A valuation model for the GLWB option in a Variable Annuity contract

Mariangela Scorrano, PhD *

Abstract

This paper proposes a valuation model for the GLWB option in variable annuity contracts using tractable financial and stochastic mortality processes in a continuous time framework. The policy has been analyzed assuming a static approach, in which policyholders withdraw each year just the guaranteed amount. Specifically we have considered the basic model proposed by Fung et al. [2014] and we have generalized it introducing more realistic assumptions. In particular, we have taken into account a CIR stochastic process for the term structure of interest rates and a Heston model for the volatility of the underlying account, analyzing their effect on the fair price of the contract. We have addressed these two hypotheses separately at first, and jointly afterwards. As part of our analysis, we have implemented the theoretical model using a Monte Carlo approach. To this end, we have created ad hoc codes based on the programming language MATLAB, exploiting its fast matrix-computation facilities.

Keywords: Variable annuities, Guaranteed Lifetime Withdrawal Benefit (GLWB), static approach, systematic mortality risk, interest rate and volatility risk, Monte Carlo approach.
JEL: C63, G13, G22, G23

1 Introduction

The past twenty years have seen a massive proliferation in insurance-linked derivative products. The public, indeed, has become more aware of investment opportunities outside the insurance sector and is increasingly trying to seize all the benefits of equity investment in conjunction with mortality protection. The competition with alternative investment vehicles offered by the financial industry has generated substantial innovation in the design of life products and in the range of benefits provided. In particular, equity-linked policies have become ever more popular, exposing policyholders to financial

*Department of Economics, Business, Mathematics and Statistics "Bruno de Finetti" - University of Trieste - mscorrano@units.it

markets and providing them with different ways to consolidate investment performance over time as well as protection against mortality-related risks. Interesting examples of such contracts are variable annuities (VAs). This kind of policies, first introduced in 1952 in the United States, experienced remarkable growth in Europe, especially during the last decade, characterized by "bearish" financial markets and relatively low interest rates. Being a quite new product class, an industry standard definition does not yet exist. Ledlie et al. [2008] describe them as *unit-linked or managed fund vehicles which offer optional guarantee benefits as a choice for the customer*. They are generally issued with a single premium (lump sum) or single recurrent premiums. The total amount of premiums is also named the principal of the contract or the invested amount. Apart from some upfront costs, premiums are entirely invested into a well diversified reference portfolio. In USA the National Association of Variable Annuity Writers explain that "with a variable annuity, contract owners are able to choose from a wide range of investment options called sub accounts, enabling them to direct some assets into investment funds that can help keep pace with inflation, and some into more conservative choices. Sub accounts are similar to mutual funds that are sold directly to the public in that they invest in stocks, bonds, and money market portfolios". Customers can therefore influence the risk-return profile of their investment by choosing from a selection of different mutual funds, from more conservative to more dynamic asset combinations.[1] During the contract's lifespan, its value may increase, or decrease, depending on the performance of the reference portfolio, thus policyholders are provided with equity participation. Under the terms and conditions specified by the contract, the insurer promises to make periodic payments to the client on preset future dates. These payments are usually determined as a fixed or variable percentage of the invested premium and are deducted from the contract's value, until the drawdown process will exhaust, sooner or later, the VA sub-account. It is well-rendered the risks underlying the policy. A prolonged negative performance of the reference portfolio during the lifespan of the contract could preempt its end and consequently reduce the total withdrawals received by the policyholder. The same happens for example if the annuitant dies few years after the contract's drafting, unlike his/her expectations. Just to face these risks that the VA market has begun to develop, insomuch as this class of annuities has achieved resounding success among investors. Many other features, in fact, contributed to make these products attractive. The success of variable annuities is no doubt due to the presence of tax incentives, introduced by governments to support the

[1]From the insurer's perspective, the buyer's portfolio choice can have a substantial impact on the profitability of the variable annuity. Individuals could increase risk and return in their portfolios to the point that the guarantee becomes unprofitable for the insurers. This is the reason why many actual prospectus of offered VAs restrict investment choices for their buyers.

development of individual pension solutions and contain public expenditure. Among them, a tax deferability of investment earnings until the commencement of withdrawals and a tax-free transfer of funds between VA investment options are allowed. But, in respect of traditional life insurance products, the main feature of variable annuities is the possibility of enjoying of a large variety of benefits represented by guarantees against investment and mortality/longevity risks. Available guarantees are usually referred to as GMxB, where "x" stands for the class of benefits involved. A first classification is between:

- Guaranteed Minimum Death Benefits (GMDB);
- Guaranteed Minimum Living Benefits (GMLB).

The GMDB rider is usually available during the accumulation period and it addresses the concern that the policyholder may die before all payments are made. If it happens, the beneficiary receives a death benefit equal to the current asset value of the contract or, if higher, the guaranteed amount, which typically is the amount of premiums paid by the deceased policyholder accrued at the guaranteed rate.
In contrast, living benefits can be described as wealth-preservation or wealth-decumulation products as they enable the policyholder to preserve wealth during the drawdown period. There are three common types of living benefit riders: the Guaranteed Minimum Accumulation Benefits (GMAB); the Guaranteed Minimum Income Benefits (GMIB) and the Guaranteed Minimum Withdrawal Benefits (GMWB). In this work we will refer to the last rider, and to be more precise, to its ultimate version, represented by the Guaranteed Lifetime Withdrawal Benefit. In fact, as a result of rising life expectancies as well as increases in lifestyle and health-care costs, retirement lifespans have become both longer and more expensive. At the same time, with the social security system under considerable stress, the idea that individuals and households need to plan for their own retirement is gaining traction. To satisfy these new needs insurance companies have started offering a lifetime benefit feature with GMWB, enabling the investor to simultaneously manage both financial as well as longevity related risks. This new rider is commonly known as "Guaranteed Lifetime Withdrawal Benefits" (GLWB) and guarantees policyholders the possibility of withdrawing an annual amount (typically 4% to 7%) of their guaranteed protection amount (GLWB Base) for their entire lifetime, no matter how the investments in the sub-accounts perform. It's the only product that combines longevity protection with withdrawal flexibility, hence it is seen as a "second-generation" guarantee. The guarantee can concern one or two lives (typically spouses). Each annual withdrawal does not exceed some maximum value, but it is evident that the total amount of withdrawals is not limited, depending on the policyholder's lifetime. Annual withdrawals of about 5% of the (single

initial) premium are commonly guaranteed for insured aged 60+. In case of death any remaining fund value is paid to the insured's dependants. In deferred versions of the contract, the product is fund linked during the deferment and the account value at the end of this period, or a guaranteed amount if greater, is treated like a single premium paid for an immediate GLWB.

Insurance companies charge a fee for the offered benefits. Guarantees and asset management fees, administrative cost and other expenses are charged typically deducting a certain percentage of the underlying fund's value from the policyholder's funds account on an annual basis. Very rarely they are charged immediately as a single initial deduction. This improves the transparency of the contract, as any deduction to the policy account value must be reported to the policyholder. Some guarantees can be added or removed, at policyholder's discretion, when the contract is already in-force. Accordingly, the corresponding fees start or stop being charged. Unlike most "good" investments, VAs' fees are quite high. For this reason, they used to receive heaps of bad press. Also investors don't look kindly upon this aspect, because of the combination of investment management and insurance expenses substantially reduces their returns.

There have been several papers devoted to the pricing and hedging of variable annuities with various forms of embedded options. In Bacinello et al. [2014] we can find a quite exhaustive classification of the papers on GMWBs and GLWs. The GLWB option has been launched in the market recently, therefore a detailed literature is not yet available. GMWB, which is a similar option except that it guarantees withdrawals over only a fixed number of years, has been analyzed initially by Milevsky and Salisbury [2006]. The authors consider two policyholder behavior strategies. Under a static withdrawal approach the contract is decomposed into a Quanto Asian Put option plus a generic term-certain annuity. Numerical PDE methods are used to evaluate the ruin probabilities for the account process and the contract value. Considering a dynamic approach where optimal withdrawals occur, instead, an optimal stopping problem akin to pricing an American put option emerges, albeit complicated by the non-traditional payment structure. The free boundary value problem is solved numerically. The authors find fees' values greater than those charged in the market. The optimal behavior approach has been then formalized in Dai et al. [2008] where a singular stochastic control problem is posed. Chen and Forsyth [2008] explore the effect of various modeling assumptions on the optimal withdrawal strategy of the policyholder, and examine the impact on the guarantee value under sub-optimal withdrawal behavior. The authors moreover propose numerical schemes for pricing various types of guaranteed minimum benefits in VAs using an impulse control formulation. Bauer et al. [2008] develop an extensive and comprehensive framework to price any of the common guarantees available with VAs. Monte Carlo simulation is used to price the contracts

