maximization of the welfare function yields corner outcomes* $W(0, 0, B, S)$ *and* $W(1, 1, B, S)$ *for all* B, S.

Proof:
See Appendix.■

Lemma 5 *Welfare of proportional equilibria is strictly decreasing in* β *until* $\beta = \frac{1}{2}$*, the welfare worst proportional (quasi-)equilibrium.*

Proof:
See Appendix.■

As seen above, holding the relative advantages of buyers and sellers constant, sellers prefer large markets (where the expected equilibrium price is higher) and buyers prefer small, less efficient markets. The previous Proposition shows that aggregating these welfare differentials from an overall welfare perspective, a single platform is optimal in the set of *all* SPNE. From the above Proposition in conjunction with the previous Lemma we can conclude that if $\Delta_B = 0$ then an exogenous charge difference $\Delta_S > 0$ will *always* be welfare improving.

As noted by EFM, the multiplicity of equilibria of the game cannot be disposed of by simple equilibrium refinement as outcomes cannot be Pareto-ranked. Hence, for example, a coalition proof Nash equilibrium has no bite here. A single Pareto-optimal equilibrium does not exit and thus we may not reduce the set of SPNE set to some focal point. This observation makes the implication of Proposition 3 even more valuable as it allows us to restrict the SPNE set without further refinements and we will make ample use of the result below.

7 Endogenous price competition

In order to discuss price formation in the above platform game we now introduce a *pricing game* in a stage prior to the two-stage game above.

In order to tackle the issue of multiple equilibria in this context previous research has resorted to "bad-expectation" *beliefs* (used in Caillaud and Jullien (2003), and refined in Armstrong and Wright (2007)) such that for any given equilibrium in market shares, a price deviation of a platform that violates incentive constraints will lead to a new market equilibrium allocation in which the profit of the deviating platform is strictly lower.

Instead we make use of the recent concept of *insulated equilibrium* (IE), allowing firms to chose allocations directly, developed in a monopoly context in Weyl (2010) while investigating a hybrid approach to two-sided markets based on Rochet and Tirole (2006) and Armstrong (2006).

An extension of IE to competition is in White and Weyl (2011). Their theorem finds:

Theorem 6 *At an IE allocation, the total price platform j charges to side I consumers satisfies*

$$P^{I,j} = C^{I,j} + \mu^{I,j} - N^{J,j} \left(\left[-\frac{\partial \mathbf{N}^J}{\partial \mathbf{P}^J} \right]^{-1} \frac{\partial \mathbf{N}^J}{\partial \mathbf{N}^I} \right)_{j,\cdot} \left[-\mathbf{D}^I_{\cdot,j} \right] \tag{18}$$

Proof:

See White and Weyl (2011) for details.■

Here $\mu^{I,j} = N^{I,j} / \left(-\frac{\partial N^{I,j}}{\partial P^{I,j}} \right)$ is a market power markup and $-\mathbf{D}^I_{\cdot,j}$ is a matrix of diversion ratios which for two platforms is

$$D^I_{k,j} \equiv \frac{\partial N^{I,k}}{\partial P^{I,j}} / \left(-\frac{\partial N^{I,j}}{\partial P^{I,j}} \right) \tag{19}$$

i.e. the fraction of side I demand that goes to platform k when it increases its own side I price keeping fixed $\mathbf{N}^J$. The two sides of each platform are buyers and sellers in our case, hence $I = S$ and $J = B$. The matrix component in the middle of (18) results from their key assumption that firms can "insulate" and thus fix an initial *coarse* allocation. The resulting equation for a coarse allocation is

$$\frac{d\mathbf{N}^J}{d\mathbf{N}^I} = \frac{\partial \mathbf{N}^J}{\partial \mathbf{N}^I} + \frac{\partial \mathbf{N}^J}{\partial \mathbf{P}^J} \frac{\partial \mathbf{P}^J}{\partial \mathbf{N}^I} = \mathbf{0} \tag{20}$$

so that the first order condition for profit maximization becomes (18).

An allocation is coarse, so that marginal consumers have positive mass. Denote by $f^B_{1,\phi}$ the mass of buyers who are indifferent between joining platform 1 and no platform and $\beta^B_{1,\phi}$ as their average interaction value. Symmetrically for $f^B_{2,\phi}$. Denote by $f^B_{1,2}$ the mass of sellers who are indifferent between platform 1 and platform 2 and by $\beta^S_{1,2}$ their average interaction value.

Allowing for a *market expansion margin* a growth parameter for buyers and sellers is given by ε. Thus the total number of buyers and sellers are replaced by $S(1+\varepsilon)$ and $B(1+\varepsilon)$ so that we have

$$f^S_{1,\phi} = f^B_{2,\phi} = \frac{\varepsilon B}{2} \tag{21}$$

We also know from the calculations underlying Proposition 3 (see (63)) that the horizontal difference between two buyer constraints is

$$\beta_{(B1)} - \beta_{(B2)} = \frac{1}{B} = f^B_{1,2} \tag{22}$$

which is the mass of buyers indifferent between the two platforms for any s and hence by definition equal to $f^B_{1,2}$. We then find that

Proposition 7 *The IE price on platform j is given by*

$$P^{S,j} = C^{S,j} + \mu^{S,j} - N^{B,j}\left(\frac{1}{\varepsilon B^2 + 2}\left(2\beta^B_{1,2} + B^2\varepsilon\beta^B_{1,0}\right)\right)$$

and

$$P^{S,j} \approx C^{S,j} - \left(\frac{1}{2\left(B\beta + 1\right)}\right) \tag{23}$$

if the market expansion margin is small.

Proof:
See Appendix.■

Given that the original EFM model does not allow for market expansion we focus on the case of a small margin. Thus the equilibrium seller prices on platform 1 under IE are

$$P^{S,1*} \approx C^{S,1} - \left(\frac{1}{2\left(\beta B + 1\right)}\right) \tag{24}$$

and on platform 2

$$P^{S,2*} \approx C^{S,1} - \left(\frac{1}{2\left((1-\beta)B + 1\right)}\right) \tag{25}$$

where the approximations can be made as precise as necessary by choosing the margin sufficiently small. Note that these results are robust to the exact introduction of expansion in demand and the speed of convergence.

These equilibrium prices will be strictly higher if there are relatively more buyers on the platform on which the seller operates.

