Survival in Speculative Markets*

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Abstract

In a stochastic exchange economy where, due to beliefs' heterogeneity, agents engage in speculative trade, I investigate the Market Selection Hypothesis that speculation rewards the agent with the most accurate beliefs. Assuming that agents have Epstein-Zin preferences and that markets are complete, I derive sufficient conditions for agents' survival in terms of saving and portfolio decisions, and show that the Market Selection Hypothesis fails generically. Beliefs heterogeneity may persist in the long-run or speculation may cause the agent with the most accurate beliefs to vanish. Failures occur because agents' portfolio returns depend not only on beliefs accuracy but also on risk preferences, through the comparison with the optimal growth portfolio. The latter plays no role in CRRA economies because, due to the interdependence of relative risk aversion and intertemporal elasticity of substitution, portfolio returns not related to beliefs accuracy are compensated by the component of saving induced by speculation.

Keywords: Heterogeneous Beliefs; Speculation; Market Selection Hypothesis; Asset Pricing; Optimal Growth Portfolio; Epstein-Zin Preferences; Saving under Uncertainty.
JEL Classification: D53, D01, G12, G14, G11, E21

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1 Introduction

The dominant academic view of financial markets is that they facilitate hedging and risk diversification. Indeed, benchmark equilibrium models of asset pricing, such as Lucas' model and the CAPM, characterize the relation between assets' equilibrium returns and agents' characteristics under the assumption that hedging and risk diversification are the only motives for holding risky assets.

A complementary view is that trade also occurs due to agents' disagreement about assets' return distributions. In fact, also in standard asset market models, agents' beliefs heterogeneity make them willing to hold risky positions different from those they would have held under pure hedging. These positions are speculative, in that they include a bet on the future dynamics of assets' fundamentals.

Although speculative incentives are certainly present in financial markets, a widespread position of financial economists is that speculation cannot have long-run consequences, and thus its investigation cannot help to characterize assets' returns in equilibrium. As suggested by the Market Selection Hypothesis (MSH) of Friedman (1953) applied to financial markets, investors with accurate beliefs should earn superior returns by 'betting' against those with inaccurate beliefs and thus drive them out of the market. Indeed, if markets are complete, each agent trades to allocate his future consumption on the path which he believes as more likely. In equilibrium, everything else being equal, the agent whose beliefs assign the highest likelihood to paths that are actually realized should hold everything in the long run. The argument, rigorously established by Sandroni (2000) and Blume and Easley (2006), shows that speculation validates the MSH: investors with inaccurate beliefs are driven out of the market, leaving no room for further speculation, and bringing prices close to their fundamental values.[1]

Despite the importance of the result, the exact role of portfolio and saving decisions for its validity is still unclear. In bounded economies where agents maximize

[1]Note, however, that under specific assumptions on agents' preferences, vanishing traders may have a price and/or portfolio impact, see Kogan et al. (2006, 2009) and Cvitanić and Malamud (2011).

a time separable expected utility, Sandroni (2000) and Blume and Easley (2006) show that an agent's long-run consumption share is determined by a survival index that depends both on beliefs accuracy and discount factors. For log-economies beliefs accuracy corresponds to portfolio expected log-returns and the discount factor is a saving rate, but other preferences lead to different optimal saving and portfolio decisions, and thus to different speculative positions. However, the survival index and long-run outcomes remain the same.[2]

In this paper, by studying saving and portfolio decisions under beliefs heterogeneity, I show that there exists economies where speculation, by failing to support the MSH, has asset pricing implications also for the long-run. I consider a discrete-time exchange economy with a complete asset market, and assume that agents have heterogeneous beliefs and maximize (possibly heterogeneous) Epstein-Zin recursive utilities. Although Epstein-Zin preferences are not key for my results, they are flexible enough to disentangle intertemporal and intra-states consumption decisions. Moreover, under some restrictions, they allow for a characterization of optimal saving and portfolio decisions in equilibrium, so that sufficient conditions for agents' survival based upon them can be established.[3,4]

I find that selection of agents' speculative portfolios does not work as the MSH suggests. In betting against each-others, both beliefs and preferences play a role, causing average returns of a portfolio derived under correct beliefs not to be necessarily higher that those of a portfolio derived under incorrect beliefs. Portfolios are not the only determinants of long run survival. Saving plays also a big role, and I show that it may or may not correct for the underporfomance of a portfolio derived under accurate beliefs. When it doesn't, MSH failures occur. For example, there exists two-agent Epstein-Zin economy where, with full probability:

F1 both agents survive and have a positive consumption share in the long-run;

F2 either the agent with the most accurate (even correct) beliefs vanishes or he dominates;

F3 the agent with the most accurate (even correct) beliefs vanishes.

Under **F1**, which typically occurs when agents' Relative Risk Aversion (RRA) coefficient is larger than 1, speculation is both a short-run and a long-run phenomenon.

[2]Yan (2008) finds similar results in an unbounded economy with Constant Relative Risk Aversion (CRRA) agents where the survival index depends also on the coefficient of RRA.

[3]The restriction on the economy parameters limits the choice of risk aversion and intertemporal substitution coefficients. Despite this limitation, the considered parametrizations are sufficient to show that the MSH fails and to shed light on the role played by saving and portfolio decisions. As surveyed in the next section, Borovička (2015) finds similar results in a related continuous time economy for a different parameter restriction.

[4]The formal definitions of an agent survival, vanishing, and dominance, as well as of beliefs accuracy, are given in Section 2.

Beliefs heterogeneity is persistent and state prices keep fluctuating between agents' evaluations. The result could help explaining stock market anomalies, as suggested in Hong and Stein (2007).

Under **F3** (**F2**), the market does not (may not) select for the most accurate beliefs. Nevertheless, a single agent is rewarded in the long-run, and determines asset prices. The relative importance of portfolio and saving decisions, which is relevant for how long-run outcome are achieved, depends both on agents' preferences and beliefs. Failure to reward the most accurate trader may occur in economies where all agents hold the same portfolio, so that only saving is relevant, as well as in economies where the saving decision is homogeneous across agents, so that only portfolios matter.