assuming a deterministic behavior strategy for the policyholders. In order to price the contracts assuming an optimal withdrawal strategy, a quasi-analytic integral solution is derived and an algorithm is developed by approximating the integrals using a multidimensional discretization approach via a finite mesh. In all these papers the guarantees are priced under the assumption of constant interest rates. Peng et al. [2012] derive the analytic approximation solutions to the fair value of GMWB riders under both equity and interest rate risks, obtaining both the upper and the lower bound on the price process. Allowing for discrete withdrawals, Bacinello et al. [2011] consider a number of guarantees under a more general financial model with stochastic interest rates and stochastic volatility in addition to stochastic mortality. In particular for GMWBs, a static behavior strategy is priced using standard Monte Carlo whereas an optimal lapse approach is priced with a Least Squares Monte Carlo algorithm. The pricing models of GLWB can be considered as extensions of those concerning the GMWB guarantee together with the inclusion of mortality risk. Shah and Bertsimas [2008] analyze the GLWB option in a time continuos framework considering simplified assumptions on population mortality and adopting different asset pricing models. Holz et al. [2012] price the contract for different product design and model parameters under the Geometric Brownian Motion dynamics of the underlying fund process. They also consider various forms of policyholder withdrawal behavior, including deterministic, probabilistic and stochastic models. Other papers investigate the impact of volatility risk, for example Kling et al. [2011]. Piscopo and Haberman [2011] assess the mortality risk in GLWB but not the other risks and their interactions. Fung et al. [2014], in particular, deal with these aspects, analyzing equity and systematic mortality risks underlying the GLWB, as well as their interactions. The valuation, however, has been performed in a Black and Scholes economy: the sub-account value has been assumed to follow a geometric Brownian motion, thus with a constant volatility, and the term structure of interest rates has been assumed to be constant.

In the following section we will introduce briefly the valuation model proposed in Fung et al. [2014] with the aim of generalizing it later on. The backing hypotheses, indeed, do not reflect the situation of financial markets. In order to consider a more realistic model, we have sought to weaken these misconceptions. Specifically we have taken into account a CIR stochastic process for the term structure of interest rates and a Heston model for the volatility of the underlying account, analyzing their effect on the fair price of the contract. We have addressed these two hypotheses separately at first, and jointly afterwards. As part of our analysis, we have implemented the theoretical model using a Monte Carlo approach. To this end, we have created ad hoc codes based on the programming language MATLAB, exploiting its fast matrix-computation facilities. Sensitivity analyses have been conducted in order to investigate the relation between the fair price of the contract and

important financial and demographic factors.

2 Basic model

In this section we briefly describe the valuation model proposed in Fung et al. [2014], introducing its components: the financial market and the mortality intensity. We will first describe them separately, and then successively we will combine them into the insurance market model.

2.1 The financial component

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, where $\mathbb{P}$ is the real world or physical probability measure and $\mathbb{F} \doteq (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions of right continuity, i.e. $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$, and $\mathbb{P}$-completeness, i.e. $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.
Let P be the upfront single premium paid at the inception of the contract, $t = 0$. No initial sales charge is applied, so the deposited amount is entirely used for immediate investment in the available sub accounts. Let x the age of the policyholder at time $t = 0$, and suppose that ω is the maximum attainable age (or limiting age), i.e. the age beyond which survival is assumed to be impossible. The limiting age ω allows for a finite time horizon $T = \omega - x$.
Suppose that the investment portfolio has both equity and fixed income exposure.
Under the real world probability measure $\mathbb{P}$, we assume that the riskless component (fixed income investment) is modelled by the money market account $B(t)$ with the following ordinary differential equation:

$$dB(t) = rB(t)dt \tag{1}$$

where $r \geq 0$ is the instantaneous interest rate. Setting $B(0) = 1$, we have $B(t) = e^{rt}$ for $t \geq 0$.
The risky component is a stock (or stock index) whose price under $\mathbb{P}$ follows the usual Geometric Brownian motion:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW_S(t), \qquad S(0) > 0 \tag{2}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and W_S is a standard Brownian motion.
We initially assume interest rate r and equity volatility σ to be constant.
Let us define the pair $\varphi(t) = (\xi(t), \eta(t))$ as the *portfolio* held at time t, where $\xi(t)$ is the number of stocks held at time t and $\eta(t)$ denotes the deposit on the savings account at time t.
Therefore the reference investment fund $V(\cdot)$ can be written as:

$$V(t) = \xi(t)S(t) + \eta(t)B(t)$$

and so its dynamics is given by:

$$\begin{aligned} dV(t) &= \xi(t)dS(t) + \eta(t)dB(t) \\ &= [\mu\xi(t)S(t) + r\eta(t)B(t)]dt + \sigma\xi(t)S(t)dW_S(t) \end{aligned} \tag{3}$$

Let $\pi(t) = \frac{\xi(t)S(t)}{V(t)}$ denote the proportion of the retirement savings being invested in the equity component. All the usual assumptions on the perfect markets hold: there are no arbitrage opportunities (i.e., there is no way to make a riskless profit), it is possible to borrow and lend any amount, even fractional, of cash at the fixed riskless rate, it is possible to buy and sell any amount, even fractional, of the stock (this includes short selling) and the above transactions do not incur any fees or costs (i.e., frictionless market). However, in our model, we assume $0 \leq \pi(\cdot) \leq 1$. In addition we consider $\pi(\cdot)$ constant, say equal to π, that is the policyholder invests a fixed proportion of his/her retirement savings in equity and fixed income markets throughout the investment period. We can therefore rewrite equation (3) as:

$$dV(t) = [\mu\pi + r(1-\pi)]V(t)dt + \sigma\pi V(t)dW_S(t) \tag{4}$$

Therefore the dynamics of the relative returns can be written as:

$$\frac{dV(t)}{V(t)} = [\mu\pi + r(1-\pi)]dt + \sigma\pi dW_S(t) \tag{5}$$

As results from the description of the policy, the VA sub-account held by the policyholder is influenced by the variable market performance, the guarantee fees charged by the insurance company and the periodic withdrawals provided by for the contract.

Denote with $A(t)$ the VA account value at time t.
Since the initial premium is invested in the market, it is subject to daily fluctuations (at least considering the equity component), the size and extent of which remain a priori uncertain. Therefore, also the balance of the VA account at a given point in time t, $A(t)$, could be either positive or negative. Should market performance result in low or negative returns, $A(\cdot)$ may reduce to zero or even fall below this value.
The other two elements (fees and withdrawals) are deducted from the VA sub-account, so they reduce its value.
Let α be the annual fee rate applied by the insurance company for activating the GLWB option. Fees are deducted from the account value as long as the contract is in force and the account value is positive.
Let $\gamma(t)$ be the withdrawals made by the policyholder at time t.
The above considerations imply that the dynamics of the VA sub-account can be described using the following stochastic differential equation (SDE):

$$dA(t) = -\alpha A(t)dt - \gamma(t)dt + A(t)\frac{dV(t)}{V(t)} \tag{6}$$

or equivalently, from equation (5), as:

$$\begin{aligned} dA(t) &= -\alpha A(t)dt - \gamma(t)dt + A(t)\{[\mu\pi + r(1-\pi)]dt + \sigma\pi dW_S(t)\} \\ &= (\mu\pi + r(1-\pi) - \alpha)A(t)dt - \gamma(t)dt + \sigma\pi A(t)dW_S(t) \end{aligned} \tag{7}$$

This equation holds as long as $A(\cdot) \geq 0$. In fact, once $A(\cdot)$ hits the zero value, it remains to be zero forever afterwards. That is, the zero value is considered to be an absorbing barrier of $A(\cdot)$. Furthermore, being P the amount originally paid by the policyholder, we have:

$$A(0) = P$$

In other words, upon contract signature (at time $t = 0$), the balance of the VA sub-account exactly matches the initial investment made by the policyholder.

Using $g(t)$ to define the withdrawal rate allowed by the insurance company at time t, the withdrawals guaranteed at time t are given by $g(t)P$. In our valuation analysis we adopt a static approach (Bacinello et al. [2011]), in which the policyholder withdraws exactly the guaranteed amount each year. Therefore, the possibility of increasing or reducing the amount withdrawn depending on the financial needs of the policyholder is not considered. Important reasons support our choice. First of all, VA providers can influence the behavior of policyholders through imposing penalty charges on the amount of withdrawal that exceeds the guaranteed amount. In practice, additional high indirect costs in terms of taxes on the excess distributions make taking large strategic withdrawals even more unattractive. Moreover, we have to consider that these options are being introduced in pension plans, in order to ensure a constant income during retirement and provide protection against market downside risk (Piscopo and Haberman [2011]). In addition, Holz et al. [2012] note that the value of a lifetime GMWB and so the fair guarantee fee under optimal customer behavior differs only slightly from that assuming deterministic behavior. In closing, a typical individual insured is unable to hedge risks due to his/her own longevity and less equipped than large institutions like insurance companies to hedge financial risks. Hence in our analysis we consider a typical investor with a more simplistic deterministic withdrawal behavior compared with an arbitrageur.

