Analytical cyclical price-dividend ratios

Fausto Mignanego*

Alessandro Sbuelz†

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Abstract

How non-linear are (log) price-dividend ratios in the fundamental state variables? The accuracy of log-linear approximations lies at the heart of much contemporary asset pricing. We analytically solve a parsimonious partial-equilibrium valuation model and uncover preliminary evidence suggesting that some caution be in order only if the fundamentals are extremely persistent. We also provide a closed-form setup for investigating how non-linearities affect optimal market timing for a non-myopic investor.

KEYWORDS: Price-dividend ratio, equity premium, non-linearity, stochastic discount factor, market price of risk, short rate, persistence, mean-reversion, affine models.

JEL: E32, E43, G11, G12.

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*Department of Mathematics, Quantitative Finance, and Econometrics, Catholic University, Milan, Italy. *E-mail address*: fausto.mignanego@unicatt.it.

†Department of Mathematics, Quantitative Finance, and Econometrics, Catholic University, Milan, Italy. Corresponding author. *E-mail address*: alessandro.sbuelz@unicatt.it. We thank for their comments and suggestions Anna Battauz, Mascia Bedendo, Paola Biffi, Gian Paolo Clemente, John Cochrane, Francesco Corielli, Alessandra Cornaro, Bernard Dumas, Stefano Giglio, Massimo Guidolin, Michele Longo, Alessandra Mainini, Fulvio Ortu, Giovanni Puopolo, Andrea Tamoni and Claudio Tebaldi. The usual disclaimer applies.

1 Introduction

Log-linear approximations of the price-dividend ratios are at the core of recent asset pricing at least since the Campbell and Shiller (1988) present-value model, as they assist in estimating the expected returns and expected dividend growth rates of the aggregate stock market. Log-linear approximations have been often used (see for instance the seminal work of Bansal and Yaron (2004)) to study the importance of long-run risks for asset valuation in a momentous strand of literature.

How non-linear are (log) price-dividend ratios in the fundamental state variables? Linear formulae in asset prices are in e.g. Bhattacharya (1978), Buraschi and Jiltsov (2007), Corradi, Distaso and Mele (2010), Veronesi (2000), Menzly, Santos and Veronesi (2004), Santos and Veronesi (2006). Gabaix (2009a,b) introduces a tractable linearity-generating valuation setup, which yields closed-form stock prices that are linear in factors[1]. Binsbergen and Koijen (2009) and Binsbergen and Koijen (2011) estimate a linearity-generating stock model with time-varying equity premium and dividend growth rate.

In contrast, we work out analytical non-linear log price-dividend ratios under the continuous-time Intertemporal Capital Asset Pricing Model (ICAPM) proposed by Brennan, Wang, and Xia (2004). They assume a streamlined partial-equilibrium valuation model in which the fundamentals are the two ingredients of the pricing kernel dynamics, that is, the riskfree rate and the market price of systematic risk (the two follow possibly correlated Ornstein-Uhlenbeck processes). Brennan, Wang, and Xia (2004) use the affine nature of their setup to analytically recover dividend strip values, which are log-linear in the fundamentals, but fall short of finding stock pricing formulae. We do find exact analytical formulae for the stock prices and, importantly, for their semielasticities with respect to the fundamentals, whose variation in the fundamentals quantifies the non-linearity of log price-dividend ratios. Our formulae improve tractability and testability of the popular Gaussian subclass of the continuous-time affine models identified by Duffie and Kan (1996), which so far endured stock-pricing difficulties while being otherwise broadly applicable.

We employ our closed-form results to provide preliminary evidence that is suggestive of vigilance in the use of log-linear approximations only if the fundamental state variables are particularly

[1] See Farhi and Gabaix (2009) for a linearity-generating equilibrium model of exchange rates.

persistent. Our formulae for the stock price semielasticities fully characterize the endogenous stock return heteroscedasticity that emerges from the homoscedastic fundamentals. The formulae confirm the crucial insight of Mele (2007) that, to induce countercyclical return volatility, risk premia must increase more in bad times than they decrease in good times.

The ICAPM of Brennan, Wang, and Xia (2004) assumes that the investment opportunity set is completely described by the riskfree rate and by the market price of risk (the maximum Sharpe ratio). We contribute by analytically assessing the impact of ICAPM stock price non-linearities on optimal market timing for a non-myopic investor. We show that, if the fundamentals are strongly persistent, long-horizon investors' optimal hedging demands become distinctly non-linear in the fundamentals themselves.

The recent vast literature on present-value models has been enriched by the works of e.g. Ang and Liu (2004), Bekaert, Engstrom, and Grenadier (2001), Bekaert, Engstrom, and Grenadier (2005), Binsbergen and Koijen (2010), Brennan and Xia (2010), Burnside (1998), Cochrane (2007), Favero, Gozluklu and Tamoni (2010), Lettau and Van Nieuwerburgh (2008), Pástor and Veronesi (2003), Pástor and Veronesi (2006), Pástor, Sinha, and Swaminathan (2007). We contribute a user-friendly analytical gauge of the link between fundamentals' persistence and stock price non-linearities and an analytical assessment of optimal dynamic portfolio choice in the presence of such non-linearities.

The paper is organized as follows. Section 2 discusses our assumptions about the pricing kernel and the dividend stream provided by the stock. Within a univariate ICAPM, Section 3 discusses stock prices, their semielasticities, and dynamic wealth allocation to the stock. Section 4 discusses stock prices and their semielasticities in a simple bivariate ICAPM. An appendix collects proofs and technical details.

2 The pricing kernel and the dividend stream

We assume the existence of an exogenous state-price density process $\{\xi\}$ that prices any traded asset in the economy and its dynamics is

$$\frac{d\xi}{\xi} = -rdt - \lambda dW_\xi^{\mathbb{P}},$$

where r is the riskfree rate, λ is the market price of systematic risk, $\mathbb{P}$ denotes the objective probability measure, and $W_\xi^{\mathbb{P}}$ is a Wiener process under $\mathbb{P}$ (its innovations represent the systematic shocks in the economy). As assumed in the ICAPM of Brennan, Wang, and Xia (2004), the market price of risk follows a mean-reverting Gaussian process,

$$d\lambda = -k_\lambda (\lambda - l)\, dt + \sigma_\lambda dW_\lambda^{\mathbb{P}}, \qquad d\left[W_\xi^{\mathbb{P}}, W_\lambda^{\mathbb{P}}\right] = \rho_{\lambda\xi} dt \quad ,$$

where the bracket operator $[\cdot,\cdot]$ denotes the covariation between two diffusive processes. We realistically posit that unexpected increases in λ tend to be accompanied by unexpected increases in ξ, that is, $\rho_{\lambda\xi} < 0$. This implies the systematic nature of λ's risk, which will command a distinct risk premium.

Given a constant volatility parameter $\sigma > 0$, the continuously-paid dividend flow per unit time is the diffusion process $\{D\}$:

$$\frac{dD}{D} = \mathbb{E}_t^{\mathbb{P}}\left(\frac{dD}{D}\right) + \sigma dW_D^{\mathbb{P}}, \qquad d\left[W_D^{\mathbb{P}}, W_\xi^{\mathbb{P}}\right] = \rho_{D\xi} dt, \qquad d\left[W_D^{\mathbb{P}}, W_\lambda^{\mathbb{P}}\right] = \rho_{D\lambda} dt.$$

Its innovations are related at least partially to systematic risk ($\rho_{D\xi} \neq 0$) to support the emergence of a nontrivial dividend-risk compensation.