7.1 Switching constraints with endogenous prices

We can further characterize the equilibrium set qualitatively and quantitatively. On platform 1 seller net utilities are

$$u_S(B,S) - P^{S,1} = \frac{\beta B - sS}{\beta B + 1} - \left(C^{S,j} - \left(\frac{1}{2\,(\beta B + 1)} \right) \right) \tag{26}$$

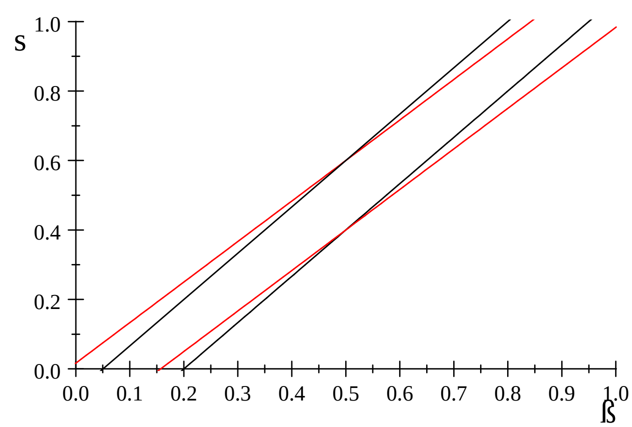
so that the switching constraint (S1) becomes

$$\frac{\beta B - sS}{\beta B + 1} + \frac{1}{2\,(B\beta + 1)} \geq \frac{(1-\beta)B - ((1-s)S + 1)}{(1-\beta)B + 1} + \frac{1}{2\,(B(1-\beta)+1)} \tag{27}$$

and the switching constraint (S2) becomes

$$\frac{(1-\beta)B - (1-s)S}{(1-\beta)B + 1} + \frac{1}{2\,(B(1-\beta)+1)} \geq \frac{\beta B - (sS+1)}{\beta B + 1} + \frac{1}{2\,(B\beta + 1)} \tag{28}$$

For our example ($B = 10, S = 5$) the effect of endogenizing prices as compared to the case where prices play no role, i.e. the setting of EFM can be depicted graphically as



so that now platforms can coexist *independently* of their size.

This will hold if there are sufficiently many buyers (relative to sellers) as in our example. The exact condition is given by:

Lemma 8 *Given* $\Delta_S = \Delta_S^*$ *and* $\Delta_B = 0$ *the set of SPNE is connected if*

$$\frac{1}{3} < \frac{B}{2S+2} < 1$$

Proof:
See Appendix.■

Hence endogenizing the seller pricing decision using the equilibrium refinement developed in White and Weyl (2011) allows also for very small platforms to be viable and hence for the implementation of very asymmetric allocations.

8 Large platforms

The above analysis finds that equilibria of this game may have non-Bertrand outcomes (despite homogeneity of the product of the transaction) where pricing differences between the two platforms may prevail in subgame perfect equilibrium. We now investigate the robustness of this property of the model for large platforms.

Proposition 9 *On large platforms any equilibrium is proportional and charges satisfy* $\Delta_B = \Delta_S = 0$.

Proof:

The buyer constraints are given above as (9) and (10). Letting the share of buyers to sellers on each platform be fixed at some $\bar{x}_i = S_i/B_i$ $i = 1, 2$ we find that the first constraint becomes

$$\frac{\bar{x}_1(\frac{1}{B_1} + \bar{x}_1)}{2(\frac{1}{B_1} + 1)} \geq \frac{\bar{x}_2(\frac{1}{B_2} + \bar{x}_2)}{2(1 + \frac{1}{B_2})(\frac{2}{B_2} + 1)} - \Delta_B \tag{29}$$

and on large platforms where $B_1, B_2 \to \infty$ we find that this reduces to

$$\frac{(\bar{x}_1)^2}{2} \geq \frac{(\bar{x}_2)^2}{2} - \Delta_B \tag{30}$$

for any share $\bar{x}_1$ as $u_B(1, \bar{x}_1) \to (\bar{x}_1)^2/2$. The second constraint can similarly be reduced to

$$\frac{(\bar{x}_2)^2}{2} - \Delta_B \geq \frac{(\bar{x}_1)^2}{2} \tag{31}$$

so that the only outcome that satisfies these constraints has $\Delta_B = 0$ and $\bar{x}_1 = \bar{x}_2$. Similarly for sellers we have from (15) that

$$1 - \bar{x}_1 \geq 1 - \bar{x}_2 - \Delta_S \tag{32}$$

and (16)

$$1 - \bar{x}_2 - \Delta_S \geq 1 - \bar{x}_1 \tag{33}$$

which again can only be satisfied for $\Delta_S = 0$ and $\bar{x}_1 = \bar{x}_2$ as $u_S(1, \bar{x}_1) \to 1 - \bar{x}_1$. The conclusion follows from noting that $\bar{x}_1 = \bar{x}_2 \Leftrightarrow \beta = s.$■

This finding mirrors Proposition 3 in Brown and Morgan (2009) who show that with vertical differentiation (i.e. a charge difference $\Delta_S > 0$ in our case) equilibrium in very large markets is impossible.

A version of their Proposition 4 holds that in addition:

Proposition 10 *In any quasi-equilibrium in which the sites coexist and eBay (here 1) enjoys an* exogenous *vertical differentiation advantage for sellers (here $\Delta_S > 0$) and a more than 50% market share, relatively more sellers are attracted to a given eBay auction than an Yahoo! auction for sufficiently many buyers.*

Proof:

The seller constraint (S2) from (16) can be transformed into

$$s_{S2} \geq \frac{1}{S\left(B+2\right)}\left(S-2B+\Delta_S+B\Delta_S-1\right)+B\beta\frac{S+B\Delta_S-B\Delta_S\beta+3}{S\left(B+2\right)} \quad (34)$$

Also given participation constraints hold the maximal advantage for sellers is bounded by

$$\Delta_S < Max_{\beta,s}\left\{\frac{(1-\beta)B-(1-s)S}{(1-\beta)B+1}\right\} \quad (35)$$

Now we show that if $\Delta_S > 0$ and $\beta > 1/2$ then $s > \beta$ for sufficiently many buyers.