From now on we assume that the guaranteed withdrawal rate does not vary over time but remains constant:

$$g(t) = g$$

as well as withdrawals, hence we have:

$$\gamma(t) = gP = G$$

With these considerations in mind, we can write the dynamics of the VA sub-account as:

$$\begin{cases} dA(t) = (\mu\pi + r(1-\pi) - \alpha)A(t)dt - Gdt + \sigma\pi A(t)dW_S(t) \\ A(0) = P \\ A(\cdot) \geq 0 \end{cases} \tag{8}$$

The GLWB option is activated and has a positive value only if the process hits zero before the death date of the policyholder. If, due to declining stock markets combined with the reducing effect of fees and withdrawals, the account value of the policy becomes zero while the insured is still alive, then the GLWB guarantee becomes effective and the insured can continue to withdraw the same guaranteed amount annually until death. In this case, the account balance is not sufficient to fund the guaranteed withdrawals and intervention by the insurance company is necessary. If, on the contrary, the dynamics of the VA sub-account is such that "ruin" never occurs (or occurs after the policyholder has passed away), then the GLWB guarantee has a zero payout. Indeed, in this case, the account balance is in itself sufficient to assure the policyholder of all the withdrawals until his/her death and the guarantee therefore does not need to be activated.

2.2 The mortality component

An important requirement for the GLWB's activation is the survival of the policyholder. For this reason it is important to consider the uncertainty related to the random residual lifetime of insureds (mortality risk) in addition to that related to financial factors (financial risk). Traditionally, a central role in the definition of a mortality model has been played by the force of mortality (or mortality intensity), defined as the instantaneous rate of mortality at a given age x. In particular, among the plausibile features that such a model would meet, the term structure of mortality rates, considering the usual ages at which VA policies are underwritten, should only be increasing to reflect the biologically reasonableness for age-specific pattern of mortality, and mean-reversion should not be a desirable property for mortality dynamics. The inclusion of mean reversion entails that if mortality improvements have been faster than anticipated in the past then the potential for further mortality improvements will be significantly reduced in the future. Such property is difficult to justify on the basis of previous observed mortality changes and with reference to our perception of the timing and impact of, for example, future medical advances. Therefore, given the criteria described in Cairns et al. [2006] and according to Fung et al. [2014], we adopt a one-factor, non mean-reverting and time homogeneous affine process for modeling the mortality intensity, $\mu_{x+t}(t)$, of a person aged x at time $t = 0$,

as follows:

$$\begin{cases} d\mu_{x+t}(t) = (a + b\mu_{x+t}(t))dt + \sigma_\mu \sqrt{\mu_{x+t}(t)} dW_M(t) \\ \mu_x(0) > 0 \end{cases} \tag{9}$$

with $a \neq 0$, $b > 0$ and $\sigma_\mu > 0$ being the volatility of the mortality intensity. It is reasonable to assume the independence of the randomness in mortality and that in interest rates, so W_M denotes a standard Brownian motion independent of W_S.

2.3 The combined model

Recall the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ introduced above. The filtration $\mathcal{F}_t$ describes the total information available at time t and it has to be large enough to support the processes representing the evolution of financial variables and of mortality. Formally we write:

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$$

where $\mathcal{G}_t \vee \mathcal{H}_t$ is the σ-algebra generated by $\mathcal{G}_t \cup \mathcal{H}_t$, with

$$\mathcal{G}_t = \sigma(W_S(s), W_M(s) : 0 \leq s \leq t)$$

$$\mathcal{H}_t = \sigma(\mathbb{I}_{\{\tau \leq s\}} : 0 \leq s \leq t)$$

Thus, $\mathcal{G}_t$ is generated by the two independent standard Brownian Motions, W_S and W_M, which describe the uncertainties related to equity and mortality intensity, respectively, and $\mathcal{H}_t$ describes the information set that indicates if the death of the policyholder has occurred before time t.
It is a well-known result in asset pricing theory that, under reasonable economic assumptions, the market price of a security is given by its expected discounted cash-flows. Discounting takes place at the risk-free rate and the expectation is taken with respect to a suitably risk-adjusted probability measure. The incompleteness of insurance markets implies that infinitely many such probabilities exist. We assume henceforth that the insurer has picked out a specific probability for valuation purposes, say $\mathbb{Q}$. In particular we define $W_S^{\mathbb{Q}}(t)$ and $W_M^{\mathbb{Q}}(t)$ as:

$$dW_S^{\mathbb{Q}}(t) = \frac{\mu - r}{\sigma} dt + dW_S(t) \tag{10}$$

$$dW_M^{\mathbb{Q}}(t) = \lambda \sqrt{\mu_{x+t}(t)} dt + dW_M(t) \tag{11}$$

By the Girsanov Theorem these are standard Brownian motions under the $\mathbb{Q}$ measure with $\frac{\mu - r}{\sigma}$ and $\lambda\sqrt{\mu_{x+t}(t)}$ representing the market price of equity risk and systematic mortality risk, respectively.

If we consider the new probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, the evolutions of the VA sub-account and of mortality intensity become:

$$dA(t) = (r - \alpha)A(t)dt - Gdt + \pi\sigma A(t) dW_S^{\mathbb{Q}}(t) \tag{12}$$

$$d\mu_{x+t}(t) = (a + (b - \lambda\sigma_\mu)\mu_{x+t}(t))dt + \sigma_\mu \sqrt{\mu_{x+t}(t)} dW_M^{\mathbb{Q}}(t) \tag{13}$$

2.4 The valuation formula

There are two perspectives from which to view the GLWB rider [Hyndman and Wenger, 2014]. A policyholder usually considers the VA and the GLWB rider as one combined instrument and he/she is interested in the total payments received over the duration of the contract. On the other hand, although the rider is embedded into the VA, the insurer might want to consider it as a separate instrument, being interested in mitigating and hedging the additional risk attributed to the rider.

2.4.1 The policyholder's perspective

Viewing the policy from a policyholder's perspective, the risk-neutral value at time t of the GLWB can be seen as the sum of the no-arbitrage values of the living and death benefits.

Living benefits are represented by static withdrawals made by the policyholder during the lifetime of the contract while he/she is alive. The income from these withdrawals can be regarded as an immediate life annuity, whose no-arbitrage value at time t is equal to:

$$LB^{pol}(t) = \mathbb{I}_{\{\tau>t\}} G \int_0^{\omega-x-t} {}_sP_{x+t} e^{-rs} ds \tag{14}$$

where $0 \leq t \leq \omega - x$, $\mathbb{I}_{\{\tau>t\}}$ is an indicator function taking value of one if the individual is still alive at time t, and zero otherwise and ${}_sP_{x+t}$ is the $\mathbb{Q}$-survival probability at time $t+s$ of an individual alive and aged $x+t$ at time t.

Death benefits can be calculated considering the payoff that the beneficiary will receive at the random time of policyholder's death, τ. Therefore we can write[2]:

$$DB^{pol}(\tau) = A(\tau) \tag{15}$$

The market value at time t of the death benefit is given by:

$$DBV^{pol}(t) = \mathbb{I}_{\{\tau>t\}} \int_0^{\omega-x-t} f_{x+t}(s) E_t^{\mathbb{Q}}[e^{-rs} A(t+s)] ds \tag{16}$$

where $f_{x+t}(s) = -\frac{d}{ds}({}_sP_{x+t})$ is the density function under $\mathbb{Q}$ of the remaining lifetime of an individual aged $x+t$ at time t and $E_t^{\mathbb{Q}}$ denotes conditional expectation.

Both $LB^{pol}(t)$ and $DBV^{pol}(t)$ are values of cash inflows, while the amount in the investment account $A(t)$ is viewed as a cash outflow to the VA provider.

[2]Recall that $A(t) \geq 0 \ \forall t$ because, once the account process hits the zero value, it remains to be zero forever afterwards. That is, the zero value is an absorbing barrier of $A(\cdot)$. Hence, we don't need to take its positive part.