3 The univariate ICAPM

The riskfree rate is constant. The expected dividend growth rate is also constant:

$$\mathbb{E}_t^{\mathbb{P}}\left(\frac{dD}{D}\right) = \mu dt.$$

The stock price is the expected value of the discounted dividend stream,

$$S(D,\lambda) = \mathbb{E}_t^{\mathbb{P}}\left(\int_0^\infty \frac{\xi_{t+\tau}}{\xi_t} D_{t+\tau} d\tau\right),$$

and solves the equilibrium pricing equation

$$\mathbb{E}_t^{\mathbb{P}}\left(\frac{dS}{S}\right) + \frac{D}{S} dt = r dt - \mathbb{E}_t^{\mathbb{P}}\left(\frac{dS}{S}\frac{d\xi}{\xi}\right)$$

(see for instance Equations 1.28 and 1.35 in Cochrane (2005)) with the absorption-at-zero boundary condition

$$S(0,\lambda) \;=\; 0.$$

Proposition 1 *If λ's risk-neutral speed of mean reversion is positive,*

$$h = k_\lambda + \sigma_\lambda \rho_{\lambda\xi} > 0,$$

and if the risk-free rate dominates the adjusted expected growth rate of dividends[2],

$$x = r - \mu + u\frac{y}{h} - 2hq > 0,$$

then the stock price $S(D,\lambda)$ can be represented as a uniformly convergent series for any bounded interval of λ:

$$S(D,\lambda) = \frac{D}{2h} e^{-(w-q)-\frac{y}{h}\lambda} q^{-\frac{x}{2h}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{\sqrt{q}}\left(w + \frac{y}{h}\lambda\right)\right)^n}{n!} \Gamma\left(q; \frac{n}{2} + \frac{x}{2h}\right) \quad ,$$

where

$$w = -u\frac{y}{h^2} + 4q,$$

$$q = \frac{1}{4h^3}\sigma_\lambda^2 y^2 \qquad (\text{signum}(q) = \text{signum}(h)),$$

$$u = k_\lambda l + \sigma \sigma_\lambda \rho_{D\lambda},$$

$$y = \sigma \rho_{D\xi}.$$

$\Gamma(q;s)$ *is the incomplete Gamma function (which is en suite in any standard software):*

$$\Gamma(q;s) = \int_0^q e^{-v} v^{s-1} dv,$$

$$\Gamma(s) = \int_0^\infty e^{-v} v^{s-1} dv.$$

[2] This is the ICAPM equivalent of the requirement in the Gordon stock pricing model that the discount rate top the growth rate of dividends.

Proof. See the Appendix. ■

The above pricing formula can be graphically shown to have intuitive limiting results. If persistence vanishes (k_λ grows large), the stock price $S(D,\lambda)$ approaches $\frac{D}{r+\sigma l\rho_{D\xi}-\mu}$, that is the Gordon growth formula with risk adjustment and constant market price of risk l (the condition $x>0$ transmutes into $r+\sigma l\rho_{D\xi}-\mu>0$). If the product $\sigma\rho_{D\xi}$ vanishes, $S(D,\lambda)$ approaches $\frac{D}{r-\mu}$, that is the Gordon growth formula without risk adjustment (the condition $x>0$ transmutes into $r-\mu>0$).

Even if the fundamentals are homoscedastic, we get an endogenous stochastic volatility for the stock price as it can be seen by inspecting the total instantaneous return on the stock:

$$\frac{dS}{S}+\frac{D}{S}dt=\left(r+\sigma\lambda\rho_{D\xi}+\frac{S_\lambda}{S}\sigma_\lambda\lambda\rho_{\lambda\xi}\right)dt+\sigma dW_D^{\mathbb{P}}+\frac{S_\lambda}{S}\sigma_\lambda dW_\lambda^{\mathbb{P}}\ .$$

Importantly, such a stochastic volatility is related to the stock price semielasticity $\frac{S_\lambda}{S}$ and can be also characterized in closed form, as the following corollary shows.

Corollary 2 *Given the assumptions in Proposition 1, we have*

$$S_\lambda(D,\lambda)\ =\ \frac{D}{2h}e^{-(w-q)}q^{-\frac{x}{2h}}\sum_{n=0}^{\infty}\frac{\partial}{\partial\lambda}\left[e^{-\frac{y}{h}\lambda}\frac{\left(\frac{1}{\sqrt{q}}\left(w+\frac{y}{h}\lambda\right)\right)^n}{n!}\right]\Gamma\left(q;\frac{n}{2}+\frac{x}{2h}\right)\ ,$$

which is a uniformly convergent series for any bounded interval of λ.

Proof. See the Appendix. ■

The parameter values in Table 1 are selected to graphically calibrate the model (see Figures 1A, 1B, and 1C) around the key sample moments reported by Zhou and Zhu (2009), which are based on US monthly data from February, 1947 to March, 2007 (see their Table II, p. 36). The average log price-dividend ratio is 3.31%, the market risk premium is 6.2% (log returns are used). The average log riskfree rate is 0.99%. The average log-dividend increase is 1.02% and its volatility is 10.69%. We fix the unconditional mean l at 0.7 and, in the case with the strongest persistence ($k_\lambda=0.075$), the market price of risk λ lies with 95% probability between

$$l-1.96\sqrt{\frac{\sigma_\lambda^2}{2k_\lambda}}=0.194\qquad\text{and}\qquad l+1.96\sqrt{\frac{\sigma_\lambda^2}{2k_\lambda}}=1.206\ ,$$

with $\frac{\sigma_\lambda}{\sqrt{2k_\lambda}} = 25.82\%$. The calibration is consistent with the conditions $h > 0$ and $x > 0$.

Table 1

Dividend growth		Price of risk			Riskfree rate	Correlations		
μ	σ	k_λ	l	σ_λ	r	$\rho_{D\lambda}$	$\rho_{D\xi}$	$\rho_{\lambda\xi}$
2%	12%	0.150 0.075	0.7	10%	1%	-0.5	0.5	-0.5

Figures 1- capitalize on the analytical formulae just obtained to graphically explore how the stock's log price-dividend ratio, risk premium, and exposure to λ's risk (the absolute value of the semielasticity $\frac{S_\lambda}{S}$) implied by the ICAPM vary with the state variable λ. Figure 1 patently shows that λ's persistence exacerbates the exposure to λ's shocks as well as the convexity of the log price-dividend ratio in λ (the equity exposure to λ's risk becomes sizeable and markedly decreasing in λ if the speed of mean reversion k_λ is slow). Given a realistic sign for the correlation between innovations in D and λ ($\rho_{D\lambda} \leq 0$), equity returns' volatility will be also decreasing in λ. This confirms the main finding of Mele (2007). Countercyclical risk premia do not imply countercyclical return volatility. Countercyclical equity volatility only occurs if equity exposures to λ's risk increase more in bad times (when λ is high) than they decrease in good times (when λ is low).