The reason why speculation may fail to validate the MSH, and even allow for long-run heterogeneity, is as follows. In Section 3, I show that, given that each agent consumption follows a multiplicative process, the dynamics of consumption shares depends on two key quantities: the ratio between the current value of next period consumption and current consumption, which I call the intertemporal substitution rate, and the expected log return of the speculative portfolio that allocates next period consumption among the different states. When an agent's portfolio is not log-optimal (RRA coefficient not 1), his speculative returns depend both on the accuracy of his beliefs and, through his risk preferences, on the comparison between his portfolio and the log-optimal portfolio computed under his beliefs. It is this second term that makes MSH failures of the type $\mathbf{F1}-\mathbf{F3}$ possible. The term is positive when beliefs and RRA preferences are such that, at the prevailing market prices, the chosen portfolio is closer to the log-optimal portfolio derived under correct beliefs than the log-optimal portfolio derived under the agent beliefs.[5] In such cases it is as if the agent is using a log-optimal portfolio and has 'effective' beliefs that are more accurate than his original beliefs. Effective beliefs can also be less accurate than the original beliefs, making this term of the expected log-return negative. Saving plays also a role in that the comparison of agents' intertemporal substitution rates may or may not compensate for the accuracy of effective beliefs. For example CRRA economies are special because, due to the interdependence of RRA and intertemporal elasticity of substitution (IES), the difference of accuracy between beliefs and effective beliefs is compensated by the saving component due to speculation.

To shed light on the role of portfolio returns for market selection purposes, in Section 4.1, I concentrate on cases where only portfolio decisions matter for agents' relative performance.[6] When all agents have log-optimal portfolios, the

[5]The log-optimal portfolio derived using correct beliefs is the portfolio with maximal growth, see Kelly (1956) and the literature surveyed in Section 1.1.

[6]This amounts to assume that all agents employ the same saving decision in equilibrium. The

comparison of their returns depends only on beliefs accuracy. The presence of agents with inaccurate beliefs implies that the agent with the most accurate beliefs has positive expected log-returns in every period. By speculating, he wins enough bets to eventually gain all the aggregate endowment. Outside of the log framework, however, effective beliefs accuracy and beliefs accuracy differ. Moreover, effective beliefs accuracy depends on assets' equilibrium returns. As a result, given two agents, it can happen that an agent's effective beliefs are the most accurate when the returns are set by the other agent, and the other way round. In this case speculation does not support the dominance of the agent with the most accurate beliefs but the outcome is long-run heterogeneity. Alternatively, it could occur that an agent has the most accurate effective beliefs for all possible equilibrium asset returns, even when his beliefs are inaccurate. Depending on all agents' risk preferences and beliefs, all types $\mathbf{F1}-\mathbf{F3}$ of MSH failures might occur.

For a more general characterization of the relative consumption dynamics, the comparison of intertemporal substitution rates, related to saving behavior, plays also an important role. Differentiated saving may or may not counterbalance the performance of speculative portfolios. In Sections 4.2-4.3, I study how both saving and portfolio decisions matter for long-run outcomes. MSH failures, in particular long-run heterogeneity, remain possible. CRRA economies are instead a special case in that, due to the interdependence of risk and intertemporal preferences, there exists an exact compensation between the over or under saving due to uncertainty and the difference of accuracy between beliefs and effective beliefs. Generically, only one agent holds the aggregate endowment in the long-run. An agent with accurate beliefs may still vanish, failure of type **F3**, but neither long-run heterogeneity, **F1**, nor path dependency, **F2**, is possible.

The role of saving is confirmed in Section 4.4, where I analyze Epstein-Zin economies where agents' beliefs and risk preferences are such that everybody holds the market portfolio in equilibrium. I derive the intuitive result that the agent who sets the lowest interest rate when alone in the market dominates in the long-run. Beliefs, together with discount factors and IES coefficients, still play a role for long-run outcomes but only because, through the saving under uncertainty channel, they determine saving rates. The accuracy of beliefs is instead irrelevant because, given that all agents hold the same portfolio in equilibrium, the relative consumption equilibrium dynamics is deterministic. The truth plays no role. As in CRRA economies, only MSH failures of type **F3** are possible.

The rest of the paper is organized as follow. In Section 2, I describe the economy. In Section 3, I show that long-run outcomes depend on the comparison of intertemporal substitution rates and portfolio expected log-returns. The main findings are in Section 4, where I analyze the MSH in (some) Epstein-Zin

latter occurs when the IES parameter is one and discount factors are homogeneous.

economies. In Section 5, I give simple examples and illustrate why results still hold beyond the case of Epstein-Zin preferences with i.i.d. aggregate endowment and beliefs considered here. Section 6 contains the conclusions. Appendixes A-C collect the proofs. In the next section I discuss the relation between my results and the literature.

1.1 Related Literature

Investors speculate when they take long and/or short positions that they would have not otherwise taken if they had agreed on the underlying state process.[7] A number of contributions investigate the effect of speculation on asset prices and the volume of trade (see e.g Varian, 1985, 1989; Harris and Raviv, 1993; Kandel and Pearson, 1995) or the relation between speculation and financial innovations (see e.g. Zapatero, 1998; Brock et al., 2009; Simsek, 2013). For example, Simsek (2013) studies a two-period economy with mean-variance optimizing agents and decomposes their portfolio risk as the sum the variance that remains after hedging and the variance due to speculation. The key result is that the speculative variance always increases when new assets are introduced. Here, I am instead interested in whether speculation can have long-run consequences. Indeed one could argue that financial innovation is needed to enable accurate traders to dominate by speculating against inaccurate traders. In this case, speculation would have short-run but not long-run effects (at least in a closed economy). My work shows instead that, also in the idealized framework of complete markets and general (intertemporal) equilibrium, speculation may have long-run consequences: disagreement may persist or the trader with accurate beliefs may vanish.