Note that s_{S2} is strictly concave in β given that $\Delta_S > 0$. The difference between s_{S2} and the 45° line (where $\beta = s$) is

$$d \equiv s_{S2} - \beta = \frac{1}{S\left(B+2\right)}\left(S-2B+\Delta+B\Delta-1\right)+\beta\frac{3B-2S+B^2\Delta-B^2\Delta\beta}{S\left(B+2\right)} \quad (36)$$

and still strictly concave in β. Thus the difference d attains a maximum at

$$\beta_{\max} = \frac{3B-2S+B^2\Delta_S}{2B^2\Delta_S} > 0 \quad (37)$$

at $\beta = 0$ the difference is

$$d = \frac{1}{S\left(B+2\right)}\left(S-2B+\Delta+B\Delta-1\right)$$

which is strictly negative given (35). As derivatives are smooth there exists a unique intermediate value of β such that $d = 0$. This value can be found as

$$\beta_k = \frac{S-B(2-\Delta_S)+\Delta_S-1}{2S-B(B\Delta_S+3)} \quad (38)$$

Note that

$$\frac{\partial\beta_k}{\partial\Delta_S} = -\frac{\left(3B-2S+B\left(B+2\right)\left(2B-S\right)\right)}{\left(\Delta B^2+3B-2S\right)^2} \quad (39)$$

which given non-triviality $B > S+1$ is negative. Hence the difference is falling in Δ_S. Then there is a critical level of seller advantage such that the critical level of the intersection of s_{s2} and the 45° line is exactly at $\beta = 1/2$. This level is

$$\Delta_k = \frac{B+2}{2B+B^2+2} \tag{40}$$

and is *falling* in B. With sufficiently many buyers for any $\Delta_S > \Delta_k (\rightarrow 0)$ we have given $\beta > 1/2$ that $s > \beta$, i.e. the seller switching constraint ($S2$) can only be satisfied strictly above the 45° line. Thus platform 1 faces $s > \beta$ and so a seller buyer ratio of

$$\frac{sS}{\beta B} > \frac{S}{B} \tag{41}$$

and by adding up

$$\frac{(1-s)S}{(1-\beta)B} < \frac{S}{B} < \frac{sS}{\beta B}. \blacksquare \tag{42}$$

The original Proposition 4 in Brown and Morgan claims that with an *exogenous* vertical differentiation advantage for sellers and a more than 50% market share, relatively more buyers are attracted to a Yahoo! than an eBay (platform 1) auction which contradicts their data. Once prices are endogenized using Proposition 7 we note that the theoretical implication is *exactly reversed* as sellers on eBay will actually face relatively more favourable buyer seller ratios as they will have to pay the higher seller charge and thus face an endogenous vertical differentiation *dis*advantage (i.e. $\Delta_S < 0$).

It turns out that endogenizing prices to sellers is also sufficient to reverse the result in Brown and Morgan (2009) about the relative *transaction prices* on both platforms and hence bring the model in line with their data in this respect too. We can show that:

Proposition 11 *In any quasi-equilibrium in which the sites coexist and eBay (here* 1*) enjoys an* endogenous *vertical differentiation* disadvantage for sellers *(here* $\Delta_S^* < 0$*) and a more than 50% market share, for sufficiently many buyers the transaction price on eBay is higher than that on Yahoo!.*

Proof:
See Appendix.■

Hence we find that with endogenous seller charges the transaction prices on eBay will be larger than those of Yahoo!, in line with the data findings in Brown and Morgan (2009).

Alternatively we can also use an exogenous vertical differentiation advantage for *buyers* to similarly show that:

Proposition 12 *In any quasi-equilibrium in which the sites coexist and eBay (here* 1*) enjoys an* exogenous *vertical differentiation* advantage for buyers *(here* $\Delta_B > 0$*) and a more than 50% market share, relatively more buyers are attracted to a given eBay auction than an Yahoo! auction for sufficiently many buyers and the transaction price on eBay is higher than on Yahoo!.*

Proof:
See Appendix.■

We have thus shown two alternatives by which the empirical results reported in Brown and Morgan (2009) can be brought in line with the theory. The first implies that the liquidity effects of a large market will dominate the effect of the endogenous seller charges on large platforms leading to a higher expected transaction price to the detriment of its buyers. Alternatively one may argue that if eBay has an exogenous vertical differentiation advantage for buyers in addition to being the dominant platform in a liquid market this is also sufficient to explain the more favourable buyers-seller ratio for its buyers and for it to have larger transaction prices than at Yahoo! auctions.

Note that endogenous seller charges satisfy

$$\lim_{B\to\infty}(\Delta_s^*) = \lim_{B\to\infty}(\frac{B}{2}\frac{1-2\beta}{(B(1-\beta)+1)(B\beta+1)}) = 0. \tag{43}$$

The intuition for this limit result is straightforward: The possibility that the switching of either buyer or seller has a tangible impact on expectations decreases as the number of buyers and sellers increases so that in the limit as markets get very large all friction disappears from the model and we get a Bertrand type outcome with regard to the charge differences and proportional equilibria. This Proposition can be easily extended to an unspecified distribution of valuations and is thus robust.

We also have a result for welfare on large platforms: As total welfare of a platform goes out of bounds if the platform gets very large we look at total welfare per buyer and seller respectively

$$\frac{w(B,S)}{B} = u_B(B,S) + \bar{x}u_S(B,S) = \bar{x}(1-\frac{\bar{x}}{2}) \tag{44}$$

and

$$\frac{w(B,S)}{S} = \frac{1}{\bar{x}}u_B(B,S) + u_S(B,S) = 1-\frac{\bar{x}}{2} \tag{45}$$

where $\bar{x}$ is the limit of the total seller to buyer ratio. By the non-triviality assumption the per capita welfare contribution of a buyer is thus always lower than that of a seller.

9 Conclusion

Often buyers cannot be charged for participating on a platform. For example on eBay seller-fee-shifting is not allowed. Alternatively the final transactions may not be observable as on used-car platforms. In these cases the strictness of the buyer switching constraints implies that independently of whether or not there are charges to the sellers, the equilibrium market structure of the platform duopoly will imply proportional equilibria. This strongly restricts the set of equilibria of the game compared to that in EFM.

The original Proposition 4 in the paper by Brown and Morgan (2009) exactly contradicts their data which finds that: "eBay sellers enjoy higher prices and more favourable buyer-seller ratios than do Yahoo! sellers." Endogenizing the platform's pricing decision for sellers by using the concept of "insulating equilibrium" in Proposition 7 we are able to show that theory and practice actually reveal an endogenous vertical *dis*advantage for sellers on eBay being the dominant platform. This observation exactly reverses their theoretical findings bringing them in line with the data from their field experiments.

A similar finding pertains with respect to the predicted relative transaction prices on both platforms. Once charges to sellers are endogenized, being the dominant platform implies that transaction prices will indeed be larger on eBay, the more liquid platform, again as observed it their field experiments. An alternative theoretical derivation of these results can be derived for an exogenous vertical differentiation advantage for buyers on eBay.