3.1 Market timing

We pursue a closed-form dynamic portfolio analysis along the lines set by Kim and Omberg (1996) and generalized by Liu (2007). The investor allocates her wealth W to two assets, the risk-free asset and the stock. There is no consumption or income during the investment horizon. The investor has Constant-Relative-Risk-Aversion (CRRA) utility from terminal wealth,

$$U(z) = \begin{cases} z^{1-\psi}/(1-\psi) & \text{for} \quad z > 0 \quad , \\ -\infty & \text{for} \quad z \leq 0 \quad , \end{cases}$$

where the level of relative risk aversion equals the parameter $\psi > 0$.

Let us assume that

$$\rho_{D\xi} = 1 \qquad \text{and} \qquad \rho_{\lambda\xi} = \rho_{D\lambda} = -1,$$

which implies that both dividend risk and price-of-risk risk are utterly systematic. Under such an assumption, $\sigma - \frac{S_\lambda}{S}\sigma_\lambda$ represents the local volatility of the stock returns. The current market price of risk λ supplies all currently available information on current and future investment opportunities. Indeed, Nielsen and Vassalou (2006) show that, in typical continuous-time portfolio problems, the only time-variation that matters for portfolio choice is the time-variation in the slope (the market price of risk) and the intercept (the riskfree rate) of the instantaneous capital market line.

Let $\tau > 0$ be the investor's time horizon and Y be the monetary investment in the stock. The optimal portfolio problem is:

$$\max_{\{y_t\}_{t=0}^{\tau}} \mathbb{E}\left(U(W_\tau) \;\mid\; W,\ \lambda \right) \quad \text{s.t.} \quad dW \;=\; Wrdt + Y\left(\frac{dS}{S} + \frac{D}{S}dt - rdt\right) \;. \qquad (1)$$

The following proposition qualifies optimal dynamic portfolios.

Proposition 3 *If $\eta > 0$, the optimal fraction of wealth allocated to the stock is:*

$$\frac{Y^*(W,\lambda,\tau)}{W} \;=\; \frac{1}{\psi}\frac{\lambda}{\sigma - \frac{S_\lambda}{S}\sigma_\lambda} \;-\; \frac{1}{\psi}\left(B(\tau) + C(\tau)\lambda\right)\frac{\sigma_\lambda}{\sigma - \frac{S_\lambda}{S}\sigma_\lambda} \quad ,$$

$$C(\tau) = \frac{2a\left(1-\exp\left(-\sqrt{\eta}\tau\right)\right)}{2\sqrt{\eta}-\left(b+\sqrt{\eta}\right)\left(1-\exp\left(-\sqrt{\eta}\tau\right)\right)} \ ,$$

$$B(\tau) = \frac{4ak_\lambda l\left(1-\exp\left(-\frac{\sqrt{\eta}}{2}\tau\right)\right)^2}{\sqrt{\eta}\left[2\sqrt{\eta}-\left(b+\sqrt{\eta}\right)\left(1-\exp\left(-\sqrt{\eta}\tau\right)\right)\right]}.$$

$$a = \frac{1}{\psi}-1, \quad b = 2\left(\frac{\psi-1}{\psi}\sigma_\lambda - k_\lambda\right), \quad c = \frac{\sigma_\lambda^2}{\psi},$$

$$\eta = b^2 - 4ac.$$

The non-nirvana condition that, for any horizon τ, grants a bounded optimal demand for the stock is

$$\sqrt{\eta} > b.$$

Proof. See the Appendix. ■

The above proposition makes clear how market timing in the optimal myopic demand

$$\frac{1}{\psi}\frac{\lambda}{\sigma - \frac{S_\lambda}{S}\sigma_\lambda}$$

as well as in the optimal hedging demand

$$-\frac{1}{\psi}\left(B(\tau)+C(\tau)\lambda\right)\frac{\sigma_\lambda}{\sigma-\frac{S_\lambda}{S}\sigma_\lambda}$$

is characterized by non-linearities via the endogenous stochastic volatility. The optimal demand for the stock in Kim and Omberg (1996) is linear in λ as, in their setting, stock returns' volatility is utterly exogenous and typically constant. In contrast, Figures 2A and 2B show that long-horizon investors' optimal market timing for hedging purposes ($\psi \neq 1$) has a markedly non-linear nature if λ is strongly persistent. The parameter values but the correlation levels are taken from Table 1 and meet the non-nirvana condition.

4 A simple bivariate ICAPM

The riskfree rate also follows a mean-reverting Gaussian process:

$$dr = -k_r\left(r-\theta\right)dt + \sigma_r dW_r^{\mathbb{P}} \quad \text{with} \quad d\left[W_D^{\mathbb{P}}, W_r^{\mathbb{P}}\right] = \rho_{Dr}dt \quad \text{and} \quad d\left[W_r^{\mathbb{P}}, W_\lambda^{\mathbb{P}}\right] = \rho_{r\lambda}dt \ .$$

We now assume

$$\mathbb{E}_t^{\mathbb{P}}\left(\frac{dD}{D}\right) = (\mu + \phi(r-\theta))\, dt \quad ,$$

which keeps at μ the unconditional per-annum expected dividend growth rate. Such an assumption departs from the bivariate ICAPM presented in Brennan, Wang, and Xia (2004). The novel inclusion of the term $\phi(r-\theta)$ in the conditional expected dividend growth rate is not difficult to be justified within equilibrium asset pricing models of long-run risks and contributes flexibility. In particular, it is meant to enable any desired direction in the link between price-dividend ratios and interest rates.

Proposition 4 *If the innovations in the riskfree rate are not systematic,*

$$d\left[W_\xi^{\mathbb{P}}, W_r^{\mathbb{P}}\right] = 0 \quad ,$$

if the risk-neutral speeds of mean reversion for r and for λ are equal and positive,

$$k_r = h > 0 \quad ,$$

and if the long-run mean of the risk-free rate dominates the corrected unconditional expected growth rate of dividends,

$$x_2 = \theta - \mu + u\frac{y}{h} - 2hq_2 - \sigma\sigma_r\rho_{Dr}\frac{\phi-1}{h} > 0 \quad ,$$

then the stock price $S(D,r,\lambda)$ can be represented as a uniformly convergent series for any bounded set of λ and r:

$$S(D,r,\lambda) = \frac{D}{2h} e^{-(w_2-q_2)-\frac{y}{h}\lambda+\frac{(\phi-1)}{h}r} q^{-\frac{x_2}{2h}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{\sqrt{q_2}}\left(w_2+\frac{y}{h}\lambda-\frac{(\phi-1)}{h}r\right)\right)^n}{n!} \Gamma\left(q_2; \frac{n}{2}+\frac{x_2}{2h}\right) \quad ,$$

where

$$\begin{aligned}
w_2 &= u_2\frac{\phi-1}{h^2} - u\frac{y}{h^2} + 4q_2 \quad , \\
q_2 &= \frac{1}{4}\left(\frac{\phi-1}{h}\right)^2\frac{\sigma_r^2}{h} + \frac{1}{4}\left(\frac{y}{h}\right)^2\frac{\sigma_\lambda^2}{h} - \frac{1}{2}\frac{(\phi-1)\,y}{h^2}\frac{\sigma_r\sigma_\lambda\rho_{r\lambda}}{h} \qquad (\mathrm{signum}\,(q_2) = \mathrm{signum}\,(h)) \quad , \\
u_2 &= h\theta + \sigma\sigma_r\rho_{Dr} \quad .
\end{aligned}$$

Proof. See the Appendix. ■

The total instantaneous return on the stock becomes

$$\frac{dS}{S}+\frac{D}{S}dt=\left(r+\sigma\lambda\rho_{D\xi}+\frac{S_\lambda}{S}\sigma_\lambda\lambda\rho_{\lambda\xi}\right)dt+\sigma dW_D^{\mathbb{P}}+\frac{S_r}{S}\sigma_r dW_r^{\mathbb{P}}+\frac{S_\lambda}{S}\sigma_\lambda dW_\lambda^{\mathbb{P}}$$

and its conditional expected level in excess of r (i.e. the conditional risk premium) is not entered by r's risk, given the assumption that the innovations in the riskfree rate are not systematic. Again, stock price semielasticities can be characterized in closed form, as the following corollary shows.