The relation between speculation and the MSH for financial economies has received increasing attention at least since the works of DeLong et al. (1990, 1991) and Blume and Easley (1992). DeLong et al. (1991) investigate whether noise traders, i.e. traders with inaccurate beliefs, might survive or even dominate against rational traders by bearing more risk. The answer is positive but the analysis is based on a partial equilibrium model. Blume and Easley (1992) study the same question in a model where asset prices are set in equilibrium by all traders. They investigate a sequence of temporary equilibria where agents save at a constant rate and can use Arrow securities to transfer wealth across states. Controlling for the saving rate, they find that when the trader with the most accurate beliefs purchases a log-optimal portfolio, he gains all the wealth in the long run and brings asset prices to reflect his beliefs. The result provides a support for the growth optimal

[7]The term speculation is also refereed to the purchase of an asset for the purpose of re-selling it at a higher price to those who value it more, see Harrison and Kreps (1978) for a formal model. See also Morris (1996) and Scheinkman and Xiong (2003).

Kelly rule (Kelly, 1956) in equilibrium models. However, when the trader with the most accurate beliefs does not use the log-optimal rule, Blume and Easley are able to construct examples in which this trader vanishes.[8]

Subsequent work by Sandroni (2000) and by Blume and Easley (2006) extend the analysis to general equilibrium models with endogenous saving. Under the assumption that markets are complete, that the aggregate endowment is bounded, and that agents maximize an expected time-separable utility, the MSH holds: provided that all traders discount future utility at the same rate, only the trader with the most accurate beliefs dominates. The market does not select against traders whose portfolios are not log-optimal, provided that their beliefs are accurate. Dominance of inaccurate beliefs, a failure of type **F3**, can still occur but it depends on discounting future utility too much. Results are derived from the comparison of a survival index that takes into account discount factors and beliefs accuracy. The role of saving and portfolio positions is not explicit ed.

A related contribution is Yan (2008), where the MSH is investigated in a continuous-time economy where the aggregate endowment follows a Brownian motion and agents have CRRA preferences. Agents agree on the volatility of the aggregate endowment process but disagree on its drift. The findings by Sandroni (2000) and Blume and Easley (2006) are confirmed, provided that the survival index takes also into account the RRA coefficient. When the economy is growing, the agent with the lowest RRA coefficient (the highest IES) has, all else equal, a higher survival index and thus dominates in the long-run. Also here the role of speculation is not evident from the survival index.[9]

By finding a failure of the MSH even in a general equilibrium framework, my results reconcile the findings of the earlier studies by DeLong et al. (1991) and Blume and Easley (1992) with those of the later literature. Traders with inaccurate beliefs might survive or even dominate in equilibrium, and saving does not always offset this result. The following quote from DeLong et al. (1991) nicely summarizes my findings: noise traders (agents with inaccurate beliefs) survive when "misperceptions make them unwittingly hold portfolio closer to those that would be held by investors with log-utility" -and correct beliefs- (p. 3). In particular, I find that whether a noise trader survives, dominates, or vanishes depends, other than on all agents' saving rate, also on a trade-off between misperceptions and risk attitudes. Moreover, I show that full dominance can occur only in market with

[8]Blume and Easley (1992) work in an i.i.d. economy with Arrow securities. The studies surveyed in Evstigneev et al. (2009) propose a generalization of the Kelly rule for more complicated asset structures.

[9]Other studies by Mailath and Sandroni (2003), Sandroni (2005),Jouini and Napp (2006, 2007), Cvitanić et al. (2012), Muraviev (2013), and Massari (2014), consider related cases. The main conclusion still holds, only failures of the type of **F3** are possible.

aggregate risk, confirming a finding of Blume and Easley (1992) (Th. 5.4).[10]

For CRRA economies, I confirm the results of Sandroni (2000), Blume and Easley (2006) and Yan (2008): neither failures of type **F1** nor **F2** can occur generically. Moreover, contrary to the previous literature, I decompose the survival index in its fundamental components and I clarify how both portfolio returns and intertemporal substitution rates depends on agents' preferences and beliefs. This decomposition sheds light on the trade-off between saving and portfolio decisions and it is thus helpful for pinning down the reason for an agent dominance or survival.[11]

In analyzing the MSH in Epstein-Zin economies, this paper is closely related to Borovička (2015). He investigates the MSH in continuous-time exchange economies with two agents having homogeneous Epstein-Zin preferences. My results, in particular that failures of the type $\mathbf{F1}-\mathbf{F3}$ are possible and generic, confirm his findings.[12] Discrete time economies allow for more freedom in the modeling of the economy and could be more amenable for applications. Moreover, my approach clarifies that Epstein-Zin preferences are not key for the result. MSH failures $\mathbf{F1}-\mathbf{F3}$ can occur whenever, for at least one trader, the component of expected log-returns that is not related to beliefs accuracy does not match the saving component due to speculation.

Within the market selection literature, there are other studies that find long-run beliefs heterogeneity. Beker and Chattopadhyay (2010) focus on two-agent economies with incomplete markets. Guerdjikova and Sciubba (2015) study economies where investors are ambiguity averse. Bottazzi and Dindo (2014) and Bottazzi et al. (2015) extend the temporary equilibrium analysis of Blume and Easley (1992) to general asset structures, short-lived and long-lived respectively, and possibly incomplete markets.

[10]Under no aggregate risk, a trader with correct beliefs has also correct effective beliefs in the limit when he consumes most of the endowment and sets assets' returns. This holds regardless of his preferences. Thus, noise traders can never dominate almost surely. The classical distinction between no aggregate risk versus aggregate risk economies is important also for market selection purposes.

[11]Sandroni (2000) and Blume and Easley (2006) work with time general expected time-separable utilities. Despite the fact that I concentrate on the special case of CRRA preferences, in Appendix B I discuss when the same link between saving and portfolio decisions holds more in general.

[12]Borovička derives market-selection outcomes for a larger region of parameters than I do here. Whether his approach is possible also in discrete-time economies and in economies with more than two agents is still an open issue. Note also that there exists parametrizations that I consider and Borovička excludes.