10 Appendix

Proof of Lemma 0:

Under the uniform distribution on [0,1] the ith *lowest* order statistic out of n draws is distributed Beta$(i, n-i+1)$ with probability density function

$$f^{i,n-i+1}(x) = \begin{cases} \frac{x^{i-1}(1-x)^{n-i}}{\int_0^1 u^{i-1}(1-u)^{n-i}du}, x \in (0,1) \\ 0 \text{ otherwise} \end{cases} \tag{46}$$

and expectation

$$\int_0^1 x f^{i,n-i+1}(x)dx = \frac{i}{n+1} \tag{47}$$

As the order statistic of the $S+1$ highest of B draws is also that of the $B-S$ *lowest,* the expectation of the price given by the $S+1$ highest buyer valuation can be rewritten as

$$\int_0^1 x f^{S+1,B}(x)dx = \frac{B-S}{B+1} \tag{48}$$

which is also expected seller surplus due to the normalized reservation value. Thus the density of the order statistic $v^{S+1,B}$ is

$$f^{S+1,B}(x) = \begin{cases} \frac{x^{B-S-1}(1-x)^{S}}{\int_0^1 u^{B-S-1}(1-u)^{S}du}, x \in (0,1) \\ 0 \text{ otherwise} \end{cases} \tag{49}$$

Total welfare $w(B,S)$ on one platform given uniformly distributed valuations can thus be written as

$$w(B,S) = S\int_0^1 \left(\int_x^1 vf(v\,|v > x)dv)\right) f^{S+1,B}(x)dx =$$

$$\frac{S}{\int_0^1 u^{B-S-1}(1-u)^S du} \int_0^1 \left(\int_x^1 v\left(\frac{1}{1-x}\right)dv\right)(x^{B-S-1}(1-x)^S)dx =$$

$$\frac{S}{2\int_0^1 u^{B-S-1}(1-u)^S du} \int_0^1 (x+1)(x^{B-S-1}(1-x)^S)dx =$$

$$= \quad S(1 - \frac{1}{2}\frac{1+S}{B+1}) = S\left(\frac{B-S}{B+1}\right) + B\left(\frac{S(1+S)}{2B(B+1)}\right). \blacksquare$$

Proof of Proposition 3:

We look at the seller constraint and the buyer constraint in turns and show whether it is possible to have them satisfied for non-proportional equilibria. Constraint (S1) is

$$\frac{B\beta - Ss}{B\beta + 1} = \frac{B(1-\beta) - (S(1-s)+1)}{B(1-\beta)+1} - \Delta_S \tag{50}$$

or

$$s_{(S1)} = \frac{3B\beta - B + SB\beta + S + 1 + \Delta_S B^2\beta - \Delta_S B^2\beta^2 + \Delta_S(B+1)}{S\,(B+2)} \tag{51}$$

and (S2) is

$$\frac{B(1-\beta) - S(1-s)}{B(1-\beta)+1} - \Delta_S = \frac{B\beta - (Ss+1)}{B\beta + 1} \tag{52}$$

or

$$s_{(S2)} = \frac{-2B + 3B\beta + SB\beta + S + \Delta_S B^2\beta - \Delta_S B^2\beta^2 + \Delta_S(B+1) - 1}{S\,(B+2)} \tag{53}$$

with vertical difference between the two seller constraints

$$s_{(S1)} - s_{(S2)} = \frac{1}{S} \tag{54}$$

for any Δ_S. Thus it may be possible to have the seller constraint strictly satisfied at a non-proportional equilibrium by 'squeezing in' non-proportional equilibrium candidate vertically.

For $\Delta_S = 0$ we have

$$\beta_{(S1)} = \frac{SsB + 2Ss + B - S - 1}{B\,(3+S)} \tag{55}$$

and

$$\beta_{(S2)} = \frac{2B - S + 2Ss + SsB + 1}{B\,(3+S)} \tag{56}$$

so that the horizontal difference between the two seller constraints is

$$\beta_{(S2)} - \beta_{(S1)} = \frac{B+2}{B\,(3+S)} > \frac{1}{B} \tag{57}$$

as $B > S + 1$. Thus it may be possible to have the seller constraint satisfied at a non-proportional equilibrium by 'squeezing in' non-proportional equilibrium candidate horizontally.

We now look at the critical buyer constraints: Now (B1) is

$$\frac{sS(1+sS)}{2\beta B(1+\beta B)} \geq \frac{(1-s)S(1+(1-s)S)}{2((1-\beta)B+1)(1+(1-\beta)B+1)} - \Delta_B \tag{58}$$

and (B2) is

$$\frac{(1-s)S(1+(1-s)S)}{2(1-\beta)B(1+(1-\beta)B)} - \Delta_B \geq \frac{sS(1+sS)}{2(\beta B+1)(1+\beta B+1)} \tag{59}$$

For $\Delta_B = 0$ the solution to (B1) is

$$\begin{aligned}\beta_{(B1)} &= \frac{1}{2\,(1+S)\,(2s-1)} \times \\ &\quad \frac{(4S+2SB)\,s^2 + (-2S+2+2B)\,s + 1 + S - \sqrt{\Psi}}{B}\end{aligned} \tag{60}$$

and the one for (B2) is

$$\begin{aligned}\beta_{(B2)} &= \frac{1}{2\,(1+S)\,(2s-1)} \times \\ &\quad \frac{(4S+2SB)\,s^2 + (2B-2-6S)\,s + 3 + 3S - \sqrt{\Psi}}{B}\end{aligned} \tag{61}$$

with

$$\begin{aligned}\Psi &= 4S^2\,(B+2)^2\,s^4 - 8S^2\,(B+2)^2\,s^3 + \\ &\quad \left(-16SB - 12 + 4S^2B^2 - 4SB^2 - 8S + 20S^2 - 16B + 16S^2B - 4B^2\right) s^2 - \\ &\quad 4\,(1+S)\left(S - 4B - B^2 - 3\right) s + 1 + S^2 + 2S\end{aligned} \tag{62}$$

The horizontal difference is then

$$\beta_{(B1)} - \beta_{(B2)} = \frac{1}{B} \tag{63}$$

Thus given that the two constraints with $\Delta_B = 0$ are always on opposite sides of the $\beta = s$ diagonal there will always be proportional equilibrium candidates and it is impossible to 'squeeze in' another non-proportional equilibrium candidate horizontally. Note that the result does not hold for $\Delta_B > 0$ although the horizontal difference remains the same.