The risk-neutral value at time t of the complete contract (VA plus GLWB rider), net of the outflow to the VA provider, is therefore defined as:

$$V^{pol}(t) = LB^{pol}(t) + DBV^{pol}(t) - \mathbb{I}_{\{\tau>t\}}A(t) \tag{17}$$

In particular, at time $t = 0$ it is:

$$V^{pol}(0) = LB^{pol}(0) + DBV^{pol}(0) - A(0) \tag{18}$$

Since

$$_sP_x = E^{\mathbb{Q}}[e^{-\int_0^s \mu_{x+u}(u)du}]$$

we have

$$f_x(s) = -\frac{d}{ds}{}_sP_x = E^{\mathbb{Q}}[e^{-\int_0^s \mu_{x+u}(u)du}\mu_{x+s}(s)]$$

The contract value at time $t = 0$ is therefore given by:

$$V^{pol}(0) = G\int_0^{\omega-x} {}_sP_x e^{-rs}ds + \int_0^{\omega-x} E^{\mathbb{Q}}[e^{-\int_0^s \mu_{x+u}(u)du}\mu_{x+s}(s)]E^{\mathbb{Q}}(e^{-rs}A(s))ds - A(0)$$

The independence between $W_S^{\mathbb{Q}}$ and $W_M^{\mathbb{Q}}$ implies that:

$$V^{pol}(0) = G\int_0^{\omega-x} {}_sP_x e^{-rs}ds + \int_0^{\omega-x} E^{\mathbb{Q}}[e^{-\int_0^s \mu_{x+u}(u)du}\mu_{x+s}(s)e^{-rs}A(s)]ds - A(0)$$

Equivalently:

$$V^{pol}(0) = E^{\mathbb{Q}}\left[\int_0^{\omega-x}\left(Ge^{-rs}e^{-\int_0^s \mu_{x+u}(u)du} + A(s)e^{-rs}e^{-\int_0^s \mu_{x+u}(u)du}\mu_{x+s}(s)\right)ds\right] - A(0)$$

or, in more compact terms:

$$V^{pol}(0) = E^{\mathbb{Q}}\left[\int_0^{\omega-x} e^{-\int_0^s \mu_{x+u}(u)du}e^{-rs}\left(G + A(s)\mu_{x+s}(s)\right)ds - A(0)\right] \tag{19}$$

The guarantee is considered fair to both, policyholder and insurer, at time $t = 0$, if it holds:

$$V^{pol}(0) = 0 \tag{20}$$

As a consequence, the fair fee rate is defined as the rate $\alpha^* \geq 0$ that solves (20):

$$\alpha^* : V^{pol}(0;\alpha^*) = 0 \tag{21}$$

This equation does not have a closed form solution and numerical methods must be used to find α^*.

It is possible to obtain the risk neutral value of the contract also in terms of the policyholder's random time of death. Recall that we are modeling the policyholder's random residual lifetime as an $\mathbb{F}$-stopping time τ admitting a

random intensity μ_x. Specifically, we regard τ_x as the first jump-time of a nonexplosive $\mathbb{F}$-counting process N recording at each time $t \geq 0$ whether the individual has died ($N_t \neq 0$) or not ($N_t = 0$) (Biffis [2005]). To improve analytical tractability, we further assume that N is a doubly stochastic (or Cox) process driven by a subfiltration $\mathbb{G}$ of $\mathbb{F}$, with $\mathbb{G}$-predictable intensity μ. We assume that the nonnegative predictable process μ satisfies $\int_0^t \mu_s ds < \infty$ a.s. for all $t > 0$. We then fix an exponential random variable Φ with parameter 1, independent of $\mathcal{G}_\infty$. Under these assumptions, Biffis [2005] defines the random time of death τ as the first time when the process $\int_0^{\cdot} \mu_{x+s}(s)ds$ is above the random level Φ, so we set:

$$\tau = \inf\left\{ t \in \mathbb{R}_+ : \int_0^t \mu_{x+s}(s)\, ds > \Phi \right\} \tag{22}$$

With these considerations, we can express the risk-neutral value of the GLWB option as:

$$V^{pol}(0) = E^{\mathbb{Q}}\left[G \int_0^\tau e^{-rs} ds + e^{-r\tau} A(\tau) \right] - A(0) \tag{23}$$

and consequently, the fair fee rate as:

$$\alpha^* : E^{\mathbb{Q}}\left[G \int_0^\tau e^{-rs} ds + e^{-r\tau} A(\tau) - A(0) \right] = 0 \tag{24}$$

2.4.2 The insurer's perspective

The alternative valuation prospective, concerning the insurer, considers the GLWB rider as a standalone product.
Recall that the trigger time defined by Milevsky and Salisbury [2006] is the first passage time of the process $A(t)$ hitting the zero value, that is

$$\zeta = \inf\{t \geq 0 : A(t) = 0\} \tag{25}$$

Once $A(t)$ hits the zero value, it remains to be zero forever afterwards. That is, the zero value is considered to be an absorbing barrier of $A(t)$ as we have already explained earlier. We use the convention $\inf(\emptyset) = \infty$. If $\zeta \leq T$ we say that the option is triggered (or exercised) at trigger time ζ. Therefore, under $\mathbb{Q}$, the value process of the VA sub-account is given by:

$$\begin{cases} dA(t) = (r - \alpha)A(t)dt - Gdt + \pi\sigma A(t) dW_S^{\mathbb{Q}}(t) \\ A(0) = P \end{cases} \qquad \text{for } 0 \leq t < \zeta \tag{26}$$

and

$$A(t) = 0 \qquad \text{for } t \geq \zeta$$

Under this approach, the rider value process can be defined as the risk-neutral expected discounted difference between future rider payouts and future fee

revenues, or the expected discounted benefits minus the expected discounted premiums.
At time ζ, if the policyholder is still alive, the rider guarantee entitles the policyholder to receive an annual payment of G until his/her death. The expected discounted benefits are therefore calculated as

$$B^{ins}(t) = \mathbb{I}_{\{\tau>t\}} \int_0^{\omega-x-t} f_{x+t}(s) E_t^{\mathbb{Q}} \left(\int_{t+\zeta}^{t+s} gA(0) e^{-r(v-t)} \mathbb{I}_{\{s>\zeta\}} dv \right) ds \quad (27)$$

$$= \mathbb{I}_{\{\tau>t\}} \int_0^{\omega-x-t} f_{x+t}(s) \left(\frac{gA(0)}{r} \right) E_t^{\mathbb{Q}}((e^{-r\zeta} - e^{-rs})^+) ds \quad (28)$$

Fee revenue is received up to the depleting time of the account value, of course if the policyholder is alive. In other terms, the insurer charges a certain percentage of the account value up to the earliest between policyholder's death and VA account value's depleting. Hence, the expected discounted premiums are:

$$P^{ins}(t) = \mathbb{I}_{\{\tau>t\}} \int_0^{\omega-x-t} f_{x+t}(s) E_t^{\mathbb{Q}} \left(\int_t^{t+(\zeta \wedge s)} e^{-r(v-t)} \alpha A(v) dv \right) ds \quad (29)$$

where $x_1 \wedge x_2 = \min\{x_1, x_2\}$. Denote by $V^{ins}(t)$ the value at time t of the GLWB contract. It is defined as:

$$V^{ins}(t) = B^{ins}(t) - P^{ins}(t) \quad (30)$$

The fair guarantee fee rate can be calculated, again, as:

$$\alpha^* : V^{ins}(0; \alpha^*) = 0 \quad (31)$$

Fung et al. [2014] show the equivalence of the two approaches. While the first one is computationally more efficient, the second approach highlights the theoretical result that the market reserve of a payment process is defined as the expected discounted benefits minus the expected discounted premiums under a risk- adjusted measure (Dahl and Moller [2006]).
In the implementation of the valuation model we will refer to the policyholder's approach.

2.5 Numerical results

Since for SDEs involved in the valuation model previously described there are no explicit solutions, numerical methods have to be used. In particular, we have adopted a Monte Carlo approach: random variables have been simulated by MATLAB high level random number generators, while for the approximation of expected values, scenario-based averages have been evaluated by exploiting MATLAB fast matrix-computation facilities. These two MATLAB specific properties have allowed to break down computational costs, in

Table 1: Parameters for the financial model

P=100	$r = 4\%$	$\sigma = 25\%$	$\pi = 0.70$	$g = 5\%$

Table 2: Calibrated parameters for the mortality model

a=0.001	b=0.087	σ_μ=0.021	$\mu_{65}(0)$=0.01147	λ=0.4

terms of complexity and time. In addition, among the numerical approaches proposed in literature for the approximate numerical solution of SDEs we have chosen the Euler-Maruyama method (Kloeden and Platen [1999]). In the described model we consider a representative individual aged 65 at the inception of the contract, $t = 0$, and whose limiting age is set to be 120. Therefore, we focus on the time interval $[0, 55]$. We require to use a number of samples sufficiently large and a time step sufficiently small to make numerical results more accurate. Thus, we have chosen to simulate 100000 trajectories of the Wiener process using a step-size $\Delta t = 0.02$, so 2750 points for the discretization of the interval $[0, 55]$.
Table 1 summarizes the parameters' values for the financial component of the model used in our simulation as a base case, unless stated otherwise. The values of the parameters a, b, and σ_μ in the intensity mortality dynamics are those reported in Fung et al. [2014] and obtained by calibrating the survival curve implied by the mortality model to the survival curve obtained from population data using the Australian Life Tables 2005-2007. Data are reported in Table 2.

We have computed the fair fee rates using both valuation formulae (19) and (23), creating ad hoc MATLAB codes (called respectively Algorithm1 and Algorithm 2). Results are illustrated in Table 3. The two valuation formulae have been tested to be equivalent also computationally. The reader, in effect, can note the negligible gap between the fair fee rates computed through the two Algorithms. However, since Algorithm 2 is a lot more efficient in terms of computing time, all the next experiments have been carried out with this procedure. In addition, we have conducted a sensitivity analysis investigating the relationship between the fair fee rate and important financial and demographic factors, such as interest rates, the volatility of the reference fund, the market price coefficient of the systematic mortality risk and the volatility of the mortality intensity. Moreover, each experiment has been fulfilled considering the effect of varying guaranteed withdrawal rates. Results are reported in Table 3.

