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Ordinal utility and the market prices of semi-moments

ABSTRACT. We define an ordinal utility in mean, downside volatility and upside volatility and show that it generalizes the classical mean-variance portfolio model, in which it collapses in case of symmetric distributions. This approach can provide a generalized Capital Asset Pricing Model and it can be used to estimate the market prices of upside and downside risks using market data on option contracts. The model is applied over the recent stock market turbulence, showing the contributions of price and quantity effects of risks on market valuations.

KEYWORDS: non-expected utility, asset pricing, downside risk, mean-variance, option prices

JEL: G11, G12, G13
MSC: 91B16, 91B25, 91B30, 91G10, 91G20

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1. INTRODUCTION

Mean-variance is certainly one of the fundamental approaches to decision under uncertainty, with important applications in portfolio choice and equilibrium asset pricing.
It is not difficult to show (Cesari and D'Adda, 2010) that mean-variance is a special case of a general non-expected, ordinal utility approach in which no independence axiom is used and the main results of asset pricing theory can be derived and straightforwardly generalized to higher moments.
The line of reasoning is as follows: starting from preferences over probability distributions, replace the latter with vectors of moments and show the existence of an ordinal utility of moments. This utility function is used in the classical (microeconomic) constrained optimization problem and the Capital Asset Pricing Model (CAPM, Sharpe, 1964) is easily obtained, in the usual mean-variance form as well as in more general forms.
In this paper we try to make a step farther, simplifying and deepening the model in the attempt to identify the fundamental sources of asset value. The idea is to define an ordinal utility in three (semi) moments: mean, downside risk and upside risk, obtaining, as a consequence, the market prices of the three basic moments entering the fundamental valuation equation (section 2).
The analysis is worked out in the two-asset case (section 3) and implemented making use of the index option contracts quoted at the Chicago Board Option Exchange (section 4). A final section concludes.

2. AN ORDINAL UTILITY OF SEMI-MOMENTS

Let $(\Omega,\Im,\wp)$ be a standard probability space, Ω being the set of elementary events (states of the world), $\Im$ the set (σ-algebra) of subsets of Ω (events), $\wp$ a (subjective) probability measure of the events. Given the set A of all possible actions or decisions, all couples (ω,a) with $\omega\in\Omega$ and $a\in A$, are mapped onto a real vector of monetary consequences $c\in R^n$, the Euclidean space of n-dimensional real vectors, so that $X(\omega,a)=c$ or $X_a(\omega)=c$ is a random vector and F_a is its multivariate probability distribution function. Clearly, the preferences over actions in A are, equivalently, preferences over the set of random variables X_a as well as preferences over the set of distribution functions. Let us confine ourselves, for ease of exposition, to the case of univariate distributions (n=1) and assume that the essential information concerning any distribution F is contained in the three-dimensional vector of (semi-)moments $\mathbf{M}\equiv(M,\Sigma^+,\Sigma^-)$ where M is the mean, Σ^+ is the upside standard deviation (upside vol) and Σ^- is the downside vol (see below for a formal definition). Note that semi-moments were already considered in Markowitz (1959) seminal work and Hicks (1962).
Under the usual assumption of preferences (Cesari e D'Adda, 2010) it can be shown that there exists a continuous, ordinal utility of moments, $H(\mathbf{M})$, representing the given preferences (see also Fishburn, 1970, pp. 27-29 and Debreu, 1964).
The fact that no independence axiom is required (as, instead, in the expected utility approach of Von Neumann and Morgenstern, 1944) implies that the model is free form a number of behavioral 'paradoxes' spawned by the empirical violation of independence (e.g. Machina, 1987 and Kahneman and Tversky, 1979).

3. MARKET EQUILIBRIUM IN THE TWO-ASSET CASE

Let us consider a two-asset world, with a risk-free bond (the endogenous default-free, zero-coupon bond) and a risky asset (the stock or market portfolio), with no dividends for simplicity. Following Sharpe (1964), endogeneity means that the risk-free bond has zero net supply in the aggregate of individual investors. Assume also an ordinal utility function in mean, upside vol and downside vol: $H(M,\Sigma^+,\Sigma^-)$.
These moments enter the utility function of the (representative) investor because assets are not interesting per se but for the mean and up/down side tilts they can induce in the investor's future, unknown wealth. In analogy with Lancaster (1966) theory of consumption, moments are essential characteristics of assets,

(some positive, some negative) and a financial security is a bundle of good and bad characteristics just like a production process generating both a desired product and pollution.
In particular, we assume non satiation and risk aversion: non satiation is defined as $H(M+\delta,\Sigma^+,\Sigma^-) > H(M,\Sigma^+,\Sigma^-)$ for any $\delta>0$; risk aversion means $H(M,0,0) > H(M, ,\Sigma^+,\Sigma^-)$ or, equivalently, M greater than the certainty equivalent amount, C, implicitly defined by the relation $H(C,0,0)\equiv H(M, ,\Sigma^+,\Sigma^-)$.
If W is current wealth, the representative investor's portfolio problem consists in choosing the optimal quantities x_0 for the risk free asset and x_m for the market portfolio (i.e. the optimal portfolio of the two assets) maximizing the ordinal utility function H in mean M, downside vol, Σ^- and upside vol, Σ^+ of future wealth under the budget constraint that current wealth W equals the amount spent for the two assets, with prices $P_0 = \frac{1}{1+r}$ and P_m respectively, and semi-vols defined as:

$$\Sigma^- \equiv \sqrt{E\left[\left(\tilde{P} - M\right)^2 I_{P<M}\right]}$$
$$\Sigma^+ \equiv \sqrt{E\left[\left(\tilde{P} - M\right)^2 I_{P>M}\right]}$$
$$\Sigma \equiv \sqrt{E\left[\left(\tilde{P} - M\right)^2\right]} = \sqrt{\left(\Sigma^-\right)^2 + \left(\Sigma^+\right)^2} \tag{3.1}$$

where $\tilde{P}$ is the unknown, future portfolio value and $I_{P<M}$ is the indicator function of the event $\{\tilde{P} < M\}$. Note that under the (implausible) case of symmetry around the mean $\Sigma^+=\Sigma^-$.