Corollary 5 *Given the assumptions in Proposition 4, we have*

$$\begin{aligned} S_r(D,r,\lambda) &= \frac{D}{2h}e^{-(w_2-q_2)-\frac{y}{h}\lambda}q^{-\frac{x_2}{2h}}\sum_{n=0}^{\infty}\frac{\partial}{\partial r}\left[e^{\frac{(\phi-1)}{h}r}\frac{\left(\frac{1}{\sqrt{q_2}}\left(w_2+\frac{y}{h}\lambda-\frac{(\phi-1)}{h}r\right)\right)^n}{n!}\right]\Gamma\left(q_2;\frac{n}{2}+\frac{x_2}{2h}\right), \\ S_\lambda(D,r,\lambda) &= \frac{D}{2h}e^{-(w_2-q_2)+\frac{(\phi-1)}{h}r}q^{-\frac{x_2}{2h}}\sum_{n=0}^{\infty}\frac{\partial}{\partial \lambda}\left[e^{-\frac{y}{h}\lambda}\frac{\left(\frac{1}{\sqrt{q_2}}\left(w_2+\frac{y}{h}\lambda-\frac{(\phi-1)}{h}r\right)\right)^n}{n!}\right]\Gamma\left(q_2;\frac{n}{2}+\frac{x_2}{2h}\right), \end{aligned}$$

which are uniformly convergent series for any bounded set of λ and r.

Proof. See the Appendix. ■

Conveniently, our bivariate ICAPM enables new stock pricing formulae as well as familiar bond pricing formulae.

Proposition 6 *Given the assumptions in Proposition 4, if $B(r,\tau)$ denotes the price of a zero-coupon bond maturing in τ years from now, the Vasiceck (1977) term structure of interest rates* *is*

$$
\begin{aligned}
-\frac{\ln B(r,\tau)}{\tau} &= -\frac{\zeta(\tau)-\epsilon(\tau)r}{\tau}, \\
\epsilon(\tau) &= \frac{1}{h}\left(1-e^{-h\tau}\right), \\
\zeta(\tau) &= -\frac{1}{h^2}\epsilon(\tau)\left(\sigma_r^2-h^2\theta\right)+\frac{1}{4h^3}\sigma_r^2\left(1-e^{-2h\tau}\right)+\frac{1}{2h^2}\left(\sigma_r^2-2h^2\theta\right)\tau.
\end{aligned}
$$

Proof. The bivariate ICAPM restricted as in Proposition 4 embeds the Vasiceck (1977) model with non-systematic interest rate risk. ■

Given the moderate level $k_\lambda = 0.15$ for λ's persistence, Figure 3 shows the stock pricing features of the bivariate ICAPM. The coefficient σ_r has been set to 1% in order to match the standard deviation of 4.28% for the log riskfree rate reported by Zhou and Zhu (2009). The correlations with r's innovations have been given sensible size and sign ($\rho_{Dr} = 0.5$ and $\rho_{D\lambda} = -0.5$). The 'leverage' coefficient has been fixed comfortably above 1 ($\phi = 2$) to go with a possible joint procyclicality of price-dividend ratios and interest rates ($S_r > 0$). Figure 3 brings bivariate substantiation that state variables' persistence exacerbates the stock price exposure to their shocks as well as the non-linearity of the log price-dividend ratio in the variables themselves. The second column of Figure 3 reveals the stronger and more non-linear impact on $\ln \frac{S}{D}$ of the riskfree rate r, which is the most persistent variable ($k_r < k_\lambda$).

5 Preliminary conclusions

We provide a closed-form stock pricing analysis of the continuous-time affine ICAPM proposed by Brennan, Wang, and Xia (2004). Preliminary evidence based on our pricing formulae show how linearity in log dividend strip values can anyway yield non-linearities in log price-dividend ratios that are increasing in the persistence of fundamental factors. The formulae also confirm the key findings of Mele (2007) about the link between countercyclical stock volatility and convex countercyclical risk premia. Finally, we empower the analytical study of optimal non-myopic market timing in the presence of non-linear log price-dividend ratios. We show that, if the factors are remarkably persistent, long-horizon investors' optimal hedging demands become visibly non-linear in the factors themselves.

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6 Appendix

6.1 Proof of Proposition 1

The per-unit-time price $P(D,\lambda,\tau)$ of a claim on the dividend flow paid in τ years from now (a dividend strip) is given by

$$P(D,\lambda,\tau) = \mathbb{E}_t^{\mathbb{P}}\left(\frac{\xi_{t+\tau}}{\xi_t} D_{t+\tau}\right)$$

and satisfies the equilibrium pricing equation

$$\mathbb{E}_t^{\mathbb{P}}(dP) = Prdt - \mathbb{E}_t^{\mathbb{P}}\left(dP\frac{d\xi}{\xi}\right), \quad \text{with } P(D,\lambda,0) = D \text{ and } P(0,\lambda,\tau) = 0,$$

that is,

$$\begin{aligned}
& -P_\tau + P_D D\mu + P_\lambda(-k_\lambda(\lambda - l)) + \\
\frac{1}{2}P_{DD}D^2\sigma^2 + P_{D\lambda}D\sigma\sigma_\lambda\rho_{D\lambda} + \frac{1}{2}P_{\lambda\lambda}\sigma_\lambda^2 \quad = \quad & Pr - \\
& P_D D\sigma(-\lambda)\rho_{D\xi} - P_\lambda\sigma_\lambda(-\lambda)\rho_{\lambda\xi},
\end{aligned}$$

$$P(D,\lambda,0) = D \quad \text{and} \quad P(0,\lambda,\tau) = 0.$$

As shown in Brennan, Wang, and Xia (2004) (Theorem 1, p. 1748), we have

$$P(D,\lambda,\tau) = D\exp\left(w\left(e^{-h\tau}-1\right) - x\tau - q\left(e^{-2h\tau}-1\right) + \frac{y}{h}\left(e^{-h\tau}-1\right)\lambda\right).$$

In the stock price $S(D,\lambda)$, we invoke the Fubini-Tonelli Theorem to exchange the expectation with the time integration and yield

$$S(D,\lambda) = \mathbb{E}_t^{\mathbb{P}}\left(\int_0^\infty \frac{\xi_{t+\tau}}{\xi_t} D_{t+\tau}d\tau\right) = \int_0^\infty \mathbb{E}_t^{\mathbb{P}}\left(\frac{\xi_{t+\tau}}{\xi_t}D_{t+\tau}\right)d\tau = \int_0^\infty P(D,\lambda,\tau)\,d\tau.$$

$S(D,\lambda)$ has the form

$$D\int_0^\infty e^{we^{-h\tau}-x\tau-(w-q)-qe^{-2h\tau}+\frac{y}{h}\left(e^{-h\tau}-1\right)\lambda}d\tau \quad .$$

Notice that the above time integral remains finite[3] under the conditions $h > 0$ and $x > 0$.