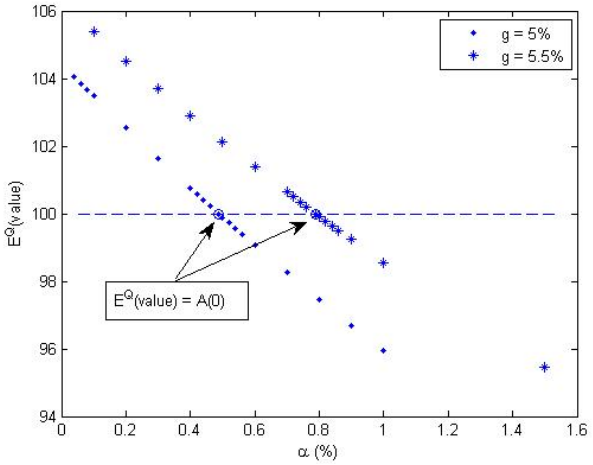

Figure 1: Sensitivity of the initial contract value with respect to the fee rate α for policyholders aged 65

2.5.1 Withdrawal rate, g

When the withdrawal rate g increases, there are two possible effects: on the one hand, the periodic amount withdrawn ($G = gA(0)$) increases and consequently also the value of the living benefits increases; on the other hand, just because the policyholder can withdraw a greater amount, the VA sub account value decreases; thus, the value of the death benefit decreases. Overall, the relationship between g and the value of the living benefit prevails (being a guaranteed amount) so that the contract becomes more valuable as g increases. Figure 1 shows the curve representing the initial contract value as a decreasing function of α. When g goes up, this curve shifts to the right. Fees charged to make the contract value fair are graphically obtained through the intersection between the curve and the horizontal line corresponding to the initial premium's value. Therefore, as the withdrawal rate increases, also fair fee rates will be greater. We can note the positive effect of the guaranteed withdrawal rate on the GLWB value (and consequently on the fair fee rates) in all the following analyses.

2.5.2 Interest rate, r

As the interest rate r increases, the discounted value of each withdrawal decreases; so the value of the living benefit decreases at each time point. Instead, concerning the value of the death benefit there are opposite effects. In fact, on the one hand, a greater risk-free rate increases the account value since r enters its drift; on the other hand, however, the discounting takes place at a higher rate, so the discounted value of the death benefit decreases.

Overall, these two effects balance out, so the contribution of death benefits disappears. A higher interest rate, therefore, results in a translation on the left of the curve reported in figure 1; consequently fair fee rates will be lower. A remark beyond the model, in economic terms, is also possible. Recall that the GLWB option allows the policyholder to withdraw a periodic amount independently from the market performance. Therefore, other things being equal, when the interest rate level is high, policyholders will prefer more profitable investments. In this case, to attract sales leads, GLWB providers will charge lower fee rates and will suffer a challenging situation. On the contrary, a low interest rate level will encourage clients to invest in these contracts; consequently their demand will increase and so will do the required fee rates.

2.5.3 Volatility of the investment account, $\pi \cdot \sigma$

In this analysis we have kept σ constant at the level of 25% and set $\pi \in \{0, 0.3, 0.5, 0.7, 1\}$, so that the study represents also the sensitivity of the fair fee rate with respect to the equity exposure π. As the volatility increases, the value of the living benefit does not change because the withdrawals are constant over time and do not depend on the account value, while the value of the death benefit increases. In fact, the higher is the volatility $\pi\sigma$ the higher is the VA account value. The positive relationship between α^* and $\pi \cdot \sigma$ can be explained with financial theory: options are more expensive when volatility is high. Recall that at inception of the contract (for some products also during the term of the contract) the insured has the possibility to influence the volatility by choosing the underlying fund from a selection of mutual funds. Since for some products offered in the market the fees do not depend on the fund choice, this possibility presents another valuable option for the policyholder. Thus, an important risk management tool for insurers offering VA guarantees is the strict limitation and control of the types of underlying funds offered within these products.

2.5.4 Market price coefficient of the systematic mortality risk, λ

We can note that, when λ is positive and increases, the effect on the mortality intensity μ is negative; so it will be an improvement in the survival probability. Higher life expectancy, so also higher probabilities of GLWB option activation, lead insurance company to increase the charged fees. Therefore, the relation between λ and α^* is positive.

2.5.5 Volatility of the mortality intensity, σ_μ

The effect of the volatility parameter of the mortality intensity σ_μ on the fair fee rate α^* is similar to that of the market price coefficient of the mortality risk λ. In fact, an increase in σ_μ leads to a decrease in the mortality intensity

μ, so to an improvement in the survival probability. Hence, higher volatility of mortality leads not only to higher uncertainty about the timing of death of an individual, but also to an increase in the survival probability. To face this situation, the insurance company, other things being equal, has to charge higher fees.

3 Extended model: stochastic interest rate and/or volatility

Until now, the theoretical model described for the pricing of the GLWB option has rested upon some assumptions that are, to some extent, "counter-factual". The valuation has been performed in a Black and Scholes economy: the sub-account value has been assumed to follow a geometric Brownian motion, thus with a constant volatility, and the term structure of interest rates has been assumed to be constant. These hypotheses, however, do not find justification in the financial markets. For the purpose of considering a model that is closer to the market, we sought to weaken these misspecifications. In particular, we consider a generalization of the proposed model, in which the volatility of the underlying portfolio and the interest rate are considered to be stochastic processes rather than a constant.

3.1 Stochastic interest rate

The assumption of deterministic interest rates, which can be acceptable for short-term options, is not realistic for medium or long-term contracts such as life insurance products. GLWB contracts are investment vehicles with a long term horizon and as such they are very sensitive to interest rate movements which are by nature uncertain. A stochastic modeling of the term structure is therefore appropriate. Many models have been developed in literature (see Shao [2012]). Among them, we will refer to the Cox-Ingersoll-Ross (CIR) model. Consider that, for our pricing purposes, in what follows we will express all the dynamics directly under the $\mathbb{Q}$ risk neutral measure. The CIR simplest version describes the dynamics of the interest rate $r(t)$ as a solution of the following stochastic differential equation:

$$\begin{cases} dr(t) = k(\bar{r} - r(t))dt + \eta\sqrt{r(t)}dW_r^{\mathbb{Q}}(t) \\ r(0) > 0 \end{cases} \tag{32}$$

where $k > 0$ determines the speed of adjustment of the interest rate towards its theoretical mean $\bar{r} > 0$, $\eta > 0$ controls the volatility of the interest rate, and W_r is a $\mathbb{Q}$-standard Brownian motion. This process has some appealing properties from an applied point of view. In particular, the condition

$$2k\bar{r} \geq \eta^2$$

Table 3: Fair guarantee fees (%) using Algorithm 1 and Algorithm 2

		Algorithm 1			Algorithm 2		
		g			g		
		4.5%	5%	5.5%	4.5%	5%	5.5%
r	1%	1.7909	3.3066	7.3373	1.7898	3.3083	7.3825
	2%	0.9593	1.6279	2.8531	0.9550	1.6246	2.8505
	3%	0.5346	0.8833	1.4477	0.5270	0.8774	1.4422
	4%	0.3009	0.4963	0.7969	0.2905	0.4874	0.7891
	5%	0.1686	0.2812	0.4517	0.1557	0.2698	0.4415
	6%	0.0932	0.1584	0.2576	0.0781	0.1444	0.2450
	7%	0.0503	0.0879	0.1458	0.0336	0.0718	0.1307
	8%	0.0260	0.0473	0.0809	0.0079	0.0296	0.0638
$\pi \cdot \sigma$	0%	0.0003	0.0040	0.0454	0.0012	0.0061	0.0492
	7.5%	0.0377	0.0991	0.2317	0.0350	0.0979	0.2326
	12.5%	0.1413	0.2691	0.4874	0.1350	0.2645	0.4847
	17.5%	0.3009	0.4963	0.7969	0.2905	0.4874	0.7891
	25%	0.6029	0.8932	1.3064	0.5820	0.8725	1.2849
λ	-0.4	0.2354	0.3896	0.6237	0.2266	0.3827	0.6181
	0	0.2662	0.4396	0.7043	0.2568	0.4321	0.6981
	0.4	0.3009	0.4963	0.7969	0.2905	0.4874	0.7891
	0.8	0.3397	0.5606	0.9039	0.3275	0.5500	0.8948
	1.2	0.3825	0.6330	1.0279	0.3722	0.6240	1.0196
	1.6	0.4283	0.7139	1.1723	0.4300	0.7166	1.1752
σ_μ	0	0.2340	0.3883	0.6236	0.2185	0.3755	0.6121
	0.0110	0.2588	0.4286	0.6881	0.2463	0.4183	0.6789
	0.0210	0.3009	0.4963	0.7969	0.2905	0.4874	0.7891
	0.0310	0.3626	0.6008	0.9715	0.3565	0.5955	0.9675
	0.0410	0.4296	0.7362	1.2232	0.4450	0.7459	1.2267
	0.0510	0.4846	0.8903	1.5576	0.5574	0.9393	1.5800

would ensure that the origin is inaccessible to the process, so that we can grant that $r(t)$ remains positive; moreover, the interest rate is elastically pulled towards the long-term constant value $\bar{r}$ at a speed controlled by k (mean-reverting). These properties are attractive in modeling real-life interest rates.
Considering a CIR model for the interest rate, the new dynamics of the VA sub-account become:

$$dA(t) = (r(t) - \alpha)A(t)dt - Gdt + \pi\sigma A(t)dW_S^{\mathbb{Q}}(t) \tag{33}$$