By assumption:

$$\frac{\partial H}{\partial M} > 0, \quad \frac{\partial H}{\partial \Sigma^-} < 0, \quad \frac{\partial H}{\partial \Sigma^+} > 0 \tag{3.2}$$

which means, respectively, nonsatiation, downside risk aversion and upside risk propension.
The optimization problem is:

$$\begin{cases} \max\limits_{x_0,x_m} H(M,\Sigma^-,\Sigma^+) \\ \quad M \equiv x_0 M_0 + x_m M_m \\ \quad \Sigma^- \equiv x_m \Sigma_m^- \\ \quad \Sigma^+ \equiv x_m \Sigma_m^+ \\ W = x_0 P_0 + x_m P_m \\ W = M P_\mu + \Sigma^- P_{\sigma-} + \Sigma^+ P_{\sigma+} \end{cases} \tag{3.3}$$

Note that the budget constraint is written in two, equivalent and alternative forms: in the usual way (quantities of assets times prices of assets) and in terms of moments: quantities of moments M, Σ-,Σ+ times prices of moments, P_μ , $P_{\sigma-}$,$P_{\sigma+}$.
This budget expression is justified by the fact that any argument of the utility function has a "price" with a market meaning in the aggregate equilibrium. In our case it will be shown that semi-moments correspond to elementary assets actually priced in the financial market and that moment prices are essentially observable.
Using the chain-rule of derivatives, the first-order conditions (FOCs) can be written as:

$$\frac{\partial H}{\partial M} M_0 - \xi P_0 = 0$$
$$\frac{\partial H}{\partial M} M_m + \frac{\partial H}{\partial \Sigma^-} \Sigma_m^- + \frac{\partial H}{\partial \Sigma^+} \Sigma_m^+ - \xi P_m = 0 \qquad (3.4)$$

so that, assuming M_0=1 (face value of the zero-coupon bond), and substituting for the multiplier ξ, we have:

$$P_m = M_m P_0 + \left(\frac{\partial H / \partial \Sigma^-}{\partial H / \partial M} P_0 \right) \Sigma_m^- + \left(\frac{\partial H / \partial \Sigma^+}{\partial H / \partial M} P_0 \right) \Sigma_m^+ \qquad (3.5)$$

Using the budget constraint in terms of moments, the FOCs become:

$$\frac{\partial H}{\partial M} - \gamma P_\mu = 0$$
$$\frac{\partial H}{\partial M} M_m + \frac{\partial H}{\partial \Sigma^-} \Sigma_m^- + \frac{\partial H}{\partial \Sigma^+} \Sigma_m^+ - \gamma (P_\mu M_m + P_{\sigma-} \Sigma_m^- + P_{\sigma+} \Sigma_m^+) = 0 \qquad (3.6)$$

and, substituting for γ, we obtain the well known relation between relative prices and marginal utilities at the optimum:

$$\frac{\partial H / \partial \Sigma^-}{\partial H / \partial M} = \frac{P_{\sigma-}}{P_\mu}$$
$$\frac{\partial H / \partial \Sigma^+}{\partial H / \partial M} = \frac{P_{\sigma+}}{P_\mu} \qquad (3.7)$$

Therefore, the marginal rates of substitution between mean and semi-vols (i.e the plane tangent to the indifference surfaces) can be obtained by:

$$dM = -\frac{\partial H / \partial \Sigma^-}{\partial H / \partial M} d\Sigma^- - \frac{\partial H / \partial \Sigma^+}{\partial H / \partial M} d\Sigma^+ = -\frac{P_{\sigma-}}{P_\mu} d\Sigma^- - \frac{P_{\sigma+}}{P_\mu} d\Sigma^+ \qquad (3.8)$$

The optimal portfolio (x_0, x_m) satisfies (3.7) which, substituted in (3.5) gives:

$$P_m = M_m P_0 + \frac{P_{\sigma-}}{P_\mu} P_0 \Sigma_m^- + \frac{P_{\sigma+}}{P_\mu} P_0 \Sigma_m^+ = M_m P_0 + \Sigma_m^- P_{\sigma-} + \Sigma_m^+ P_{\sigma+} \qquad (3.9)$$

The second equality, i.e. $P_0 = P_\mu > 0$, comes from the first one, the definition of M, Σ^- and Σ^+ and the two budget constraints in (3.3). In fact, equate the two budget constraints, replace M, Σ^- and Σ^+ with their definitions and P_m with the first equality in (3.9), obtaining an equation implying $P_0 = P_\mu$.
This means that the "price of one unit of mean" is just the unit price of a default-free, zero-coupon bond, a quantity well known and promptly available in the market. In this way the term structure of interest rates enters the valuation of any asset.
From upside and downside risk aversion and (3.7) it follows that $P_{\sigma-}$ is negative and $P_{\sigma+}$ is positive.

$$P_{\sigma-} < 0, \quad P_{\sigma+} > 0 \qquad (3.10)$$

Equation (3.9) is the fundamental pricing equation. Under the assumption that in aggregate $x_0=0$ (endogenous risk-free asset), P_m is the price of the "market portfolio" and under symmetry (3.9) collapses into Sharpe's capital market line (CML).