[3] The condition $x > 0$ concurs with Condition B in Brennan and Xia (2001), Theorem 3, p. 260. They consider a general equilibrium model with Ornstein-Uhlenbeck-type fundamentals and learning. They fall short of finding closed forms for the equilibrium stock prices and their semielasticities.

Given $h > 0$ and $\text{signum}(q) = \text{signum}(h)$, the substitution

$$v = qe^{-2h\tau}$$

yields

$$\begin{aligned} S(D,\lambda) &= De^{-(w-q)-\frac{y}{h}\lambda}\int_q^0 e^{-x\left(-\frac{1}{2h}\ln\left(\frac{1}{q}v\right)\right)+\frac{1}{\sqrt{q}}\left(w+\frac{y}{h}\lambda\right)\sqrt{v}-v}\left(-\frac{1}{2hv}\right)dv \\ &= \frac{D}{2h}e^{-(w-q)-\frac{y}{h}\lambda}q^{-\frac{x}{2h}}\int_0^q e^{\frac{1}{\sqrt{q}}\left(w+\frac{y}{h}\lambda\right)\sqrt{v}}e^{-v}v^{\frac{x}{2h}-1}dv. \end{aligned}$$

The power series expansion

$$e^{\frac{1}{\sqrt{q}}\left(w+\frac{y}{h}\lambda\right)\sqrt{v}} = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{\sqrt{q}}\left(w+\frac{y}{h}\lambda\right)\right)^n}{n!}v^{\frac{n}{2}}$$

enjoys pointwise convergence for any $v \geq 0$. This implies the uniform convergence of the series in every closed interval belonging to $[0,+\infty)$. As the closed interval $[0,q]$ is one of them, we can exchange the series and integral operators to get

$$S(D,\lambda) = \frac{D}{2h}e^{-(w-q)}q^{-\frac{x}{2h}}e^{-\frac{y}{h}\lambda}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{\sqrt{q}}\left(w+\frac{y}{h}\lambda\right)\right)^n}{n!}\int_0^q e^{-v}v^{\frac{n}{2}+\frac{x}{2h}-1}dv \quad .$$

Hence, we have

$$S(D,\lambda) = \frac{D}{2h}e^{-(w-q)}q^{-\frac{x}{2h}}e^{-\frac{y}{h}\lambda}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{\sqrt{q}}\left(w+\frac{y}{h}\lambda\right)\right)^n}{n!}\Gamma\left(q;\frac{n}{2}+\frac{x}{2h}\right) \quad ,$$

where the assumption of a positive x grants the existence of the integral

$$\int_0^q e^{-v}v^{\frac{n}{2}+\frac{x}{2h}-1}dv$$

for any n and its correspondence to the incomplete gamma function $\Gamma\left(q;\frac{n}{2}+\frac{x}{2h}\right)$. We now employ the Weierstrass M-test to show that the above series expansion of $S(D,\lambda)$ is uniformly convergent in any bounded interval for λ. A series of functions $\sum_{n=0}^{\infty} g_n(\lambda)$ converges uniformly in a certain domain if the series $\sum_{n=0}^{\infty} M_n$ converges, where

$$|g_n(\lambda)| \leq M_n$$

for any n and for all λ in that domain. Take a positive, arbitrarily large scalar R and recall that $\Gamma(q;\epsilon)$ is strictly positive for $q>0$ and $\epsilon>0$. Given

$$g_n(\lambda) \;=\; \frac{\left(\frac{1}{\sqrt{q}}\left(w+\frac{y}{h}\lambda\right)\right)^n}{n!}\Gamma\left(q;\frac{n}{2}+\frac{x}{2h}\right)\ , \tag{2}$$

we have, for any λ such that $\left|w+\frac{y}{h}\lambda\right|\leq R$,

$$\begin{aligned} |g_n(\lambda)| &\leq \frac{R^n}{n!\sqrt{q}}\Gamma\left(q;\frac{n}{2}+\frac{x}{2h}\right) \\ &\leq \frac{R^n}{n!\sqrt{q}}\frac{q^{\frac{n}{2}+\frac{x}{2h}}}{\frac{n}{2}+\frac{x}{2h}} \\ &= M_n\ , \end{aligned}$$

since

$$\int_0^q e^{-v}v^{\frac{n}{2}+\frac{x}{2h}-1}dv \;\leq\; \int_0^q v^{\frac{n}{2}+\frac{x}{2h}-1}dv=\frac{q^{\frac{n}{2}+\frac{x}{2h}}}{\frac{n}{2}+\frac{x}{2h}}\quad .$$

The ratio test substantiates the convergence of the series $\sum_{n=0}^{\infty} M_n$. Indeed,

$$\frac{M_{n+1}}{M_n} \;=\; \frac{R}{n+1}q^{\frac{1}{2}}\frac{\frac{n}{2}+\frac{x}{2h}}{\frac{n+1}{2}+\frac{x}{2h}}$$

approaches zero as $n\longrightarrow+\infty$. It follows that the series expression for $S(D,\lambda)$ uniformly converges for any λ in the arbitrarily large region $\left|w+\frac{y}{h}\lambda\right|\leq R$. This completes the proof.$\blacksquare$

6.2 Proof of Corollary 2

The partial derivative of $S(D,\lambda)$ with respect to λ is

$$S_{\lambda}=\frac{D}{2h}e^{-(w-q)}q^{-\frac{x}{2h}}\left\{\frac{\partial}{\partial\lambda}\left[e^{-\frac{y}{h}\lambda}\right]\sum_{n=0}^{\infty}g_n(\lambda)+e^{-\frac{y}{h}\lambda}\frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{\infty}g_n(\lambda)\right]\right\}\ ,$$

where the functions $g_n(\lambda)$ are defined in (2). In the proof of Proposition 1 we have just shown that the power series $\sum_{n=0}^{\infty}g_n(\lambda)$ is uniformly convergent for any λ in the arbitrarily large region $\left|w+\frac{y}{h}\lambda\right|\leq R$. Hence, it can be there differentiated term by term and the series of derivatives uniformly converges in the same domain. It follows that, in the same domain, the series

$$\frac{D}{2h}e^{-(w-q)}q^{-\frac{x}{2h}}\sum_{n=0}^{\infty}\frac{\partial}{\partial\lambda}\left[e^{-\frac{y}{h}\lambda}g_n(\lambda)\right]$$

is uniformly convergent and coincides with S_{λ}. This completes the proof.■

6.3 Proof of Proposition 3

We closely follow the steps in Kim and Omberg (1996). Let $J(W,x,\tau)$ be the optimal expected utility for the portfolio problem (1). Given λ's dynamics, the Hamilton-Jacobi-Bellman conditions result in the optimal investment function

$$Y^{*}(W,\lambda,\tau)=\left(\frac{J_W}{-J_{WW}}\right)\frac{\lambda}{\sigma-\frac{S_{\lambda}}{S}\sigma_{\lambda}}+\left(\frac{J_{W\lambda}}{-J_{WW}}\right)\frac{\sigma_{\lambda}(-1)}{\sigma-\frac{S_{\lambda}}{S}\sigma_{\lambda}}\ , \tag{3}$$