2 The Economy

In this section, I introduce the exchange economy and show how, in presence of heterogeneous agents, it is possible to characterize the dynamics of equilibrium consumption and state prices directly from agents' intertemporal substitution and portfolio decisions. The application to economies where agents have Epstein-Zin preferences is in Section 2.1.

Time begins at date $t = 0$ and it is indexed by $t \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. $\mathcal{S} = \{1, 2, \ldots, S\}$ is the set of states of the world, $2^{\mathcal{S}}$ is its power set, and $\Sigma = \times_{t=0}^{\infty} \mathcal{S}$ is the set of paths σ. $s_t \in \mathcal{S}$ denotes the state realized at date t and $\sigma_t = (s_0, s_1, \ldots, s_t) \in \Sigma_t$ the partial history till period t. To each partial history there corresponds a node of the uncertainty tree. $C(\sigma_t)$ is the cylinder set with base σ_t, $C(\sigma_t) = \{\sigma \in \Sigma | \, \sigma = (\sigma_t, \ldots)\}$ and $\mathcal{F}_t$ the σ-algebra generated by the cylinders, $\mathcal{F}_t = \sigma\left(\{C(\sigma_t) \, \forall \, \sigma_t \in \Sigma_t\}\right)$. $\mathcal{F}$ is the σ-algebra generated by the union of $\mathcal{F}$, $\mathcal{F} = \sigma\left(\cup_t \mathcal{F}_t\right)$. By construction $\{\mathcal{F}_t\}$ is a filtration. P is a probability measure on $(\Sigma, \{\mathcal{F}_t\})$ and $(\Sigma, \{\mathcal{F}_t\}, \mathrm{P})$ is the probability space on which I construct everything. All random variables are adapted to the filtration $\{\mathcal{F}_t\}$ and $x_t(\sigma_t)$ may be used in place of $x_t(\sigma)$. The dependence on a sequence σ, or on a partial history σ_t, is typically not explicited.

The economy contains I traders and a single consumption good. Trader $i \in \mathcal{I} = \{1, 2, \ldots, I\}$ consumption in period t on path σ is $c_t^i(\sigma)$. A consumption plan is a stochastic process $\{c_t\}$ and each trader i is endowed with the particular consumption plan $\{e_t^i\}$. The aggregate endowment is $\{e_t\}$ and for all t, s, and σ_t the growth rate of the economy is

$$g_{s,t}(\sigma_t) = \frac{e_{t+1}(\sigma_{t+1})}{e_t(\sigma_t)} \quad \text{when} \quad \sigma_{t+1} = (\sigma_t, s)\,.$$

For all t and σ_t, I denote with $\hat{g}_t$ the date t vector of de-trended growth rates

$$\hat{g}_{s,t} = \frac{g_{s,t}}{\exp \mathrm{E}_{\mathrm{P}}[\log g_t]}\,. \tag{1}$$

I assume that the growth process is i.i.d..

Assumption 2.1. *For all $t \in \mathbb{N}_0$, $s \in \mathcal{S}$, and $\sigma_t \in \Sigma_t$, $g_{s,t} = g_s$ and $\mathrm{P}(C(\sigma_T)) = \prod_{t=1}^{T} \mathrm{P}_{s_t}$ for a measure $\mathrm{P} = (\mathrm{P}_1, \ldots, \mathrm{P}_S)$ on $(\mathcal{S}, 2^{\mathcal{S}})$.*

With an abuse of notation, I use P to denote both the measure on $(\Sigma, \{\mathcal{F}\}_t)$ and on $(\mathcal{S}, 2^{\mathcal{S}})$. As discussed in Section 5, the assumption is without loss of generality for market selection purposes in that MSH failures can also be obtained with more complicated growth processes.

Each agent objective is to maximize a certain utility of his consumption stream. Agents may transfer their initial endowment across time and states by trading

assets in a complete market. In evaluating consumption streams $\{c_t^i\}$ agent i uses subjective beliefs Q^i, a probability measure on $(\Sigma, \{\mathcal{F}\})$. I shall assume that all agents believe that the world is i.i.d., that beliefs are absolute continue with respect to each other and the truth, and that beliefs are heterogeneous among agents.

Assumption 2.2. *For all agents $i \in \mathcal{I}$, beliefs Q^i on $(\mathcal{S}, 2^{\mathcal{S}})$ are constant and $\mathrm{Q}_s^i > 0 \Leftrightarrow \mathrm{P}_s > 0$, for all s in $\mathcal{S}$. Moreover, $\mathrm{Q}^i \neq \mathrm{Q}^j$ for all i and j in $\mathcal{I}$.*

As with Assumption 2.1, the i.i.d. part of Assumption 2.2 is without loss of generality: as long as agents' disagreement persists in the long-run, MSH failures can be shown to occur also with more complicated beliefs. The absolute continuity assumption is instead needed for the existence of an equilibrium, see Appendix A.3.

Other than assuming that the market is complete I do not further specify its structure.[13] Asset prices are determined in equilibrium. I postpone the characterization of agents' consumption and asset demand and just assume for now that there exists a no-arbitrage equilibrium where all agents' consumption plans are strictly positive.[14] The notation for state prices mimic the one for conditional probabilities: for $\tau > 0$, $q_{\sigma_{t+\tau},t}$ is the price of a unit of the consumption good after partial history $\sigma_{t+\tau}$ relative to one unit of consumption in date t. Due to the no-arbitrage condition state prices satisfy:

$$q_{\sigma_{t+\tau},t} = \frac{q_{\sigma_{t+\tau},0}}{q_{\sigma_t,0}} .$$

Using the vector of one-period states prices, q_t, one obtains the interest rate from t to $t+1$, r_t, and the corresponding discount rate δ_t. Risk neutral probabilities (normalized state prices) are $\mathrm{Q}_t^0 = r_t q_t$.