4. THE MARKET PRICES OF UPSIDE AND DOWNSIDE VOL

If this approach is correct, the prices of upside and downside vol are implicit in the market quotes of financial assets.
In order to obtain an estimate of these prices, let us consider the decomposition:

$$S(T)-M = \mathrm{Max}(0,S(T)-M) - \mathrm{Max}(0, M-S(T)) \qquad (4.1)$$

with S(T) future (time T) value of the stock market, having mean M. In the right hand side of (4.1) the payoff of European call and put can be easily recognized. In current values the above equation is called put-call parity.
Since the actual market portfolio generates dividends we assume a continuous flow of dividend at the (known) rate δ_S and rewrite the parity equation (using continuous compounding) as:

$$S(t)e^{-\delta_S(T-t)} - MP_0 = \mathrm{Call}(t,M) - \mathrm{Put}(t,M) \qquad (4.2)$$

Therefore:

$$S(t) = Me^{\delta_S(T-t)}P_0 + \left[\frac{e^{\delta_S(T-t)}\mathrm{Call}(t,M)}{\Sigma^+}\right]\Sigma^+ + \left[-\frac{e^{\delta_S(T-t)}\mathrm{Put}(t,M)}{\Sigma^-}\right]\Sigma^- \qquad (4.3)$$

where in square brackets are the upside vol and downside vol prices, $P_{\sigma-}, P_{\sigma+}$ respectively.
In equation (4.3) S(t), δ_S and P_0 are known but M has to be estimated, so that, having M, Σ^+ and Σ^- can be calculated, Call(t,M) and Put(t,M) can be identified and, finally, the prices of semi-moments are obtained.

Two approaches are available in order to estimate the expected value M: a simple, short cut and a more elaborate route.

The short cut is to estimate M, Σ^+ and Σ^- as the weighted average of quoted strikes, using the volumes exchanged as weights. If speculation were the main driver in option markets, the obtained average would be a good approximation of the market expectation.

The second, more grounded approach is to obtain the mean M from the probability density of S(T), $f_{S(T)}$. A cross-section of option prices at different strikes K is useful to estimate the implied probability function $f_{S(T)}$ on the basis of Breeden and Litzenberger (1978) result that the second derivative of the option price with respect to the strike K is proportional to the risk-neutral density at K:

$$\frac{\partial^2 \mathrm{Call}(t,K)}{\partial K^2} = \frac{\partial^2 \mathrm{Put}(t,K)}{\partial K^2} = e^{-r(T-t)}\hat{f}_{S(T)}(K) \qquad (4.4)$$

Clearly $f_{S(T)}$ differs from $\hat{f}_{S(T)}$ by the risk-aversion adjustment and in the following we shall assume the simple and flexible form (cf. Bliss and Panigirtzoglou, 2004):

$$f_{S(T)}(K) \propto \left(\frac{K}{S(t)}\right)^{\gamma}\hat{f}_{S(T)}(K) \qquad (4.5)$$

with γ increasing with the risk aversion.

Here is the procedure we followed:
1) we take into account the put and call market quotes (three-month options on S&P 500 quoted daily at the CBOE) for all available strikes (see Fig. 1 for the December, 31, 2009 quotes)
2) we transform option quotes, using the Black and Scholes (1973) model, into implied volatility (Fig. 2) and interpolate a quadratic function of the forward moneyness, defined as $K\text{-}S(t)e^{(r-\delta s)(T-t)}$ (Fig. 3 and 4)
3) we transform the interpolating volatility function back into an option pricing function and calculate the first and second numerical derivatives with respect to K, mixing call and put densities into one density function and converting the risk-neutral density into natural density using γ=3.37 as estimated by Bliss and Panigirtzoglou (Fig.5). Finally, we calculate the mean M, the upside and downside vols and their implied prices. Note that the parameter γ can be made time-varying and estimated by minimizing the forecasting error over time.