The vertical difference between (B2) and (B1) is difficult to calculate directly. However we can use the fact that the distance between (B1) and the diagonal is monotone increasing in β and the mirror image, between (B2) and the diagonal is monotone decreasing in β which follows from buyers preference for the smaller platform. Hence if we can show that this distance for (B1) at $\beta = 1$ (or the distance for (B2) at $\beta = 0$ is smaller than $\frac{1}{S}$ again we can be sure that no non-proportional equilibrium can be 'squeezed in' next to the proportional equilibria on the diagonal.

Using (B1)

$$\beta_{(B1)} = \frac{1}{2(1+S)(2s-1)} \times \frac{(4S+2SB)s^2 + (-2S+2+2B)s + 1 + S - \sqrt{\Psi}}{B} \tag{64}$$

we solve this equation for the relevant root and evaluate it at $\beta = 1$ to find

$$s_{(B1)} = -\frac{1}{2}\frac{\begin{array}{c}-2SB - 2SB^2 - B - B^2 - 2+ \\ \sqrt{(4+4B+5B^2+B^4+2B^3+16SB+8B^2S^2+8BS^2+16SB^2)}\end{array}}{S(B+B^2-2)} \tag{65}$$

The vertical distance to the diagonal is then given as

$$1 - s_{(B1)}(\beta = 1) = \frac{1}{2}\frac{\begin{array}{c}-4S - B - B^2 - 2+ \\ \sqrt{(4+4B+5B^2+B^4+2B^3+16SB+8B^2S^2+8BS^2+16SB^2)}\end{array}}{S(B+B^2-2)} \tag{66}$$

Now

$$1 - s_{(B1)}(\beta = 1) < \frac{1}{S} \tag{67}$$

will hold if

$$B < -S \tag{68}$$

which cannot hold, or if

$$B > S - 1 \tag{69}$$

which holds by the non-triviality constraint that $B > S + 1$.■

Proof of Proposition 4:
In order to obtain general results we need to maximize using the Kuhn-Tucker approach

$$Max_{s,\beta}W = \frac{1}{2}S\frac{\begin{array}{c}(2S+SB)\,s^2+(B-2S-2SB\beta-2B\beta)\,s\\ -2B+B\beta+S+SB\beta-2B^2\beta-1+2B^2\beta^2\end{array}}{(B\beta+1)\,(-B+B\beta-1)} \tag{70}$$

subject to the constraints

$$\beta \geq 0, s \geq 0, \beta \leq 1, s \leq 1 \tag{71}$$

so that we can set up the Lagrangian

$$\begin{aligned} & L(\beta, s, \lambda_1, \lambda_2, \lambda_3, \lambda_4,) = \\ & \frac{1}{2}S\frac{\begin{array}{c}(2S+SB)\,s^2+(B-2S-2SB\beta-2B\beta)\,s\\ -2B+B\beta+S+SB\beta-2B^2\beta-1+2B^2\beta^2\end{array}}{(B\beta+1)\,(-B+B\beta-1)} \\ & -\lambda_1(-\beta)-\lambda_2(-s)-\lambda_3(\beta-1)-\lambda_4(s-1) \end{aligned} \tag{72}$$

and the first order conditions can be written as

$$\frac{\partial L}{\partial \beta} = \frac{1}{2}SB\frac{\begin{array}{c}(2s-1)\left(B^2\beta^2(S+1)+S(2B\beta+1)+1\right)+\\ (1-2\beta)\,sB\,(sS(B+2)+B)+(s-\beta)2B\end{array}}{(B\beta+1)^2\,(B(1-\beta)+1)^2} + \lambda_1 - \lambda_3 = 0 \tag{73}$$

$$\frac{\partial L}{\partial s} = \frac{1}{2}S\frac{(1-2s)2S+(2\beta-1)B+(\beta-s)2SB}{B^2\beta(1-\beta)+B+1} + \lambda_2 - \lambda_4 = 0 \tag{74}$$

$$\frac{\partial L}{\partial \lambda_1} = \beta \geq 0, \lambda_1 \geq 0, \lambda_1\beta = 0 \tag{75}$$

$$\frac{\partial L}{\partial \lambda_2} = s \geq 0, \lambda_2 \geq 0, \lambda_2 s = 0 \tag{76}$$

$$\frac{\partial L}{\partial \lambda_3} = 1-\beta \geq 0, \lambda_3 \geq 0, \lambda_3(1-\beta) = 0 \tag{77}$$

$$\frac{\partial L}{\partial \lambda_4} = 1-s \geq 0, \lambda_4 \geq, \lambda_4(1-s) = 0 \tag{78}$$

Note that the first term in (73) given by

$$A \equiv \frac{1}{2}SB\frac{\begin{array}{c}(2s-1)\left(B^2\beta^2(S+1)+S(2B\beta+1)+1\right)+\\ (1-2\beta)\,sB\,(sS(B+2)+B)+(s-\beta)2B\end{array}}{(B\beta+1)^2\,(B(1-\beta)+1)^2} \tag{79}$$

is strictly negative if $\beta \in \left(\frac{1}{2}, 1\right]$ and $s \in \left[0, \frac{1}{2}\right]$ (or if $\beta \in \left[\frac{1}{2}, 1\right]$ and $s \in \left[0, \frac{1}{2}\right)$) and positive otherwise.

Note that the first term B in (74) given by

$$B \equiv \frac{1}{2} S \frac{(1-2s)2S + (2\beta-1)B + (\beta-s)2SB}{B^2\beta(1-\beta) + B + 1} \tag{80}$$

is strictly negative if $\beta \in \left[0, \frac{1}{2}\right]$ and $s \in \left(\frac{1}{2}, 1\right]$ (or if $\beta \in \left(0, \frac{1}{2}\right]$ and $s \in \left[\frac{1}{2}, 1\right]$) and positive otherwise.