In order to characterize equilibrium consumption plans, I use agents' portfolio and saving decisions as follows. Consider agent i equilibrium consumption $\{c_t^i\}$ in two subsequent periods t and $t+1$. Since for each agent i consumption is an adapted process, for every t and every history σ_t there exists a scalar $\delta_t^i > 0$ and

[13] Date $t = 0$ trading does not require agents to hold rational prices expectations but it amounts to trade infinite assets in the initial period. Sequential trading of short- or long-lived assets makes the opposite assumptions. Depending on the chosen asset structure, relevant assumptions on the budget constraint should be taken to guarantee the existence of an equilibrium. In particular, under date $t = 0$ trading no bankruptcy is allowed. Under sequential trading no bankruptcy and no Ponzi schemes are allowed, see also Araujo and Sandroni (1999).

[14] When the aggregate endowment is growing I also assume that agents are discounting the future enough so that their equilibrium value function is finite. See also Assumption A.1 in Appendix A.

a vector $\alpha_t^i \in \Delta_+^S$ such that[15]

$$\begin{cases} \delta_t^i(\sigma_t) &= \dfrac{\sum_{s\in\mathcal{S}} q_{s,t} c_{t+1}^i(\sigma_t, s)}{c_t^i(\sigma_t)}\,, \\ \\ \alpha_{s,t}(\sigma_t) &= \dfrac{q_{s,t} c_{t+1}^i(\sigma_t, s)}{\sum_{s'\in\mathcal{S}} q_{s',t} c_{t+1}^i(\sigma_t, s')}\,, \quad \text{for all } s \in \mathcal{S}\,. \end{cases} \tag{2}$$

The scalar δ_t^i is the ratio between date t value of next period contingent consumption and date t consumption, agent i intertemporal substitution decision in a stochastic context, and it is thus related to how much agent i saves. The vector α_t^i is agent i allocation decision across states, only for consumption in period $t+1$, and it is thus related to agent i portfolio decision.[16] The equilibrium value of agents' intertemporal substitution rates and portfolios is determined by a set of Euler equations. We postpone to the next section the exact specification of both δ and α for some specific parametrizations of Epstein-Zin economies.

Given the sequence of decisions $\{\alpha_t^i\}$ and $\{\delta_t^i\}$, the dynamics of consumption is thus

$$c_{t+1}^i(\sigma_t, s) = \delta_t^i \frac{\alpha_{s,t}^i}{q_{s,t}} c_t^i(\sigma_t) = \frac{\delta_t^i}{\delta_t} \frac{\alpha_{s,t}^i}{\mathrm{Q}_{s,t}^0} c_t^i(\sigma_t)\,. \tag{3}$$

If the former is valid for all $i \in \mathcal{I}$, we can also use it to characterize market clearing prices. Aggregating (3) over all the agents and dividing by the total consumption in date t we find for each s

$$g_s = \sum_{i\in\mathcal{I}} \frac{\delta_t^i}{\delta_t} \frac{\alpha_{s,t}^i}{\mathrm{Q}_{s,t}^0}\, \phi_t^i(\sigma_t)\,, \tag{4}$$

where $\phi_t^i = c_t^i / e_t^i$ is the relative consumption of agent i. The former can be rewritten as follows in order to highlights the relation between normalized state prices and agents' portfolios.

First of all note that since $\alpha_t^i \in \Delta^S$ for all agents, (4) can be used to find the economy discount rate, which depends on agents' intertemporal substitution rates,

$$\delta_t = \sum_{i\in\mathcal{I}} \frac{\delta_t^i \phi_t^i}{\sum_{s'\in\mathcal{S}} g_{s'}\, \mathrm{Q}_{s',t}^0}\,. \tag{5}$$

[15] The fact that both δ_t^i and α_t^i are positive follows from showing that consumption is positive in equilibrium.

[16] α_t^i and δ_t^i can be interpreted as the one-period portfolio and saving decisions. The expressions of the full portfolio and saving rule $\bar{\alpha}_t^i$ and $\bar{\delta}_t^i$ are given in Appendix A.1.

Discount rates can then be used with relative consumption and intertemporal substitution rates to find the portfolio impact of each agent. Define

$$\varphi_t^i = \frac{\delta_t^i \phi_t^i}{\delta_t \sum_{s' \in \mathcal{S}} g_{s'} \, \mathrm{Q}_{s',t}^0} \quad \text{for all} \quad i \in \mathcal{I}\,, \tag{6}$$

so that by construction $\varphi_t^i \in \Delta_+^I$, and the normalized value of the supply of Arrow security corresponding to state s:

$$l_s(\mathrm{Q}_t^0) = \frac{\mathrm{Q}_{s,t}^0 \, e_{t+1}(\sigma_t, s)}{\sum_{s' \in \mathcal{S}} \mathrm{Q}_{s',t}^0 \, e_{t+1}(\sigma_t, s')} = \frac{\mathrm{Q}_{s,t}^0 \, g_s}{\sum_{s' \in \mathcal{S}} \mathrm{Q}_{s',t}^0 \, g_{s'}}\,.$$

The market clearing condition (4) can be rewritten as

$$l_s(\mathrm{Q}_t^0) = \sum_{i \in \mathcal{I}} \alpha_{s,t}^i \, \varphi_t^i\,, \quad \text{for all} \quad s \in \mathcal{S}\,. \tag{7}$$

State prices are such that the aggregate portfolio, the convex combination of each agent's portfolio, equates the (normalized value of the) aggregate supply. Importantly, portfolio impacts depend, other than on relative consumption, also on the ratio of between intertemporal substitution rates and discount factors, see (6).

The system of equations (3) and (5-7) for all $t \in \mathbb{N}_0$ and $\sigma_t \in \Sigma_t$ characterizes agents' relative consumption and state prices on an equilibrium path. Thus, an equilibrium allocation and supporting prices can be computed iteratively if

C1 we know that an equilibrium exists and is interior;

C2 for all t, σ_t one-period optimal portfolio and substitution decisions of all agents can be recovered from quantities (state prices, agents' beliefs and consumption) known in σ_t;

C3 the initial equilibrium relative consumption distribution $\{\phi_0^i \,\text{for all}\, i \in \mathcal{I}\}$ is known.