FIG. 1

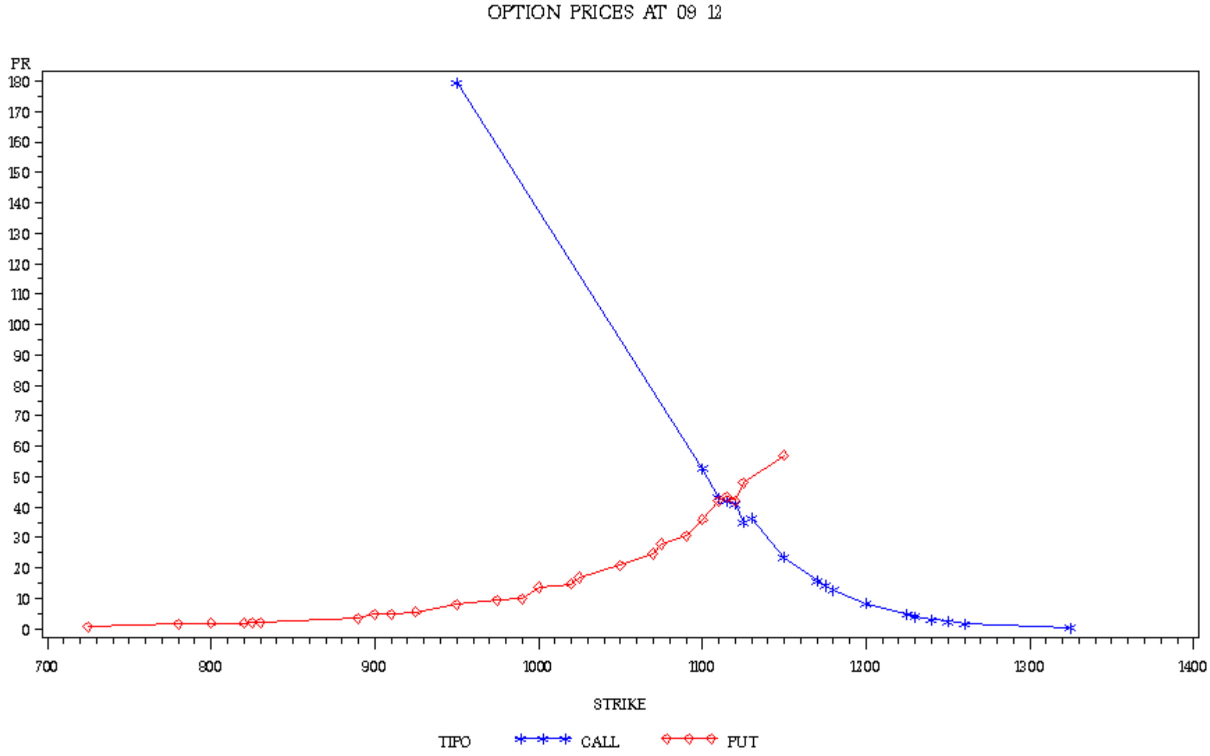


FIG. 2

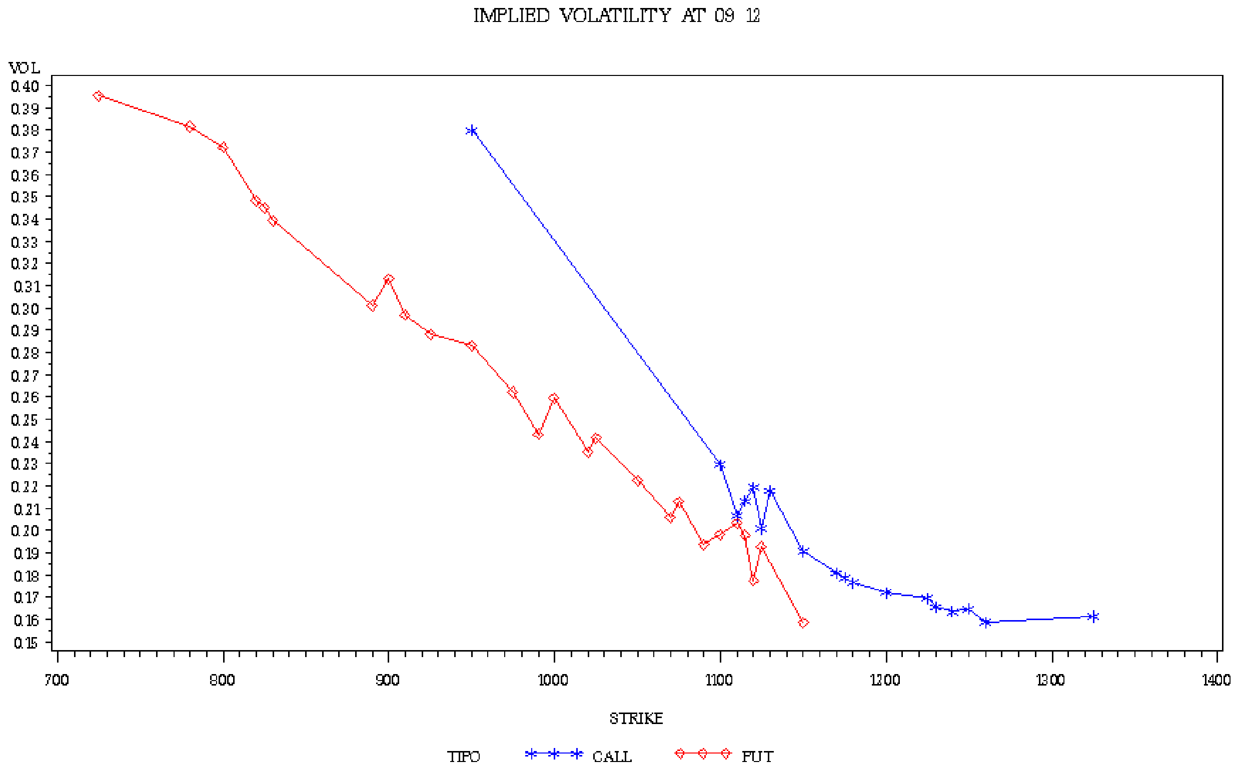

IMPLIED VOLATILITY AT 09 12
VOL
0.40
0.39
0.38
0.37
0.36
0.35
0.34
0.33
0.32
0.31
0.30
0.29
0.28
0.27
0.26
0.25
0.24
0.23
0.22
0.21
0.20
0.19
0.18
0.17
0.16
0.15
700
800
900
1000
1100
1200
1300
1400
STRIKE
TIPO
CALL
PUT