We need to consider *three cases* in turn:

a) Given

$$0 \leq \beta \leq \frac{1}{2} < s \leq 1$$

we may rewrite the constraints as:

$$\frac{\partial L}{\partial \beta} = (A > 0) + \lambda_1 - \lambda_3 = 0 \tag{81}$$

$$\frac{\partial L}{\partial s} = (B < 0) + \lambda_2 - \lambda_4 = 0 \tag{82}$$

$$\frac{\partial L}{\partial \lambda_1} = \beta \geq 0, \lambda_1 \geq 0, \lambda_1 \beta = 0 \tag{83}$$

$$\frac{\partial L}{\partial \lambda_2} = s \geq 0, \lambda_2 \geq 0, \lambda_2 s = 0 \tag{84}$$

$$\frac{\partial L}{\partial \lambda_3} = 1 - \beta \geq 0, \lambda_3 \geq 0, \lambda_3(1-\beta) = 0 \tag{85}$$

$$\frac{\partial L}{\partial \lambda_4} = 1 - s \geq 0, \lambda_4 \geq 0, \lambda_4(1-s) = 0 \tag{86}$$

As $\lambda_1 \geq 0$ we find that $\lambda_3 > 0$ and as $\lambda_4 \geq 0$ we find that $\lambda_2 > 0$. Then it follows that we need that $\beta = 1$ and $s = 0$ which cannot be the case. Contradiction.

b) Given

$$0 \leq s \leq \frac{1}{2} < \beta \leq 1$$

we may rewrite the constraints as:

$$\frac{\partial L}{\partial \beta} = (A < 0) + \lambda_1 - \lambda_3 = 0 \tag{87}$$

$$\frac{\partial L}{\partial s} = (B > 0) + \lambda_2 - \lambda_4 = 0 \tag{88}$$

$$\frac{\partial L}{\partial \lambda_1} = \beta \geq 0, \lambda_1 \geq 0, \lambda_1 \beta = 0 \tag{89}$$

$$\frac{\partial L}{\partial \lambda_2} = s \geq 0, \lambda_2 \geq 0, \lambda_2 s = 0 \tag{90}$$

$$\frac{\partial L}{\partial \lambda_3} = 1 - \beta \geq 0, \lambda_3 \geq 0, \lambda_3(1-\beta) = 0 \tag{91}$$

$$\frac{\partial L}{\partial \lambda_4} = 1 - s \geq 0, \lambda_4 \geq 0, \lambda_4(1-s) = 0 \tag{92}$$

As $\lambda_3 \geq 0$ we find that $\lambda_1 > 0$ and as $\lambda_2 \geq 0$ we find that $\lambda_4 > 0$. Then it follows that we need that $s = 1$ and $\beta = 0$ which cannot be the case. Contradiction.

c) If we assume that

$$s = \beta$$

the Lagrangian reduces to

$$L(\beta, s, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = L(s, \lambda) \tag{93}$$
$$\frac{1}{2}S\frac{\begin{array}{c}2Ss^2 - SBs^2 + 2Bs - 2sS - 2Bs^2 - \\ 2B + S + SBs - 2B^2 s - 1 + 2B^2 s^2\end{array}}{(Bs+1)(Bs-B-1)} - \lambda_1(-s) - \lambda_2(s-1)$$

$$\frac{\partial L}{\partial s} = \left(C = \frac{1}{2}S\frac{(2s-1)(B+2)(B-S)}{(Bs+1)^2 (B(1-s)+1)^2} \right) + \lambda_1 - \lambda_2 = 0 \tag{94}$$

$$\frac{\partial L}{\partial \lambda_1} = s \geq 0, \lambda_1 \geq 0, \lambda_1 s = 0 \tag{95}$$

$$\frac{\partial L}{\partial \lambda_2} = 1 - s \geq 0, \lambda_2 \geq 0, \lambda_2(1-s) = 0 \tag{96}$$

Note that C negative if $s < \frac{1}{2}$ and positive if $s > \frac{1}{2}$.
Case $s > \frac{1}{2}$:

$$\frac{\partial L}{\partial s} = (C > 0) + \lambda_1 - \lambda_2 = 0 \tag{97}$$

$$\frac{\partial L}{\partial \lambda_1} = s \geq 0, \lambda_1 \geq 0, \lambda_1 s = 0 \tag{98}$$

$$\frac{\partial L}{\partial \lambda_2} = 1 - s \geq 0, \lambda_2 \geq 0, \lambda_2(1-s) = 0 \tag{99}$$

As $\lambda_1 \geq 0$ and $C > 0$ then $\lambda_2 > 0$ then $s = 1$, a corner solution.

Case $s < \frac{1}{2}$:

As $\lambda_2 \geq 0$ and $C < 0$ then $\lambda_1 > 0$ then $s = 0$, a corner solution.

All we need to show for the second order condition to hold is that

$$S\left(B+2\right)\left(S-B\right)\frac{3B^2 s(1-s)-B-1-B^2}{\left(Bs+1\right)^3\left(B(1-s)+1\right)^3} < 0 \tag{100}$$

At the first corner solution $s = 1$ we find

$$H_{11}(s=1) = S\left(B+2\right)\left(B-S\right)\frac{B+1+B^2}{\left(B+1\right)^3} < 0 \tag{101}$$

Similarly at the second corner solution $s = 0$ we find

$$H_{11}(s=0) = S\left(B+2\right)\left(B-S\right)\frac{B+1+B^2}{\left(B+1\right)^3} < 0 \tag{102}$$

and thus both solutions are indeed maximizing the welfare function.■

Proof of Lemma 5:

Any proportional equilibrium implies that $\beta = s$ and so total welfare reduces to

$$W = \frac{1}{2}S\frac{\begin{array}{c}\left(2B+SB-2B^2-2S\right)\beta^2+\\ \left(2B^2+2S-2B-SB\right)\beta+2B-S+1\end{array}}{\left(B\beta+1\right)\left(B(1-\beta)+1\right)} \tag{103}$$

with first derivative

$$\frac{\partial W}{\partial \beta} \overset{!}{=} 0 = \frac{1}{2}S\frac{\left(2\beta-1\right)\left(B+2\right)\left(B-S\right)}{\left(B\beta+1\right)^2\left(B(1-\beta)+1\right)^2} \tag{104}$$

with solution $\beta^* = 1/2$. See that $\partial W/\partial\beta < 0$ for $\beta < 1/2$ and $\partial W/\partial\beta > 0$ for $\beta > 1/2$ and any B, S.