the partial differential equation

$$-J_{\tau}+J_W Wr-\frac{1}{2}J_{WW}(Y^{*})^2\left(\sigma-\frac{S_{\lambda}}{S}\sigma_{\lambda}\right)^2+J_{\lambda}(-k_{\lambda}(\lambda-l))+\frac{1}{2}J_{\lambda\lambda}\sigma_{\lambda}^2=0\ , \tag{4}$$

the terminal condition,

$$J(W,\lambda,0)=U(W)\,,$$

and the second-order condition for a maximum $J_{WW}<0$. The optimal investment function for the logarithmic case ($\phi=1$) is the myopic investment function, $\frac{\lambda}{\sigma-\frac{S_{\lambda}}{S}\sigma_{\lambda}}$. For $\phi\neq1$, assume trial solutions of the form

$$J(W,\lambda,\tau)=U(W)\exp\left(A(\tau)+B(\tau)\lambda+\frac{1}{2}C(\tau)\lambda^2\right)\ ,$$

$$A(0)=B(0)=C(0)=0\ .$$

Solutions of this form automatically satisfy both the terminal condition and the second-order condition for a maximum.

Substitution of the trial solutions into equation (3) yields the investment function

$$Y^*(W,\lambda,\tau) = \frac{W}{\phi}\frac{\lambda}{\sigma - \frac{S_\lambda}{S}\sigma_\lambda} + \frac{W}{\phi}\left(B(\tau)+C(\tau)\lambda\right)\left(\frac{\sigma_\lambda(-1)}{\sigma - \frac{S_\lambda}{S}\sigma_\lambda}\right).$$

Further substitution of the trial solutions into equation (4) produces a quadratic equation for λ. The three coefficients of such an equation must be zero, yielding the following system of first-order nonlinear ordinary differential equations:

$$\begin{aligned} C' &= C^2c + Cb + a, \\ B' &= BCc + B\frac{b}{2} + Ck_\lambda l, \\ A' &= B^2\frac{c}{2} + C\frac{\sigma_\lambda^2}{2} + Bk_\lambda l + (1-\phi)r, \end{aligned}$$

$$A(0) = B(0) = C(0) = 0.$$

If the quantity

$$\eta = b^2 - 4ac = \left(2\left(\frac{\phi-1}{\phi}\sigma_\lambda - k_\lambda\right)\right)^2 - 4\left(\frac{1}{\phi}-1\right)\frac{\sigma_\lambda^2}{\phi}$$

is positive, then the solutions have this form:

$$C(\tau) = \frac{2a\left(1-\exp\left(-\sqrt{\eta}\tau\right)\right)}{2\sqrt{\eta}-\left(b+\sqrt{\eta}\right)\left(1-\exp\left(-\sqrt{\eta}\tau\right)\right)},$$

$$B(\tau) = \frac{4ak_\lambda l\left(1-\exp\left(-\frac{\sqrt{\eta}}{2}\tau\right)\right)^2}{\sqrt{\eta}\left[2\sqrt{\eta}-\left(b+\sqrt{\eta}\right)\left(1-\exp\left(-\sqrt{\eta}\tau\right)\right)\right]}.$$

The non-nirvana condition

$$\sqrt{\eta} > b$$

prevents zeros in the denominators of $C(\tau)$ and $B(\tau)$ for any non-negative horizon τ. This completes the proof.■

7 Proof of Proposition 4 and Corollary 5

The per-unit-time price $P(D, r, \lambda, \tau)$ of a dividend strip maturing in τ years from now is given by

$$P(D, r, \lambda, \tau) = \mathbb{E}_t^{\mathbb{P}} \left(\frac{\xi_{t+\tau}}{\xi_t} D_{t+\tau} \right)$$

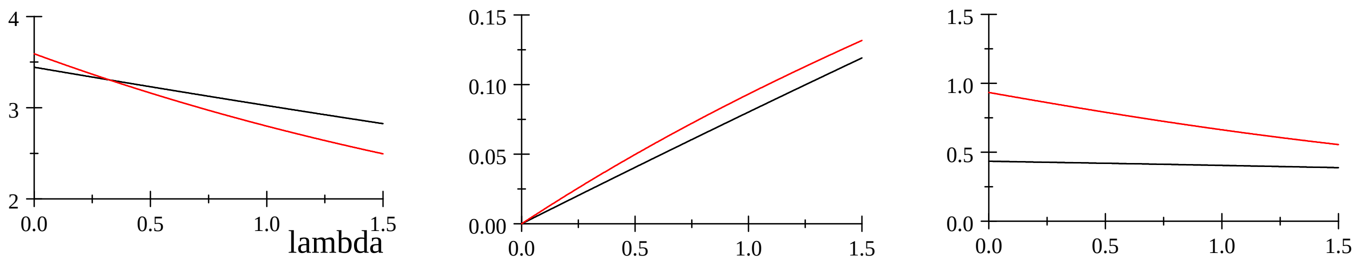
and satisfies the equation

$$\mathbb{E}_t^{\mathbb{P}}(dP) = Prdt - \mathbb{E}_t^{\mathbb{P}} \left(\frac{d\xi}{\xi} dP \right), \qquad P(D, r, \lambda, 0) = D \quad \text{and} \quad P(0, r, \lambda, \tau) = 0.$$

The rest of the proofs closely follows the steps developed in proving Proposition 1 and Corollary 2 (the interval $\left|w + \frac{y}{h}\lambda\right| \leq R$ is replaced by the region $\left\|w_2 + \frac{y}{h}\lambda - \frac{(\phi-1)}{h}r\right\| \leq R$).

Figure 1A. Stock pricing in the univariate ICAPM

The parameter k_λ is 0.15 for the black line and 0.075 for the red line.



$\ln \frac{S(D,\lambda)}{D}$ (log PD ratio) $\sigma\lambda\rho_{D\xi} + \frac{S_\lambda}{S}\sigma_\lambda\lambda\rho_{\lambda\xi}$ (risk premium) $\left|\frac{S_\lambda}{S}\right|$ (exposure to λ's shocks)

Figure 1B. Stock pricing in the univariate ICAPM

The long-run volatility $\dfrac{\sigma_\lambda}{\sqrt{2k_\lambda}}$ is fixed at 18.26%.

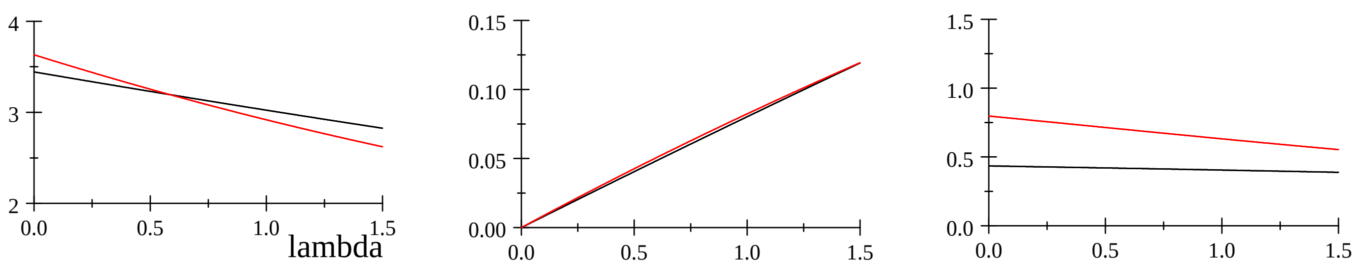


Figure 1C. Stock pricing in the univariate ICAPM

The long-run volatility $\dfrac{\sigma_\lambda}{\sqrt{2k_\lambda}}$ is fixed at 25.82%.