Therefore, our model is specified through the following system of stochastic differential equations:

$$\begin{cases} dA(t) = (r(t) - \alpha)A(t)dt - Gdt + \pi\sigma A(t)dW_S^{\mathbb{Q}}(t), & A(\cdot) \geq 0, A(0) > 0 \\ dr(t) = k(\bar{r} - r(t))dt + \eta\sqrt{r(t)}dW_r^{\mathbb{Q}}(t), & r(0) > 0 \\ d\mu_{x+t}(t) = (a + (b - \lambda\sigma_\mu)\mu_{x+t}(t))dt + \sigma_\mu\sqrt{\mu_{x+t}(t)}dW_M^{\mathbb{Q}}(t), & \mu(0) > 0 \end{cases} \tag{34}$$

where $dW_S(t)dW_r(t) = \rho_{S,r}dt$, with $|\rho_{S,r}| \leq 1$, is the correlation between the reference fund and interest rate. $W_r^{\mathbb{Q}}$ and $W_M^{\mathbb{Q}}$ are instead considered independent, as well as we took $W_S^{\mathbb{Q}}$ and $W_M^{\mathbb{Q}}$. More explicitly, we can rewrite system (34) as:

$$\begin{cases} dA(t) = (r(t) - \alpha)A(t)dt - Gdt + \pi\sigma A(t)\left(\rho_{S,r}dW_r^{\mathbb{Q}}(t) + \sqrt{1 - \rho_{S,r}^2}d\tilde{W}_S^{\mathbb{Q}}(t)\right) \\ dr(t) = k(\bar{r} - r(t))dt + \eta\sqrt{r(t)}dW_r^{\mathbb{Q}}(t) \\ d\mu_{x+t}(t) = (a + (b - \lambda\sigma_\mu)\mu_{x+t}(t))dt + \sigma_\mu\sqrt{\mu_{x+t}(t)}dW_M^{\mathbb{Q}}(t) \end{cases} \tag{35}$$

where $\tilde{W}_S$ and W_r are independent Brownian motions, $A(0), r(0), \mu(0) > 0$ and $A(\cdot) \geq 0$
The valuation formula (24) becomes:

$$\alpha^* : E^{\mathbb{Q}}\left[G\int_0^\tau e^{-\int_0^t r(u)du}dt + e^{-\int_0^\tau r(u)du}A(\tau) - A(0)\right] = 0 \tag{36}$$

A drawback of the CIR process is that the SDE (32) is not explicitly solvable. Our pricing approach, however, requires to solve the problem of simulating a CIR process. As before, we adopt an Euler-Maruyama approximation scheme. However, some problems arise. A theoretical difficulty concerns the square-root term. In fact, the square root is not globally Lipschitz. Therefore the usual theorems leading to strong or weak convergence (Kloeden and Platen [1999]), which require the drift and diffusion coefficients to satisfy a linear growth condition, cannot be applied. Hence, the convergence of the Euler scheme is not guaranteed. Various methods have been proposed to solve this problem and to prove the convergence (the interested reader can refer to Lord et al. [2010]). There exists another problem of practical nature.

Table 4: Calibrated parameters for the CIR process

$k = 0.01$	$\eta = 0.01$	$r(0) = \bar{r} = 0.02$	$\rho_{S,r} = 0.2$

In fact, despite the domain of the square root process being the nonnegative real line, the discretization is not guaranteed to be the same. For any choice of the time grid, indeed, the probability of the interest rate becoming negative at the next time step is strictly greater than zero. Practitioners have therefore often opted for a quick "fix" by either setting the process equal to zero whenever it attains a negative value (so considering only the positive part of the process), or by reflecting it in the origin, and continuing from there on (so taking advantage of the absolute value function). These fixes are often referred to as *absorption* or *reflection* (Lord et al. [2010]). In what follows, we use $x^+ = \max(x, 0)$ as fixing function. Therefore, we consider only the positive part of the process:

$$r(t + \Delta t) = [r(t) + k(\bar{r} - r(t))\Delta t + \eta\sqrt{r(t)}\Delta W_r^{\mathbb{Q}}(t)]^+ \tag{37}$$

The parameters of the CIR process are those reported in Grzelak and Oosterlee [2011] and summarized in Table 4.

As in the previous section, we have conducted sensitivity analyses in order to study the relationship between the fair fee rate and the same financial and demographic factors. Similarly we note a positive relation of the fair fee rate α^* with the market price coefficient of mortality risk λ (see figure 2), with the volatility of the mortality intensity σ_μ (see figure 3) and with the volatility of the investment account $\pi \cdot \sigma$ (see figure 4).

We have then analyzed the impact of varying the parameters of the interest rate model on the fair fee rate: the mean reversion coefficient k and the rate of diffusion η.

An increase in the mean reversion coefficient k, in general, doesn't have a clear effect on the contract fair price; its contribution, in fact, depends on the sign of the difference $\bar{r} - r(t)$. In particular, if $\bar{r} > r$, when k increases, the CIR drift factor will be greater, pushing the interest rate upwards. This will lead to a smaller value of the GLWB contract and consequently of the fair fee rate. When $r > \bar{r}$, the relation is inverse, even if interest rates are already high, so the impact on the contract fair price could be also in this case negative.

Analogue considerations hold for the diffusion coefficient η. In fact, an increase in the volatility of interest rates η would amplify the effect of the random shock on the rate. Therefore, depending on the the sign of the $dW_r(t)$ term (recall that Brownian motion's increments are normally distributed with expectation zero), its impact on the interest rate (and consequently on the fair fee rate) could be both positive and negative.

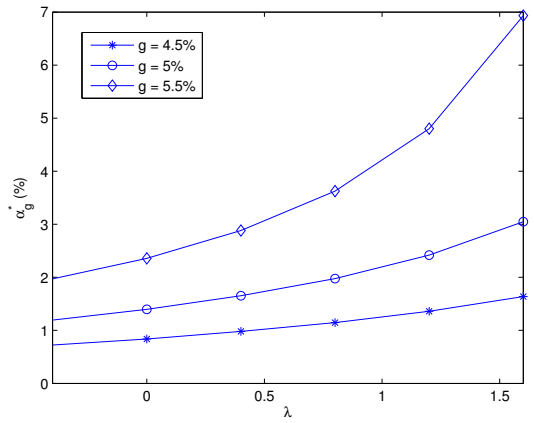

Figure 2: Sensitivity of the fair fee rate α^* with respect to the market price coefficient of systematic mortality risk λ for policyholders aged 65

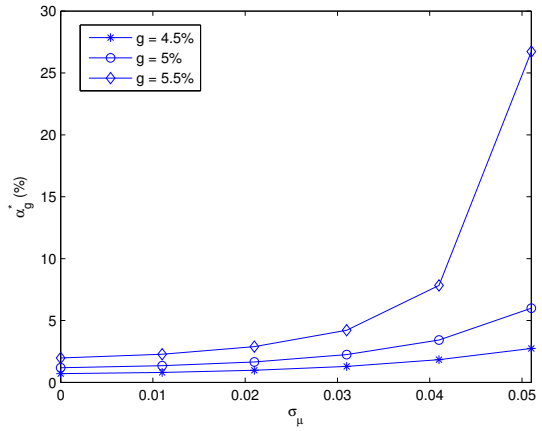

Figure 3: Sensitivity of the fair fee rate α^* with respect to the volatility parameter of mortality intensity σ_μ for policyholders aged 65

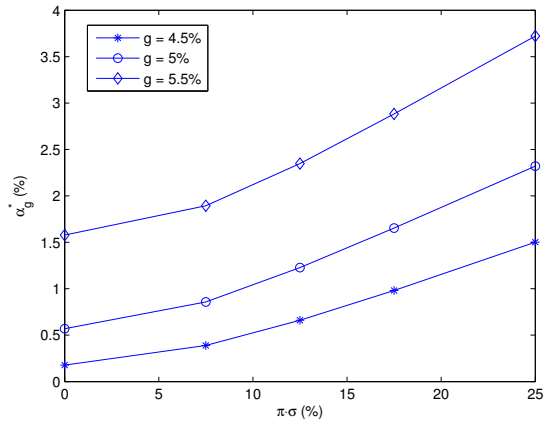

Figure 4: Sensitivity of the fair fee rate α^* with respect to the volatility of the investment account $\pi\sigma$ for policyholders aged 65

In addition, we have studied the impact on the fair fee rate of the long-run mean of r, $\bar{r}$, and of its initial value, $r(0)$, considering different values for k and η. Results are reported in Tables 5 and 6. In particular, if $k = 0$, so when only the diffusion of the CIR process is present, Table 5 confirms that the fair contract price is not dependent on the long-run mean of r. Moreover, if $\eta = 0$, so if the random shock on the rate is zero, we are in the case of deterministic interest rates, and more precisely, if in addition $\bar{r} = r$ (the diagonal of the first table in Table 6), we are considering constant interest rates.