The second order condition is

$$\frac{\partial^2 W}{\partial \beta^2}\,|_{\beta=\beta^*} = 16S\frac{B-S}{\left(B+2\right)^3} > 0 \tag{105}$$

as $B > S+1$ and hence β^* yields a minimum of the welfare function when we look at proportional equilibria, the welfare worst proportional equilibrium.■

Proof of Proposition 7:
The matrices given in (20) can be decomposed into:

$$\begin{bmatrix} \tilde{\beta}_{1,\phi}^{B} f_{1,\phi}^{B} + \tilde{\beta}_{1,2}^{B} f_{1,2}^{B} & -\tilde{\beta}_{1,2}^{B} f_{1,2}^{B} \\ -\tilde{\beta}_{1,2}^{B} f_{1,2}^{B} & \tilde{\beta}_{2,\phi}^{B} f_{2,\phi}^{B} + \tilde{\beta}_{1,2}^{B} f_{1,2}^{B} \end{bmatrix} + \begin{bmatrix} -f_{1,\phi}^{B} - f_{1,2}^{B} & f_{1,2}^{B} \\ f_{1,2}^{B} & -f_{2,\phi}^{B} - f_{1,2}^{B} \end{bmatrix} \frac{\partial \mathbf{P}^{B}}{\partial \mathbf{N}^{S}} = \mathbf{0} \tag{106}$$

which, using symmetry, becomes

$$\begin{bmatrix} \tilde{\beta}_{1,\phi}^{B} \frac{\varepsilon B}{2} + \tilde{\beta}_{1,2}^{B} \frac{1}{B} & -\tilde{\beta}_{1,2}^{B} \frac{1}{B} \\ -\tilde{\beta}_{1,2}^{B} \frac{1}{B} & \tilde{\beta}_{1,\phi}^{B} \frac{\varepsilon B}{2} + \tilde{\beta}_{1,2}^{B} \frac{1}{B} \end{bmatrix} + (-1) \begin{bmatrix} \frac{\varepsilon B}{2} + \frac{1}{B} & -\frac{1}{B} \\ -\frac{1}{B} & \frac{\varepsilon B}{2} + \frac{1}{B} \end{bmatrix} \frac{\partial \mathbf{P}^{B}}{\partial \mathbf{N}^{S}} = \mathbf{0} \tag{107}$$

or

$$\begin{bmatrix} \frac{\varepsilon B}{2} + \frac{1}{B} & -\frac{1}{B} \\ -\frac{1}{B} & \frac{\varepsilon B}{2} + \frac{1}{B} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\beta}_{1,\phi}^{B} \frac{\varepsilon B}{2} + \tilde{\beta}_{1,2}^{B} \frac{1}{B} & -\tilde{\beta}_{1,2}^{B} \frac{1}{B} \\ -\tilde{\beta}_{1,2}^{B} \frac{1}{B} & \tilde{\beta}_{1,\phi}^{B} \frac{\varepsilon B}{2} + \tilde{\beta}_{1,2}^{B} \frac{1}{B} \end{bmatrix} = \tag{108}$$

$$\frac{2}{B\varepsilon \left(B^{2}\varepsilon + 4\right)} \begin{bmatrix} B^{2}\varepsilon + 2 & 2 \\ 2 & B^{2}\varepsilon + 2 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\beta}_{1,\phi}^{B} \frac{\varepsilon B}{2} + \tilde{\beta}_{1,2}^{B} \frac{1}{B} & -\tilde{\beta}_{1,2}^{B} \frac{1}{B} \\ -\tilde{\beta}_{1,2}^{B} \frac{1}{B} & \tilde{\beta}_{1,\phi}^{B} \frac{\varepsilon B}{2} + \tilde{\beta}_{1,2}^{B} \frac{1}{B} \end{bmatrix} = \frac{\partial \mathbf{P}^{B}}{\partial \mathbf{N}^{S}}$$

The price to sellers is now

$$\begin{aligned} P^{S,j} = {} & C^{S,j} - N^{B,j} \times 2(B^{2}\varepsilon + 4)^{-1} \times \qquad (109) \\ & \left(\begin{bmatrix} \frac{1}{2}\left(2\beta_{1,\phi}^{B} + 2\beta_{1,2}^{B} + B^{2}\varepsilon\beta_{1,\phi}^{B}\right) & \left(\beta_{1,\phi}^{B} - \beta_{1,2}^{B}\right) \\ \left(\beta_{1,\phi}^{B} - \beta_{1,2}^{B}\right) & \frac{1}{2}\left(2\beta_{1,\phi}^{B} + 2\beta_{1,2}^{B} + B^{2}\varepsilon\beta_{1,\phi}^{B}\right) \end{bmatrix} \right)_{j,\cdot} \left[-\mathbf{D}_{\cdot,j}^{I}\right] \end{aligned}$$

Given

$$\begin{bmatrix} 1 \\ -D_{1,2}^{B} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{f_{1,2}^{B}}{f_{1,\phi}^{B} + f_{1,2}^{B}} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2}{\varepsilon B^{2} + 2} \end{bmatrix} \tag{110}$$

we find, expanding by the first row that prices are

$$\begin{aligned} P^{S,j} &= C^{S,j} + \mu^{S,j} - N^{B,j} \times 2(B^{2}\varepsilon + 4)^{-1} \times \\ & \left(\left[\frac{1}{2}\left(2\beta_{1,\phi}^{B} + 2\beta_{1,2}^{B} + B^{2}\varepsilon\beta_{1,\phi}^{B}\right) - \frac{2}{\varepsilon B^{2} + 2}\left(\beta_{1,\phi}^{B} - \beta_{1,2}^{B}\right) \right] \right) \qquad (111) \\ &= C^{S,j} + \mu^{S,j} - N^{B,j} \left(\frac{1}{\varepsilon B^{2} + 2}\left(2\beta_{1,2}^{B} + B^{2}\varepsilon\beta_{1,\phi}^{B}\right) \right) \end{aligned}$$

As the EFM model does not allow for a market expansion margin we let $\varepsilon \to 0$ so what matters are switchers' valuations $\beta_{1,2}^{B}$ only. As, in addition, consumers are homogenous these valuations are $\beta_{1,2}^{B} = \beta^{B} = \frac{\partial u_{B}(B,S)}{\partial S}$. We thus find that the limit equilibrium price on platform j (with buyer share βB and seller share sS) is

$$\begin{aligned}
P^{S,j} &= C^{S,j} + \mu^{S,j} - N^{B,j}\beta^{B} \\
&= C^{S,j} - sS\left(-\frac{1}{\beta B+1}\right) - \beta B\left(\frac{1}{2\beta B}\frac{2sS+1}{\beta B+1}\right) \quad (112)\\
&= C^{S,j} - \left(\frac{1}{2\left(B\beta+1\right)}\right). \blacksquare
\end{aligned}$$

Proof of Lemma 8:
The binding switching constraint (S1) can be solved as

$$s_{S1}(\mathbf{P}^{S*}) = \frac{1}{4S+2BS}\left(-B+2S+4B\beta+2BS\beta+2\right) \quad (113)$$