As it is shown in Appendix A and in the next section, conditions **C1-C2** do hold when agents' intertemporal substitution rates and portfolios come from the maximization of specific parametrizations of an Epstein-Zin recursive utility. Regarding condition **C3**, note that long-run properties can be characterized even when it does not hold. In fact, provided long-run outcomes of (3) and (5-7) are identified for every initial consumption distribution, also equilibrium long-run outcomes are characterized.

2.1 Epstein-Zin Economies

I assume that agents use assets to transfer consumption across time and states in order to maximize a recursive utility of the Epstein-Zin type, see e.g. Epstein and Zin (1989). In particular I assume that, for all $i \in \mathcal{I}$, agent i with beliefs Q^i maximizes a utility U^i that has a recursive structure of the type

$$U_t^i = \left((1-\beta^i) c_t^{1-\rho^i} + \beta^i \left(\mathrm{E}_{\mathrm{Q}^i}[(U_{t+1}^i)^{1-\gamma^i}] \right)^{\frac{1-\rho^i}{1-\gamma^i}} \right)^{\frac{1}{1-\rho^i}}, \; t \in \mathbb{N}_0 \,. \tag{8}$$

$\beta^i \in (0,1)$ is the discount factor; $\gamma^i \in (0,\infty)$ is the coefficient of Relative Risk Aversion (RRA); $\rho^i \in (0,\infty)$ is the inverse of the coefficient of Intertemporal Elasticity of Substitution (IES) on a deterministic consumption path. The utility is defined also for $\gamma^i = 1$ and $\rho^i = 1$ by taking the appropriate limits, see also Epstein and Zin (1989).

Parameters shall be chosen such that the utility of the aggregate endowment is finite, implying that to the recursive formulation there corresponds an utility over consumption streams, see Assumption A.1 in Appendix A.3.[17] In these cases one can use the Euler equations to characterize (interior) equilibrium allocation as a function of market prices, see Appendix A.2. For generic values of the discount factor β^i, the IES coefficient ρ^i, and the RRA coefficient γ^i, Euler equations involving subsequent time periods are coupled, so that the intertemporal and saving decision depends on all future state prices and beliefs , violating **C2**. However, as I show in Proposition A.1 in Appendix A, under specific preferences parametrizations **C2** is met in that optimal decisions can be recovered from contemporaneous prices and beliefs.[18]

For agent i, in date t and history σ_t one finds:

$$\delta_t^i = \delta_t \left(\frac{\beta^i}{\delta_t} \right)^{\frac{1}{\rho^i}} \left(\sum_{s' \in S} (\mathrm{Q}_{s'}^i)^{\frac{1}{\gamma^i}} (\mathrm{Q}_{s',t}^0)^{1-\frac{1}{\gamma^i}} \right)^{\frac{\gamma^i}{1-\gamma^i}\frac{1-\rho^i}{\rho^i}}, \tag{9}$$

$$\alpha_{s,t}^i = \frac{(\mathrm{Q}_s^i)^{\frac{1}{\gamma^i}} (\mathrm{Q}_{s,t}^0)^{1-\frac{1}{\gamma^i}}}{\sum_{s' \in S} (\mathrm{Q}_{s'}^i)^{\frac{1}{\gamma^i}} (\mathrm{Q}_{s',t}^0)^{1-\frac{1}{\gamma^i}}} \quad \text{for all } s \in \mathcal{S} \,. \tag{10}$$

Portfolio decisions α_t depend on beliefs, relative state prices, and the RRA coefficient. Intertemporal rates of substitution δ_t depend also on market discount rates,

[17] For example, when the aggregate endowment is growing, a sufficient condition is that $\rho^i > 1$ or that $\rho^i < 1$ and agent i discounts future expected utility fast enough.

[18] The special cases are: $\rho^i = 1$; γ^i and Q^i such that agent i holds the market portfolio in equilibrium; the CRRA limit of $\gamma^i = \rho^i$. Despite many parameter specifications are left out, these cases are enough to show that the MSH fails and to shed light on the role of saving and portfolio decisions for market selection purposes.

the IES coefficient, and the discount factor. Viewed as a function of market prices, we can define the Epstein-Zin intertemporal substitution rule, $\delta(\cdot,\cdot;\beta^i,\rho^i,\gamma^i,\mathrm{Q}^i)$ such that $\delta^i_t=\delta(\delta_t,\mathrm{Q}^0_t;\beta^i,\rho^i,\gamma^i,\mathrm{Q}^i)$, and the Epstein-Zin portfolio rule, $\alpha(\cdot;\gamma^i,\mathrm{Q}^i)$ such that $\alpha^i_{s,t}=\alpha_s(\mathrm{Q}^0_t;\gamma^i,\mathrm{Q}^i)$ for all $s\in\mathcal{S}$. The dependence of intertemporal substitution rule on beliefs and normalized state prices represents a saving under uncertainty component in that it would not be present if the economy were deterministic, i.e. if $g_s=g$ for all s and if agents shared the same beliefs, $\mathrm{Q}^i=\mathrm{Q}$ for all $i\in\mathcal{I}$. The portfolio rule is particularly simple when $\gamma=1$, leading to $\alpha^i=\mathrm{Q}^i$. Agent i 'bets his beliefs' as in the CRRA log-case ($\gamma=\rho=1$). For this reason we shall call this portfolio rule the log-optimal one also outside the CRRA framework.

Using rules, the market equilibrium condition (7) implicitly set date-t-history-σ_t state prices depending on the date-t-history-σ_t consumption distribution. Throughout this work, I shall use the system of equations (3-7), with rules (δ,α) for all $i\in\mathcal{I}$, in order to characterize long-run consumption distributions and prices iteratively.