FIG. 3

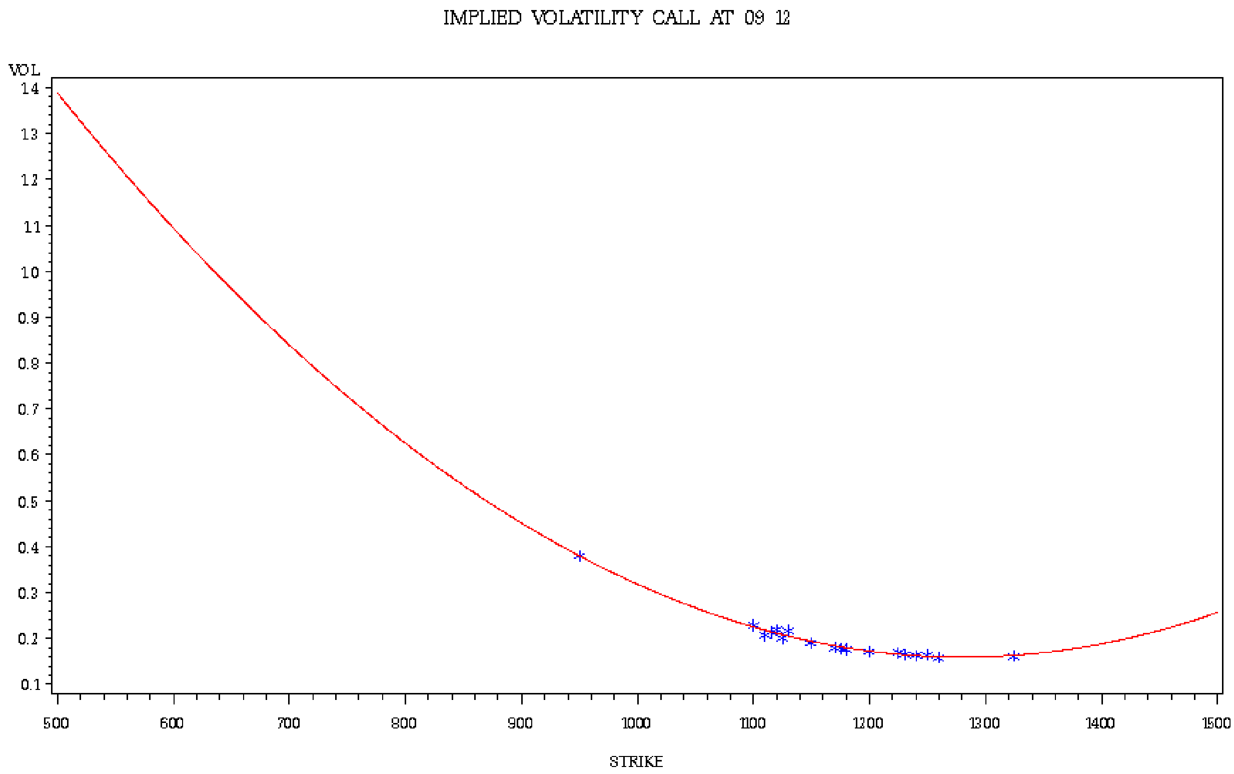

IMPLIED VOLATILITY CALL AT 09 12
VOL
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
500
600
700
800
900
1000
1100
1200
1300
1400
1500
STRIKE