3.2 Stochastic volatility

It is widely recognized that financial models which consider a constant volatility parameter (such as the Black-Scholes one) are no longer sufficient to capture modern market phenomena, especially since the 1987 crash. Empirical studies of stock price returns, in fact, show that volatility exhibits "random" characteristics. The natural extension of these models that has been pursued in the literature and in practice, suggests to modify the specification of volatility to make it a stochastic process. Stochastic volatility models predict that volatility itself follows a stochastic process (Fouque et al. [2000]):

$$\{\sigma(t), t \geq 0\} \tag{38}$$

with

$$\sigma(t) = f(v(t)) \tag{39}$$

Table 5: Sensitivity of the fair fee rate α^* with respect to the long-run mean $\bar{r}$ and to the initial value $r(0)$, with different values for the mean reversion coefficient k

$k = 0.00$		$r(0)$		
		0.01	0.02	0.04
$\bar{r}$	0.01	3.3178	1.6570	0.5244
	0.02	3.3178	1.6570	0.5244
	0.04	3.3178	1.6570	0.5244

$k = 0.01$		$r(0)$		
		0.01	0.02	0.04
$\bar{r}$	0.01	3.3138	1.7427	0.5978
	0.02	3.1057	1.6531	0.5710
	0.04	2.7427	1.4911	0.5216

$k = 0.50$		$r(0)$		
		0.01	0.02	0.04
$\bar{r}$	0.01	3.2854	2.9273	2.3605
	0.02	1.7880	1.6237	1.3489
	0.04	0.6461	0.5925	0.4998

$k = 1.00$		$r(0)$		
		0.01	0.02	0.04
$\bar{r}$	0.01	3.2832	3.0887	2.7471
	0.02	1.7044	1.6214	1.4706
	0.04	0.5680	0.5434	0.4980

Table 6: Sensitivity of the fair fee rate α^* with respect to the long-run mean $\bar{r}$ and to the initial value $r(0)$, with different values for the rate of diffusion η

$\eta = 0.000$		$r(0)$		
		0.01	0.02	0.04
$\bar{r}$	0.01	3.2809	1.7078	0.5702
	0.02	3.0729	1.6190	0.5441
	0.04	2.7104	1.4583	0.4961

$\eta = 0.005$		$r(0)$		
		0.01	0.02	0.04
$\bar{r}$	0.01	3.2935	1.7216	0.5813
	0.02	3.0856	1.6325	0.5550
	0.04	2.7230	1.4714	0.5065

$\eta = 0.010$		$r(0)$		
		0.01	0.02	0.04
$\bar{r}$	0.01	3.6268	2.0990	0.8786
	0.02	3.4304	2.0052	0.8451
	0.04	3.0778	1.8318	0.7822

$\eta = 0.020$		$r(0)$		
		0.01	0.02	0.04
$\bar{r}$	0.01	3.3138	1.7427	0.5978
	0.02	3.1057	1.6531	0.5710
	0.04	2.7427	1.4911	0.5216

$\eta = 0.050$		$r(0)$		
		0.01	0.02	0.04
$\bar{r}$	0.01	3.3760	1.8052	0.6452
	0.02	3.1668	1.7142	0.6172
	0.04	2.8019	1.5493	0.5654

Table 7: Calibrated parameters for the volatility process

$\theta = 0.3$	$\gamma = 0.6$	$v(0) = \bar{v} = 0.05$	$\rho_{S,v} = -0.3$

where $f(h) > 0 \ \forall h \in \mathbb{R}$ and $\{v(t), t \geq 0\}$ represents a stochastic process. Various alternative models have been proposed in literature, differentiated for the driving process $v(t)$ and for the function f. Among them, we have chosen to consider the Heston model. The new dynamics of the VA sub-account becomes:

$$dA(t) = (r - \alpha)A(t)dt - Gdt + \pi\sigma(t)A(t)dW_S^{\mathbb{Q}}(t) \tag{40}$$

where $\sigma(t) = \sqrt{v(t)}$ and

$$dv(t) = \theta(\bar{v} - v(t))dt + \gamma\sqrt{v(t)}dW_v(t) \tag{41}$$

Recall that for our pricing purposes, we have expressed all the dynamics directly under the $\mathbb{Q}$ risk neutral measure. As described for the interest rates CIR process, also in this case the use of an Euler discretization can give rise to a problem of practical nature. In fact, it is not guaranteed the positivity of the domain of the square root process. So, as in the interest rate case, in what follows, we use $x^+ = \max(x, 0)$ as fixing function.
The parameters of the volatility process are those reported in Grzelak and Oosterlee [2011] and summarized in Table 7. Sensitivity analyses have been conducted in order to study the relationship between the fair fee rate and the same financial and demographic factors, as well as in respect of the parameters of the Heston model. Results and considerations are analogue to those for the CIR interest rates process above described.

3.3 The Heston-CIR hybrid model

Derivatives that depend on a variety of factors can be modeled through the specification of a system of stochastic differential equations, that correspond to the involved state variables. By correlating the SDEs from the different asset classes one can define so-called hybrid models. In our case, in particular, we have combined the stochastic processes described in the previous subsections for the term structure of interest rates and the volatility of the underlying account.
The generalized model, under the $\mathbb{Q}$ measure, can be expressed in the fol-

lowing way:

$$\begin{cases} dA(t) = & [(r(t)-\alpha)A(t)-G]dt+ & \pi\sqrt{v(t)}A(t)dW_S^{\mathbb{Q}}(t) \\ dv(t) = & \theta(\bar{v}-v(t))dt+ & \gamma\sqrt{v(t)}dW_v^{\mathbb{Q}}(t) \\ dr(t) = & k(\bar{r}-r(t))dt+ & \eta\sqrt{r(t)}dW_r^{\mathbb{Q}}(t) \\ d\mu_{x+t}(t) = & [a+(b-\lambda\sigma_\mu)\mu_{x+t}(t)]dt+ & \sigma_\mu\sqrt{\mu_{x+t}(t)}dW_M^{\mathbb{Q}}(t) \end{cases} \tag{42}$$

with $A(0), v(0), r(0), \mu(0) > 0$ and $A(\cdot) \geq 0$. The various random factors may be independent, but more realistically, there is often correlation between them. In our model, we consider 4 Wiener processes: those related to VA-sub account, interest rate and volatility processes are all correlated each other, while we assume independence between financial and systematic mortality risk, so $\rho_{M,S} = \rho_{M,v} = \rho_{M,r} = 0$.
There is not a closed form solution of our hybrid model, therefore numerical approximation has to be employed.
To construct discretized correlated Wiener processes for use in SDE solvers, we begin with a desired correlation matrix that we would like to specify for the Wiener processes W_S, W_v, W_r.

$$C = \begin{bmatrix} \rho_{S,S} & \rho_{S,v} & \rho_{S,r} \\ \rho_{v,S} & \rho_{v,v} & \rho_{v,r} \\ \rho_{r,S} & \rho_{r,v} & \rho_{r,r} \end{bmatrix}$$

C is a symmetric matrix with units on the main diagonal. To simplify and lighten the notation, we set

$$\rho_1 = \rho_{S,v} \qquad \rho_2 = \rho_{S,r} \qquad \rho_3 = \rho_{v,r}$$

So, we have:

$$C = \begin{bmatrix} 1 & \rho_1 & \rho_2 \\ * & 1 & \rho_3 \\ * & * & 1 \end{bmatrix}$$

Our aim is to write the system of SDEs (42) in terms of independent Brownian motions in order to simulate the involved processes.
We make use of the Cholesky decomposition to factorize the positive definite matrix C into the product of a unique lower triangular matrix L with strictly positive entries on the main diagonal and its transpose:

$$C = LL^T$$

with

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \rho_1 & \sqrt{1-\rho_1^2} & 0 \\ \rho_2 & \frac{\rho_3-\rho_1\rho_2}{\sqrt{1-\rho_1^2}} & \sqrt{1-\rho_2^2-\left(\frac{\rho_3-\rho_1\rho_2}{\sqrt{1-\rho_1^2}}\right)^2} \end{bmatrix}$$

Table 8: Sensitivity of the fair fee rate α^* with respect to σ_μ and λ with different values for the correlation coefficient $\rho_{r,v}$