3 Market Selection

I am interested in studying whether, in terms of consumption, an agent survives, vanishes, or dominates. Since aggregate consumption can be unbounded or converge to zero, I focus on the relative consumption ϕ_t. Consistently with the literature I define:

Definition 3.1. *Agent i survives on σ if $\limsup_{t\to\infty}\phi^i_t(\sigma)>0$, he vanishes when $\lim_{t\to\infty}\phi^i_t(\sigma)=0$, he dominates when $\lim_{t\to\infty}\phi^i_t(\sigma)=1$.*

I shall show that intertemporal substitution rates and portfolio expected log-returns can be used to give sufficient conditions for an agent to vanish, survive, or dominate P-a.s.. The general idea is as follows.

Since the consumption dynamics (3) is a multiplicative process, the log-consumption follows an additive process. The same holds for agents' relative log-consumption dynamics. Defining $z^{i,j}_t=\log\frac{c^i_t}{c^j_t}$ one has

$$z^{i,j}_{t+1}(\sigma_{t+1})=z^{i,j}_t(\sigma_t)+\epsilon^{i,j}_{t+1}(\sigma_{t+1})\,, \tag{11}$$

whose innovation is

$$\epsilon^{i,j}_{t+1}(\sigma_{t+1})=\log\frac{\delta^i_t}{\delta^j_t}+\log\frac{\alpha^i_{s,t}}{\alpha^j_{s,t}}\quad\text{when}\quad\sigma_{t+1}=(\sigma_t,s)\,,\quad\text{for all}\quad s\in\mathcal{S}\,.$$

The first (deterministic) part of the innovation is the difference of agents' log-substitution rates. It is intuitive that the agent who postpones consumption is

better equipped of long-run survival. To interpret the contribution of portfolio decisions let us compute the return of the portfolio α_t^i. From (3) one finds

$$r_{s,t}^i := \frac{c_{t+1}^i(\sigma_t, s)}{\sum_{s' \in \mathcal{S}} q_{s',t} c_{t+1}^i(\sigma_t, s')} = \frac{\alpha_{s,t}^i}{q_{s,t}} = \frac{1}{\delta_t} \frac{\alpha_{s,t}^i}{\mathrm{Q}_{s,t}^0} .$$

Denoting the relative entropy of Q with respect to P, also Kullback-Leibler divergence, as

$$I_P(Q) = \sum_{s \in \mathcal{S}} \mathrm{P}_s \log \frac{\mathrm{P}_s}{\mathrm{Q}_s} ,$$

agent i expected log-return in date t is

$$\mathrm{E}[\log r_t^i | \mathcal{F}_t] = \log r_t + I_P(Q_t^0) - I_P(\alpha_t^i) .$$

The expected log-return in excess of the log risk-free rate is thus

$$\mu_t^i = I_P(Q_t^0) - I_P(\alpha_t^i) . \tag{12}$$

In view of his role for the market selection, I denote μ_t^i, as agent i *growth premium* in period t.

The drift of the relative log-consumption dynamics can thus be written as

$$E[\epsilon_{t+1}^{i,j} | \mathcal{F}_t] = \log \frac{\delta_t^i}{\delta_t^j} + \mu_t^i - \mu_t^j . \tag{13}$$

The second component is the difference of agents' growth premia. Denoting $\mu_t^{i,j} = \mu_t^i - \mu_t^j$, from the formula for growth premia (12) one finds:

$$\mu_t^{i,j} = I_P(\alpha_t^j) - I_P(\alpha_t^i) .$$

The difference of growth premia amounts to the comparison of the relative 'distance' between agents' portfolio decisions and P, the optimal growth portfolio (log-optimal portfolio derived under the truth).

In the rest of the paper we rely on the aggregate effect of intertemporal substitution rates and growth premia to characterize whether an agent survives, dominates, or vanishes. Define each agent i *generalized survival index* in node σ_t as

$$k_t^i = \log \delta_t^i - I_P(\alpha_t^i) , \tag{14}$$

the conditional drift of the relative consumption process is just

$$E[\epsilon_{t+1}^{i,j} | \mathcal{F}_t] = k_t^i - k_t^j . \tag{15}$$

If agent i has an higher survival index than agent j, the drift of the relative consumption process is in his favor and thus he gains, in expectation, consumption. This is similar to the approach that has been followed by most of the literature, where survival indexes are derived from the Euler equation of the Pareto optimal allocation problem. I shall show how the difference of the various proposed survival indexes can be derived from the difference of generalized survival indexes k as in (15).

The advantage of expliciting survival indexes in terms of substitution rates and growth premia is that one can appraise their relative importance in determining survival. As we shall see, depending on preferences and beliefs, there are economies where only growth premia (and thus portfolio decisions) matters for long-run outcomes and economies where only substitution rates (and thus saving decisions) play a role. A feature of Epstein-Zin preferences is that the ordering of survival indexes in each node might depend on prevailing market conditions. As a result different agents might be rewarded (by the combination of saving and portfolios) under different sets of market prices.

Two sets of results shall be derived from the difference of agents' survival indexes in (15). Using the Law of Large Numbers for martingales differences, one can state necessary or sufficient conditions in terms of survival indexes time averages. This is the approach proposed by Sandroni (2000) (see e.g. his Proposition 3). However, since substitution rates and growth premia depend on equilibrium prices and discount factors, these conditions can be evaluated analytically only under special assumptions. One example is homogeneous CRRA economies, where the trade-off between saving and portfolio decisions does not depend on market clearing price, see e.g. Proposition 4.2. The alternative is to analyze economies with two agents (or two group of agents) and compute substitution rates and growth premia in the limit of one agent (group) consuming all the endowment. Sufficient conditions for survival, vanishing, or dominance depend on these limits, see e.g. Proposition 4.1 or the results in Bottazzi and Dindo (2014). An advantage of this approach is that these limits can be evaluated analytically.