FIG.4

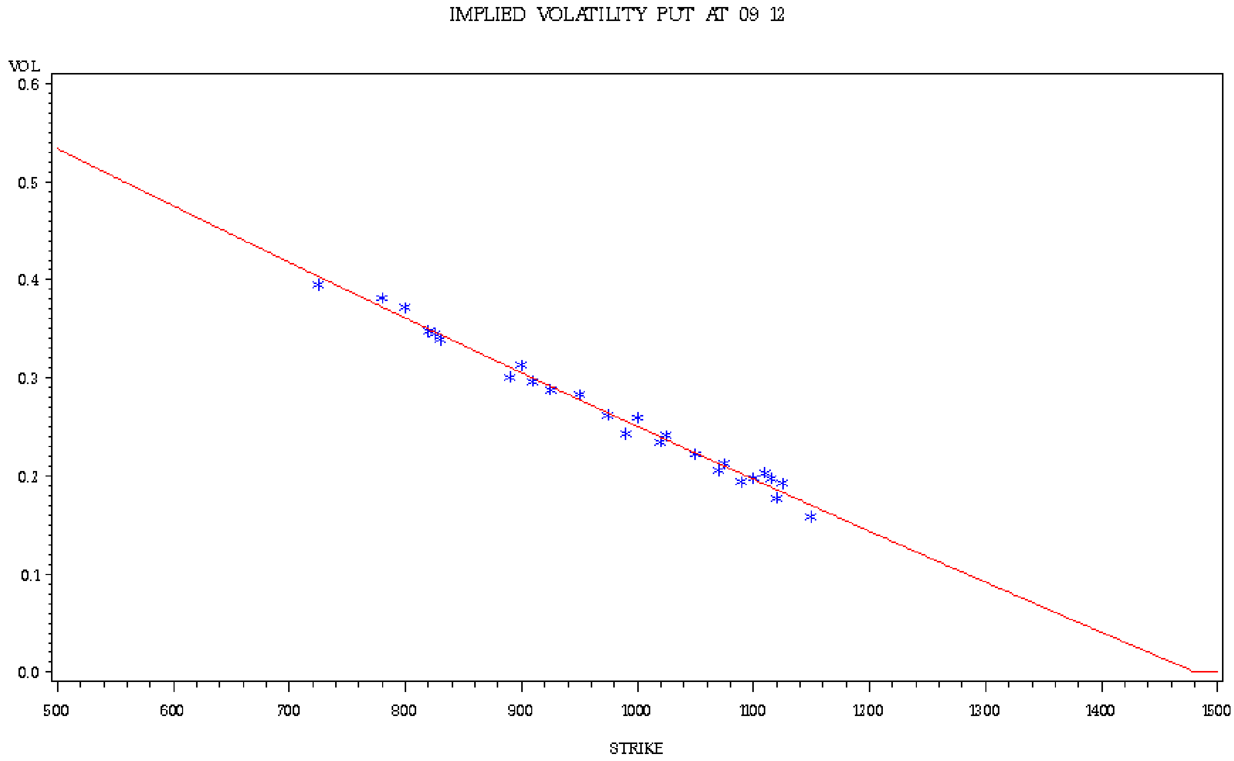
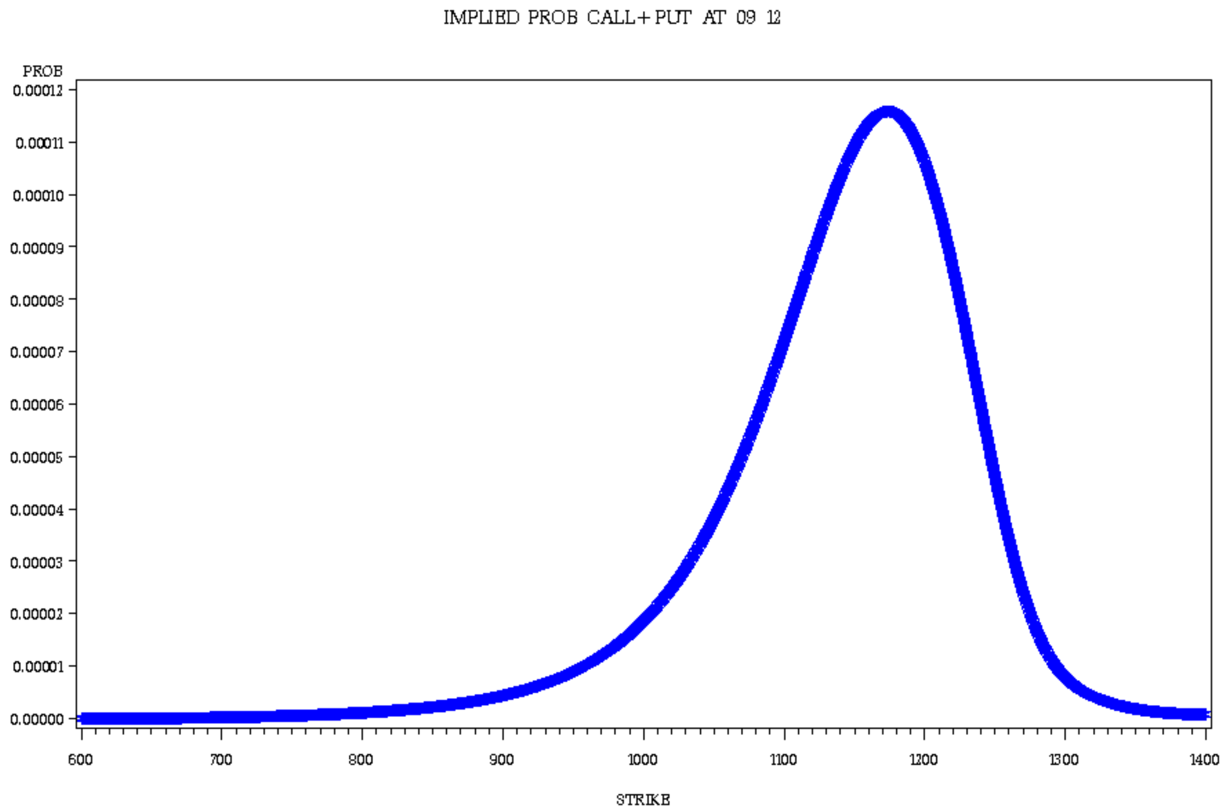

IMPLIED VOLATILITY PUT AT 09 12
VOL
0.6
0.5
0.4
0.3
0.2
0.1
0.0
500
600
700
800
900
1000
1100
1200
1300
1400
1500
STRIKE

FIG. 5


IMPLIED PROB CALL+PUT AT 09 12
PROB
0.00012
0.00011
0.00010
0.00009
0.00008
0.00007
0.00006
0.00005
0.00004
0.00003
0.00002
0.00001
0.00000
600
700
800
900
1000
1100
1200
1300
1400
STRIKE

The results for the troubled years 2007-2009 are shown in Fig. 6, 7 and 8. Notwithstanding the startling movements in the market index (Fig. 6) and the upside and downside vols (Fig.7) their prices present a remarkable steadiness (Fig. 8) implying that the ample swings in the S&P 500, from the peak of September 2007, above 1500 points, to the depth of march 2009, below 700, are largely explained by the movements of the mean and the semi-vols.