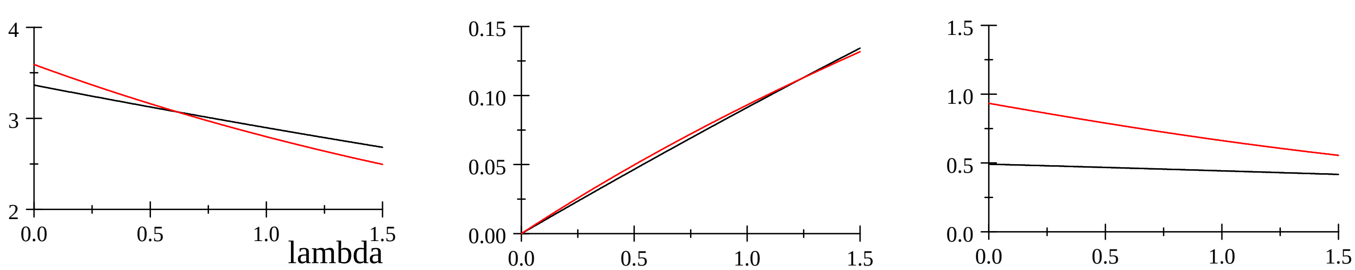
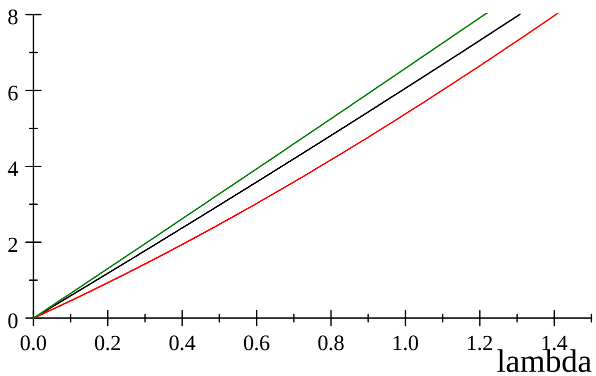


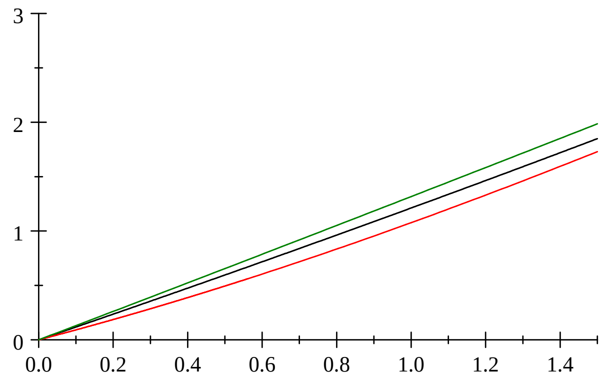
Figure 2A. Optimal stock allocation in the univariate ICAPM

The y-axis represents the optimal fraction of wealth invested in the stock.

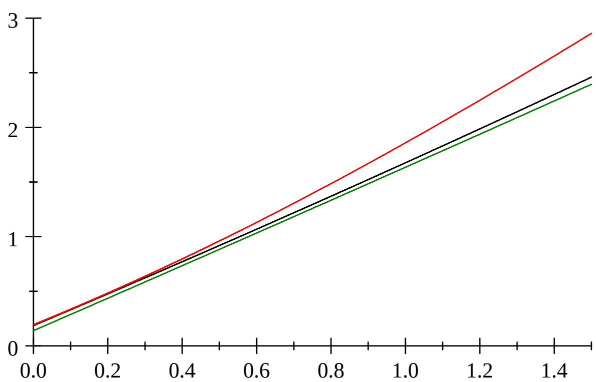
$k_\lambda = 0.125$ (red), $k_\lambda = 0.25$ (black), $k_\lambda = 0.375$ (green).



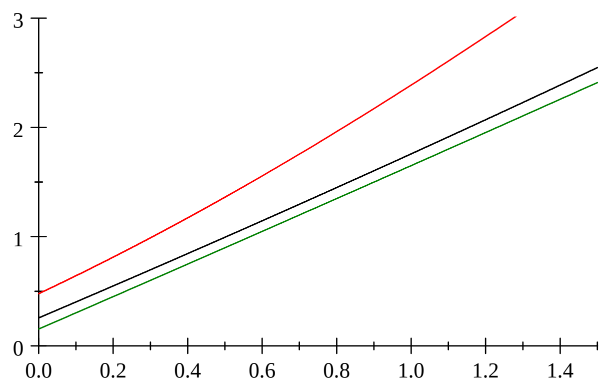
Relative risk aversion $\psi = 1$ (log utility)



$\psi = 5$, investment horizon $\tau = 0$



$\psi = 5$, investment horizon $\tau = 10$



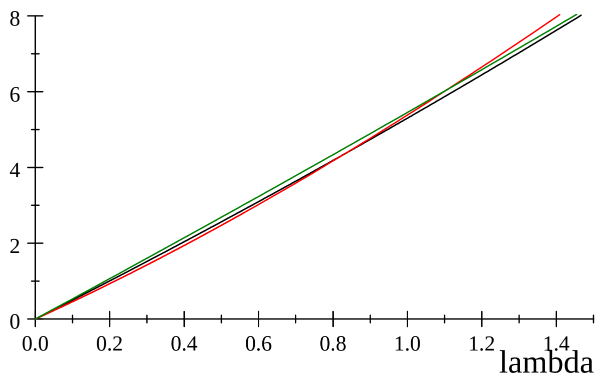
$\psi = 5$, investment horizon $\tau = 20$

Figure 2B. Optimal stock allocation in the univariate ICAPM

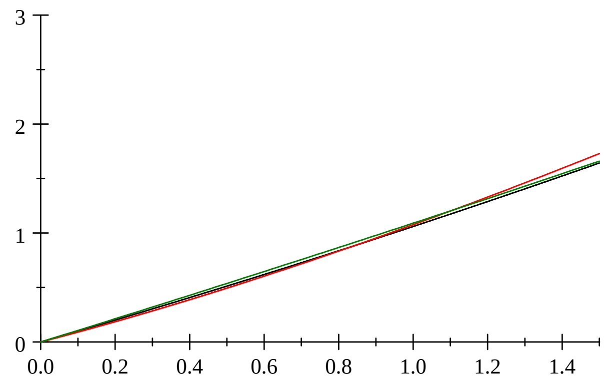
The y-axis represents the optimal fraction of wealth invested in the stock.

$k_\lambda = 0.125$ (red), $k_\lambda = 0.25$ (black), $k_\lambda = 0.375$ (green).