The intercept with the ordinate is at $\beta = 0$ hence

$$s_{S1}(\mathbf{P}^{S*}) = \frac{2S+2-B}{4S+2BS} > 0 \quad (114)$$

is the positive intercept with the ordinate if

$$2S+2 > B \quad (115)$$

Similarly the biding switching constraint (S2) can be solved as

$$s_{S2}(\mathbf{P}^{S*}) = \frac{1}{4S+2BS}\left(-3B+2S+4B\beta+2BS\beta-2\right) < 0 \quad (116)$$

The intercept is positive on the abscissa and therefore negative on the ordinate if

$$s_{S2}(\mathbf{P}^{S*}) = \frac{1}{4S+2BS}\left(-3B+2S-2\right) < 0 \quad (117)$$

or if

$$B > \frac{2S-2}{3} \quad (118)$$

Hence platforms can coexist for any size differences if

$$\frac{2S-2}{3} < B < 2S+2 \quad (119)$$

or

$$\frac{1}{3} < \frac{B}{2S+2} < 1. \blacksquare \quad (120)$$

Proof of Proposition 11:

The transaction prices difference is

$$t_y - t_e = \frac{(1-\beta)B-(1-s)S}{(1-\beta)B+1} - \frac{\beta B - sS}{\beta B+1} = \frac{B-S-2B\beta+2Ss-BS\beta+BSs}{(B(1-\beta)+1)(B\beta+1)} \tag{121}$$

which has the same sign as

$$\sigma \equiv B-S-2B\beta+2Ss-BS\beta+BSs = B-S+Ss(B+2)-B\beta(2+S) \tag{122}$$

which is increasing in s.The (S2) constraint gives

$$s_{S2} \geq \frac{1}{S(B+2)}(S-2B+\Delta_S+B\Delta_S-1)+B\beta\frac{S+B\Delta_S-B\Delta_S\beta+3}{S(B+2)} \tag{123}$$

With endogenous prices the seller charge differential is

$$\Delta_S^* = \frac{B}{2}\frac{1-2\beta}{(B(1-\beta)+1)(B\beta+1)} < 0 \tag{124}$$

and with many buyers $\Delta_S^* \to 0$. Hence what remains is the condition

$$s_{S2} \geq \frac{1}{S(B+2)}(S-2B-1)+B\beta\frac{S+3}{S(B+2)} \tag{125}$$

Substituting in the above yields

$$\sigma = -B(1-\beta)-1 < 0 \tag{126}$$

so this is not sufficient for σ to be positive we need a higher s. Still we know from Proposition 3 that the buyer switching constraints that are forcing equilibria to be *proportional* that $\beta = s$ has to hold. The transaction price differential then reduces to

$$t_y - t_e = (B-S)\frac{1-2\beta}{(B(1-\beta)+1)(B\beta+1)} \tag{127}$$

and as $B > S+1$ and $\beta > 1/2$ we find that this difference is indeed negative.■

Proof of Proposition 12:

The transaction prices difference is

$$t_y - t_e = \frac{(1-\beta)B - (1-s)S}{(1-\beta)B+1} - \frac{\beta B - sS}{\beta B + 1} = \frac{B - S - 2B\beta + 2Ss - BS\beta + BSs}{(B(1-\beta)+1)(B\beta+1)} \tag{128}$$

which has the same sign as

$$\sigma \equiv B - S - 2B\beta + 2Ss - BS\beta + BSs = B - S + Ss(B+2) - B\beta(2+S) \tag{129}$$

which is decreasing in β.

Assuming a exogenous vertical buyer advantage for eBay (1), the (B2) constraint implies

$$\frac{(1-s)S(1+(1-s)S)}{2(1-\beta)B(1+(1-\beta)B)} - \Delta_B \geq \frac{sS(1+sS)}{2(\beta B+1)(1+\beta B+1)} \tag{130}$$

solving for β implicitly yields

$$2\beta_{B2} \geq 2 + \frac{B - \sqrt{\frac{\begin{array}{c} B^2(2(1+\beta B)(2+\beta B)\Delta_B + sS + sS^2)\times \\ (2(1+\beta B)(2+\beta B)(\Delta_B + 2(-1+s)S(-1+(-1+s)S)) \\ +sS + sS^2 \end{array}}{2(1+\beta B)(2+\beta B)\Delta_B + sS + sS^2}}}{B^2} \tag{131}$$

With sufficiently many buyers $2\beta_{B2} \to 2$ so that $\beta > s$. Also

$$\sigma \equiv B - S + Ss(B+2) - B(2+S) = 2Ss - S - BS - B + BSs. \tag{132}$$

The maximum this can take (at $s = 1$) is

$$\sigma = 2S - S - BS - B + BS = -(B-S) \tag{133}$$

which by non-triviality $B > S + 1$ is *always* negative.■

11 References

Anderson, R.M., Ellison, G., & Fudenberg, D. (2010): "Location choice in two-sided markets with indivisible agents", *Games and Economic Behaviour*, 69, p.2-23.

Armstrong, M. (2006): ”Competition in Two-Sided Markets”, *Rand Journal of Economics*, 37, p.668-691.

Armstrong, M. & Wright, J (2007): ”Two-Sided Markets, Competitive Bottlenecks and Exclusive Contracts”, *Economic Theory*, 2007, 32, 353-380.

Brown, J. & Morgan, J. (2009): "How much is a Dollar Worth? Tipping versus Equilibrium Coexistence on Competing Online Auction Sites", *The Journal of Political Economy*, Vol. 117, No. 4, pp. 668-700.

Caillaud, B. & Jullien, B. (2001): ” Competing Cybermediaries”, *European Economic Review*, 45, 797-808.

Caillaud, B. & Jullien, B. (2003): ”Chicken & Egg: Competition Among Intermediation Service Providers”, *Rand Journal of Economics,* 34(2), 309-328.

Ellison, G., Fudenberg, D., & Möbius, M. (2004): ”Competing Auctions”, *The Journal of the European Economic Association*, 2(1), p.30-66.

Shaked, A. & Sutton, J. (1982): "Relaxing Price Competition Through Product Differentiation" *Review of Economic Studies*, 49(1), 3-13.

Weyl, E.G. (2010): "A Price Theory of Multi-Sided Plattforms", *American Economic Review*, 100, p.1642-1672.

White A, & Weyl, E.G. (2011): "Insulated Platform Competition", mimeo. available at http://www.glenweyl.com/

Yin, P.-L. (2004): ”Online Auction Markets”, *Harvard Business School Case Study*, 9-705-411, August 23, 2004.