FIG. 6

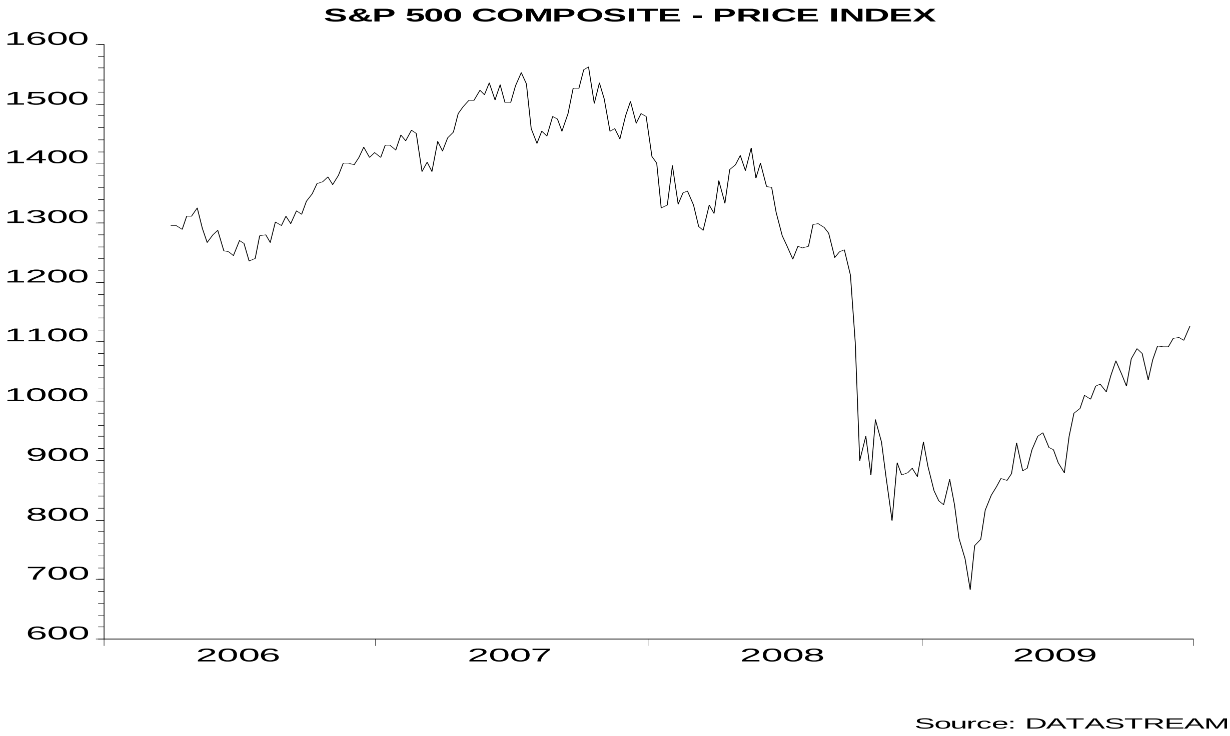
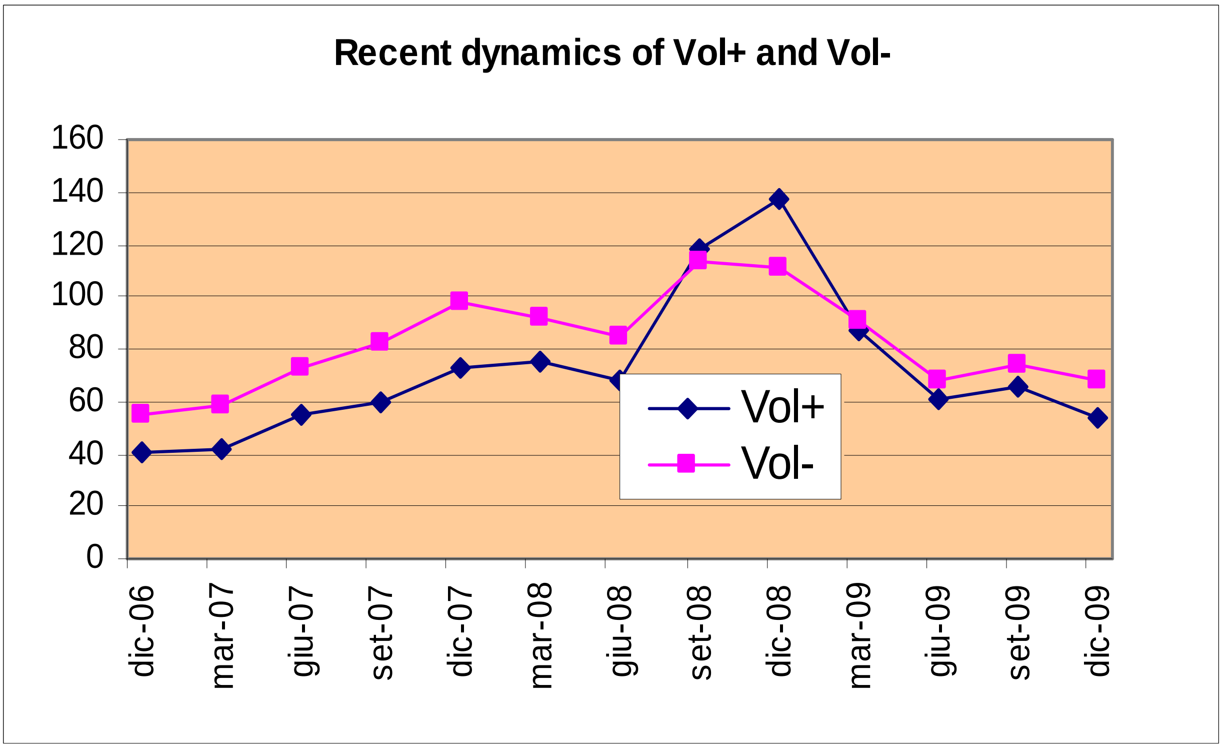
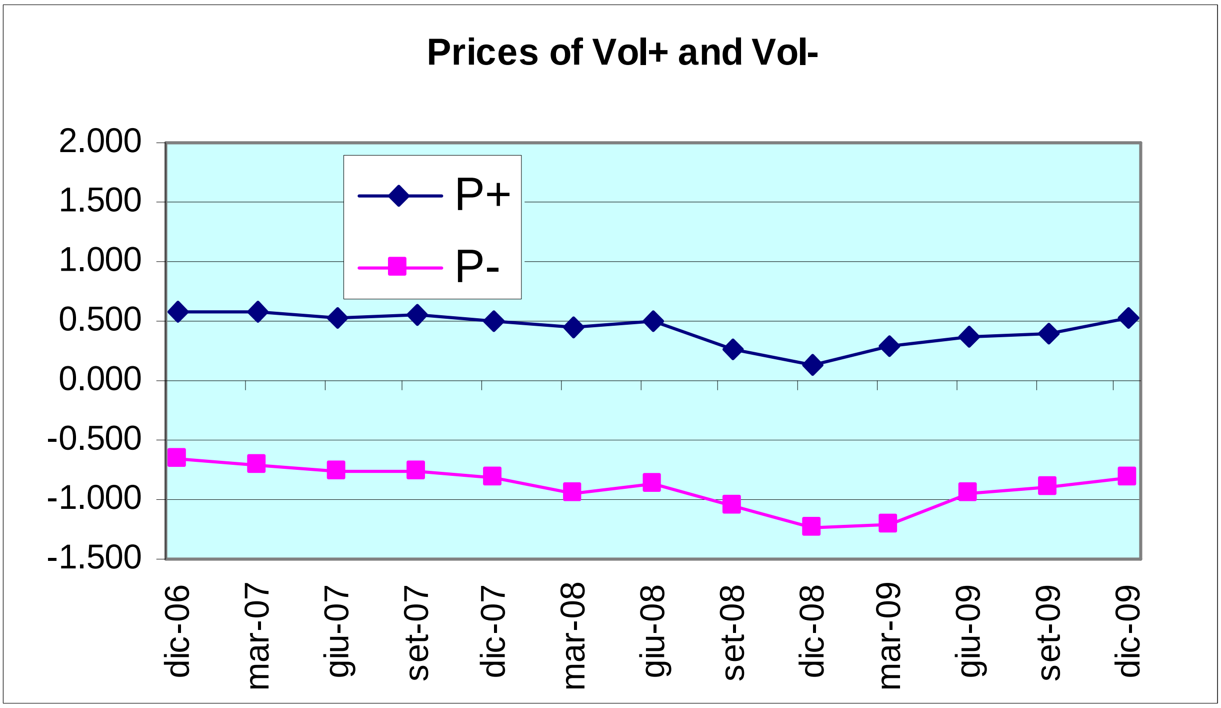