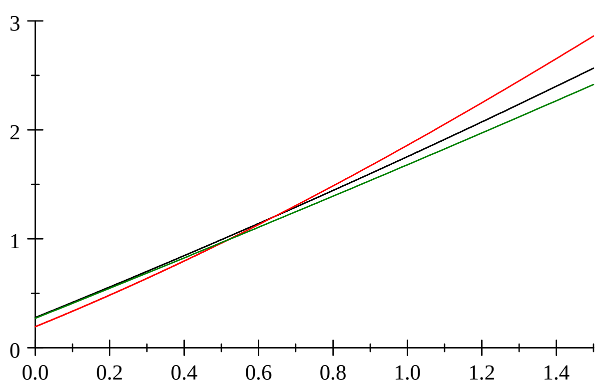
The long-run volatility $\dfrac{\sigma_\lambda}{\sqrt{2k_\lambda}}$ is fixed at 20%.



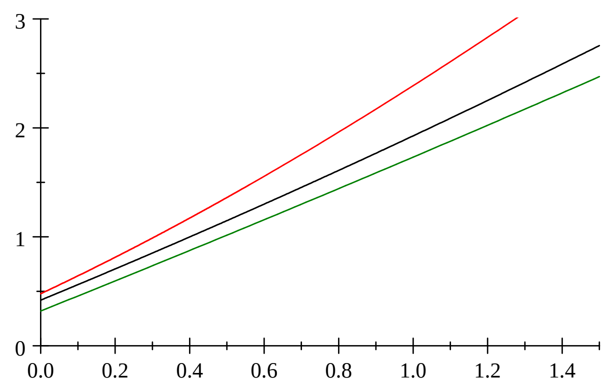
Relative risk aversion $\psi = 1$ (log utility)



$\psi = 5$, investment horizon $\tau = 0$



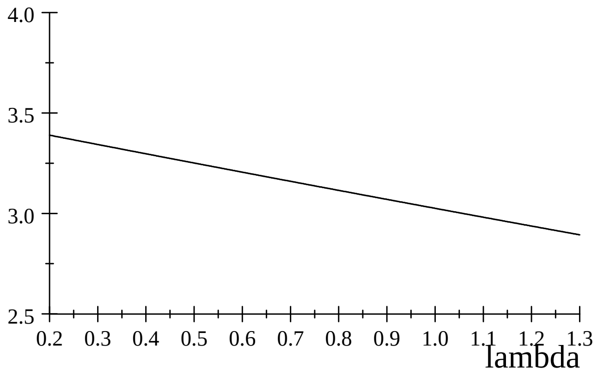
$\psi = 5$, investment horizon $\tau = 10$



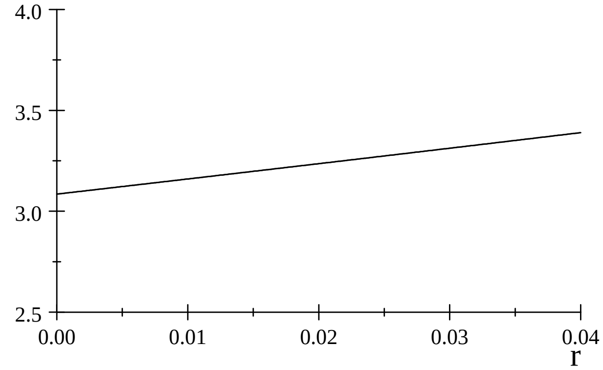
$\psi = 5$, investment horizon $\tau = 20$

Figure 3. Stock pricing in the bivariate ICAPM

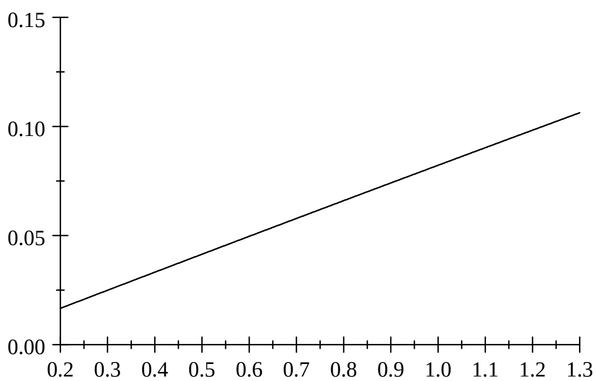
In the first column, r equals $\theta = 1\%$. In the second column, λ equals $l = 0.7$.



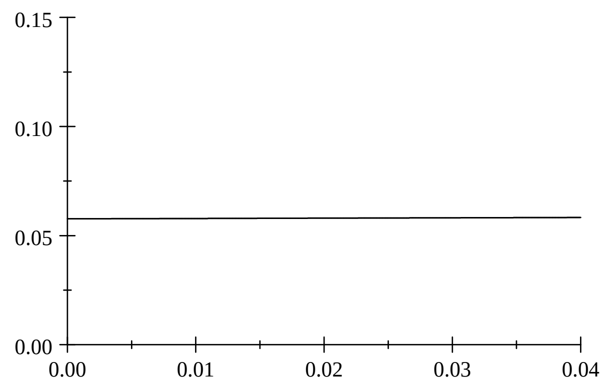
$\ln\left(\frac{S(D,r,\lambda)}{D}\right)$ (log price-dividend ratio)



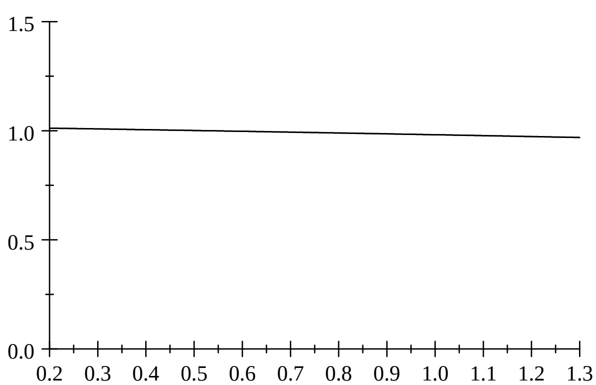
$\ln\left(\frac{S(D,r,\lambda)}{D}\right)$ (log price-dividend ratio)



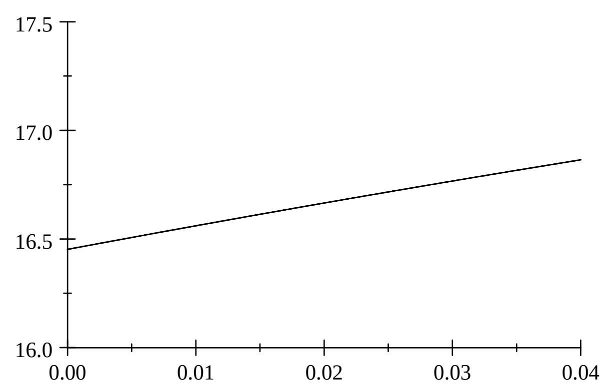
$\sigma\lambda\rho_{D\xi} + \frac{S_\lambda}{S}\sigma_\lambda\lambda\rho_{\lambda\xi}$ (risk premium)



$\sigma\lambda\rho_{D\xi} + \frac{S_\lambda}{S}\sigma_\lambda\lambda\rho_{\lambda\xi}$ (risk premium)



$\left|\frac{S_\lambda}{S}\right|$ (exposure to λ's shocks)



$\left|\frac{S_r}{S}\right|$ (exposure to r's shocks)