$\rho_{r,v} = 0.00$		g 4.5%	5%	5.5%	$\rho_{r,v} = 0.15$		g 4.5%	5%	5.5%
σ_μ	0	0.6854	1.1021	1.8295	σ_μ	0	0.6792	1.0971	1.8262
	0.0110	0.7692	1.2520	2.1275		0.0110	0.7622	1.2461	2.1220
	0.0210	0.9210	1.5367	2.7266		0.0210	0.9161	1.5317	2.7219
	0.0310	1.2178	2.1167	4.0845		0.0310	1.2150	2.1152	4.0845
	0.0410	1.7497	3.2978	7.8394		0.0410	1.7483	3.2980	7.8472
	0.0510	2.6391	5.9230	28.6995		0.0510	2.6426	5.9275	28.7127
λ	-0.4	0.6989	1.1142	1.8327	λ	-0.4	0.6928	1.1080	1.8265
	0	0.7966	1.2966	2.2051		0	0.7902	1.2906	2.1990
	0.4	0.9210	1.5367	2.7266		0.4	0.9161	1.5317	2.7219
	0.8	1.0830	1.8580	3.4897		0.8	1.0792	1.8549	3.4875
	1.2	1.2893	2.2977	4.7085		1.2	1.2835	2.2932	4.7077
	1.6	1.5520	2.9283	6.9568		1.6	1.5462	2.9230	6.9673

L is called the Cholesky factor of C and it can be interpreted as a generalized square root of C.
With these considerations in mind, with the help of the upper matrix, we can rewrite the subsystem of the first three SDEs in (42) as:

$$\begin{bmatrix} dA(t) \\ dv(t) \\ dr(t) \end{bmatrix} = \begin{bmatrix} (r(t) - \alpha)A(t) - G \\ \theta(\bar{v} - v(t)) \\ k(\bar{r} - r(t)) \end{bmatrix} dt + B \begin{bmatrix} d\tilde{W}_S^{\mathbb{Q}}(t) \\ d\tilde{W}_v^{\mathbb{Q}}(t) \\ d\tilde{W}_r^{\mathbb{Q}}(t) \end{bmatrix}$$

where

$$B = \begin{bmatrix} \pi\sqrt{v(t)} & \rho_1\pi\sqrt{v(t)} & \rho_2\pi\sqrt{v(t)} \\ 0 & \sqrt{1-\rho_1^2}\gamma\sqrt{v(t)} & \frac{\rho_3-\rho_1\rho_2}{\sqrt{1-\rho_1^2}}\gamma\sqrt{v(t)} \\ 0 & 0 & \sqrt{1-\rho_2^2-\left(\frac{\rho_3-\rho_1\rho_2}{\sqrt{1-\rho_1^2}}\right)^2}\eta\sqrt{r(t)} \end{bmatrix}$$

and $d\tilde{W}_i^{\mathbb{Q}}(t)$ $(i = S, v, r)$ are independent Brownian motions. After the Euler discretization of the involved processes, we have proceeded to price the GLWB option using Algorithm 2. We used the values reported in the previous Tables 4 and 7 for the parameters of the CIR and Heston processes. In addition, we set $\rho_{r,v}$ equal to 0.15. Numerical results are reported in Table 8.

In particular, as before, we have conducted sensitivity analyses in order to study the relationship between the fair fee rate and the demographic factors already examined, i.e. the market price coefficient of mortality risk

Table 9: Summary comparison

		$\eta = 0$ $\gamma = 0$	$\eta = 0.01$ $\gamma = 0$	$\eta = 0$ $\gamma = 0.6$	$\eta = 0.01$ $\gamma = 0.6$
σ_μ	0	0.9882	1.0122	1.0796	1.0971
	0.0110	1.1386	1.1659	1.2256	1.2461
	0.0210	1.4335	1.4669	1.5054	1.5317
	0.0310	1.9964	2.0417	2.0765	2.1152
	0.0410	3.1473	3.2212	3.2305	3.2980
	0.0510	5.6904	5.8595	5.7648	5.9275
λ	-0.4	1.0044	1.0292	1.0906	1.1080
	0	1.1914	1.2198	1.2693	1.2906
	0.4	1.4335	1.4669	1.5054	1.5317
	0.8	1.7492	1.7893	1.8217	1.8549
	1.2	2.1741	2.2237	2.2507	2.2932
	1.6	2.7910	2.8555	2.8658	2.9230

and the volatility of the mortality intensity. As in the previous experiments, we can note a positive relation of the fair fee rate α^* with λ and with σ_μ. In addition, as in many papers on these topics the correlation coefficient $\rho_{r,v}$ is set equal to zero, we have considered also this hypothesis. Numerical analyses show the stability of the results: little changes in the correlation coefficient correspond to little changes in the fair fee rates.

4 Conclusions

In conclusion, we have summarized in Table 9 all the results obtained in order to compare them. In particular we show how the fair price of the GLWB contract changes when we consider the basic model (first column), a stochastic process only for the term structure of interest rates (second column), a stochastic process only for the volatility of the reference fund (third column) or a combined stochastic model (last column). Numerical results confirm that introducing random shocks on interest rates and/or volatility increases the value of the fees that GLWB' issuers have to charge in order to fairly price the contract. Therefore, a more general stochastic approach, especially that obtained allowing both interest rates and volatility to vary randomly, makes the contract undoubtedly more expensive, but it is more able to describe the real fluctuations of the market, so it is recommended in order to avoid underestimation of the liabilities of the insurance company.

References

A.R. Bacinello, P. Millossovich, A. Olivieri, and E. Pitacco. Variable annuities: a unifying valuation approach. *Insurance: Mathematics and Economics*, 49:285–297, 2011.

A.R. Bacinello, P. Millossovich, and A. Montealegre. The valuation of GMWB variable annuities under alternative fund distributions and policyholder behaviours. *Scandinavian Actuarial Journal*, 2014.

D. Bauer, A. Kling, and J. Russ. A universal pricing framework for guaranteed minimum benefits in variable annuities. *ASTIN Bullettin*, 38(2): 621–651, 2008.

E. Biffis. Affine processes for dynamic mortality and actuarial valuations. *Insurance: Mathematics and Economics*, 37(3):443–468, 2005.

A. Cairns, D. Blake, and K. Dowd. Pricing death: framework for the valuation and securitization of mortality risk. *ASTIN Bullettin*, 36(1):79–120, 2006.

Z. Chen and P. A. Forsyth. A numerical scheme for the impulse control formulatio for pricing variable annuities with a guaranteed minimum withdrawal benefit (gmwb). *Numerische Mathematik*, 109(4):535–569, 2008.

M. Dahl and T. Moller. Valuation and hedging of life insurance liabilities with systematic mortality risk. *Insurance: Mathematics and Economics*, 39:193–217, 2006.

M. Dai, Y. K. Kwok, and J. Zong. Guaranteed minimum withdrawal benefit in variable annuities. *Mathematical Finance*, 18(4):595–611, 2008.

J.P. Fouque, G. Papanicolaou, and K. R. Sircar. *Derivatives in Financial Markets with Stochastic Volatility.* Cambridge University Press, 2000.

M.C. Fung, K. Ignatieva, and M. Sherris. Systematic mortality risk: An analysis of guaranteed lifetime withdrawal benefits in variable annuities. *Insurance: Mathematics and Economics*, 58:103–115, 2014.

L. Grzelak and K. Oosterlee. On the Heston model with stochastic interest rates. *SIAM Journal of Financial Mathematics*, 2(1):255–286, 2011.

D. Holz, A. Kling, and J. Russ. GMWB for life: an analysis of lifelong withdrawal guarantees. *Zeitschrift für die gesamte Versicherungswissenschaft*, 101(3):305–325, 2012.

C.B. Hyndman and M. Wenger. Valuation perspectives and decompositions for variable annuities with gmwb riders. *Insurance: Mathematics and Economics*, 55:283–290, 2014.

A. Kling, F. Ruez, and J. Ruß. The impact of stochastic volatility on pricing, hedging, and hedge efficiency of variable annuities guarantees. *ASTIN Bullettin*, 41(2):511–545, 2011.

P. E. Kloeden and E. Platen. *Numerical solution of stochastic differential equations*. Springer, 1999.

M. C. Ledlie, D. P. Corry, G. S. Finkelstein, A. J. Ritchie, K. Su, and D. C. E. Wilson. Variable annuities. *British Actuarial Journal*, 14(2): 327–389, 2008.

R. Lord, R. Koekkoek, and D. V.Dijk. A comparison of biased simulation schemes for stochastic volatility models. *Quantitative Finance*, 10(2):177–194, 2010.

M. A. Milevsky and T. S. Salisbury. Financial valuation of guaranteed minimum withdrawal benefits. *Insurance: Mathematics and Economics*, 38(1): 21–38, 2006.

J. Peng, K. Leung, and Y. K. Kwok. Pricing guaranteed minimum withdrawal benefits under stochastic interest rates. *Quantitative Finance*, 12 (6):933–941, 2012.

G. Piscopo and S. Haberman. The valuation of guaranteed lifelong withdrawal benefit options in variable annuity contracts and the impact of mortality risk. *North American Actuarial Journal*, 15(1):59–76, 2011.

P. Shah and D. Bertsimas. An analysis of the guaranteed withdrawal benefits for life option. *SSRN Working Paper*, 2008.

A. Shao. *A fast and exact simulation for CIR process*. PhD thesis, University of Florida, 2012.