FIG. 7



Recent dynamics of Vol+ and Vol-
160
140
120
100
80
60
40
20
0
Vol+
Vol-
dic-06
mar-07
giu-07
set-07
dic-07
mar-08
giu-08
set-08
dic-08
mar-09
giu-09
set-09
dic-09

FIG. 8



Prices of Vol+ and Vol-
2.000
1.500
1.000
0.500
0.000
-0.500
-1.000
-1.500
P+
P-
dic-06
mar-07
giu-07
set-07
dic-07
mar-08
giu-08
set-08
dic-08
mar-09
giu-09
set-09
dic-09

5. CONCLUSIONS

Assuming an ordinal utility of semi-moments, - mean, upside volatility and downside volatility - we generalize the classical mean-variance approach to portfolio selection and asset pricing. We show that the asset price depends on the quantity of semi-moments included in the asset and the market prices of semi-moments determined in the financial equilibrium. An application to market data, using three years of option quotes at the Chicago Board Option Exchange, enables us to estimate the prices of semi-moments over the recent stock marker turmoil.

REFERENCES

Black, F. and Scholes, M. (1973), The pricing of options and corporate liabilities, *Journal of Political Economy*, 81, May, 637-659

Bliss, R. R. and Panigirtzoglou, N. (2004), Option-implied risk aversion estimates, *Journal of Finance*, 59, 1, 407-446

Breeden, D. T. and Litzenberger, R. H. (1978), Prices of state-contingent claims implicit in option prices, *Journal of Business*, 51, 4, 621-651

Cesari, R. and D'Adda, C. (2010), An ordinal utility of moments for decisions under uncertainty, *Asian Journal of Mathematics and Statistics*, 3, 2, 52-68

Debreu, G., 1964, Continuity properties of Paretian utility, *International Economic Review*, 5, 285-293

Fishburn, P. C., 1970, *Utility theory for decision making*, New York, Wiley

Hicks, J. R., 1962, Liquidity, *Economic Journal*, 72, 787-802

Kahneman, D. and Tversky, A. (1979), Prospect theory: an analysis of decision under risk, *Econometrica*, 47, 263-291

Lancaster, K., 1966, A new approach to consumer theory, *Journal of Political Economy*, 74, 132-157

Machina, M. J. (1987), Choice under uncertainty: problems solved and unsolved, *Economic Perspectives*, 1,1, 121-154

Markowitz, H. M., 1959, *Portfolio selection. Efficient diversification of investments*, New York, Wiley, 2nd ed. 1991, Oxford, Blackwell

Sharpe, W. F., 1964, Capital asset prices: a theory of market equilibrium under conditions of risk", *Journal of Finance*, 19, 425-442

Von Neumann, J. and O. Morgenstern, 1944, *Theory of games and economic behavior*, Princeton, PUP, 2nd ed. 1947, 3rd ed. 1953