4 Selection in Epstein-Zin Economies

By assuming that agents maximize Epstein-Zin preferences we can consider exchange economies where, in equilibrium,

i) all agents use the same intertemporal substitution rate but hold different portfolios -so that $\log \frac{\delta_t^i}{\delta_t^j} = 0$ for all t and σ_t, but, generically, $\mu_t^{i,j} \neq 0$;

ii) agents' intertemporal and risk preferences are tight together -so that $\log \frac{\delta_t^i}{\delta_t^j}$

$\mu_t^{i,j}$ are inter-dependent;

iii) all agents hold the market portfolio but use different intertemporal substitution rates -so that $\mu_t^{i,j} = 0$ for all t and σ_t, but, generically, $\log \frac{\delta_t^i}{\delta_t^j} \neq 0$.

Given i), we can analyze the property of long-run consumption dynamics when only growth premia matters. This is the content of Section 4.1. MSH failures $\mathbf{F1} - \mathbf{F3}$ are generic. Due to ii), we can show that in CRRA economies there exists a trade off between the difference of growth premia and log substitution rates. As a result only failures of the type **F3** are possible and are due to the saving component. This is the content of Section 4.2. In Section 4.3, we show that selection still leads to all failures $\mathbf{F1} - \mathbf{F3}$ when intertemporal substitution rates are not homogeneous across agents but we are outside the CRRA framework. Given iii), we can move to analyze market selection when only saving behavior matter, see Section 4.4. Again, only failures of type **F3** occur. Despite beliefs heterogeneity matter for long-run outcomes, the relative consumption dynamics is deterministic and the truth has no role in these economies.

4.1 Selection of Portfolios

When all agents have the same IES parameter $\rho^i = 1$ and discount rate $\beta^i = \beta$, agents choose the same intertemporal substitution rate in equilibrium: $\delta_t^i = \beta$ for all i, t and σ_t. As a result market selection outcomes are determined only by portfolio decisions and the comparison of generalized survival indexes as in (15) is a comparison of growth premia:[19]

$$k_t^i - k_t^j = \mu_t^{i,j} \,. \tag{16}$$

When i is the representative agent (homogeneous preferences and beliefs economy), date t normalized state prices and market discount factors are given by

$$\begin{aligned} \mathrm{Q}^0_{s,t}\,|_i &= \frac{\mathrm{Q}^i_s\, \hat{g}_s^{-\gamma^i}}{\sum_{s' \in \mathbb{S}} \mathrm{Q}^i_{s'}\, \hat{g}_{s'}^{-\gamma^i}} \,, \quad \text{for all } s \,, \\ \delta_t|_i &= = \beta e^{-\,\mathrm{E_P}[\log g]} \frac{\sum_{s \in \mathbb{S}} \mathrm{Q}^i_s\, \hat{g}_s^{-\gamma^i}}{\sum_{s \in \mathbb{S}} \mathrm{Q}^i_s\, \hat{g}_s^{1-\gamma^i}} = \frac{\beta e^{-\,\mathrm{E_P}[\log g]}}{\sum_{s \in \mathbb{S}} \mathrm{Q}^0_{s,t}\, \hat{g}_s} \,, \end{aligned} \tag{17}$$

where $\hat{g}$ is the vector of de-trended growth rates as in (1). Date t equilibrium saving and portfolio decisions are

$$\begin{aligned} \alpha^i_{s,t}|_i &= \frac{\mathrm{Q}^i_s\, \hat{g}_s^{1-\gamma}}{\sum_{s \in \mathbb{S}} \mathrm{Q}^i_{s',t}\, \hat{g}_{s'}^{1-\gamma^i}} \quad \text{for all } s \,, \\ \delta^i_t|_i &= \beta \,. \end{aligned} \tag{18}$$

[19] When the IES parameter ρ^i is one, the one-period decisions δ_t^i and α_t^i coincide with the 'full' saving and portfolio decisions $\bar{\delta}_t^i$ and $\bar{\alpha}_t^i$, see also Appendix A.

Equilibrium discount rates, normalized state prices, and portfolio decisions are particularly simple when there is no aggregate risk, $g_s = g$ for all $s \in \mathcal{S}$, leading to and $\mathrm{Q}^0_t = \alpha^i_t|_i = \mathrm{Q}^i$ and $\delta_t|_i = \beta/g$.

Under heterogeneous beliefs and, possibly, RRA coefficients, equilibrium market discount rates and normalized state prices do instead depend on the contemporaneous consumption distribution. A simplification of (7) occurs because all agents are saving at the same rate. Saving-adjusted weights φ and relative consumption weights ϕ coincides so that normalized asset prices solve

$$l_s(\mathrm{Q}^0_{s,t}) = \sum_{i\in\mathcal{I}} \alpha^i_{s,t}(\mathrm{Q}^0)\phi^i_t \quad \text{for all } s \in \mathcal{S}\,,$$

while the market discount factor is

$$\delta_t = \frac{\beta e^{-\,\mathrm{E_P}[\log g]}}{\sum_{s\in\mathcal{S}} \mathrm{Q}^0_{s,t}\,\hat{g}_s}\,.$$

The long-run consumption dynamics is determined only by portfolio decisions and thus depends only on the comparison of agents' growth premia. It is instructive to consider the well-known case of a log-economy, $\gamma^i = \gamma = 1$ for all agents, first.

Log-Optimal Portfolios The relative consumption dynamics is particularly simple when $\gamma = 1$, as each agent 'bets' his own beliefs. The difference of agent i and agent j growth premia is determined only by beliefs accuracy:

$$\mu^{i,j}_t = I_P(Q^j) - I_P(Q^i)\,. \tag{19}$$

If agent i has more accurate beliefs, then he has a larger growth premium in every period. The dynamics of the log consumption ratio of i and j in (11) has positive drift, agent i dominates while agent j vanishes. Moreover since the agent with most accurate beliefs dominates against any other agent, then he also dominates against all of them in a I-agent economy. In a log-economy speculation enables the agent with the most accurate beliefs to play a favorable game of chance in every period.[20]. The result is well known. For equilibrium economies it goes back at least to Blume and Easley (1992).[21]

[20]Favorable refers here to having positive expected log-returns rather than positive expected returns

[21]The result is straightforward when beliefs are i.i.d.. It is more subtle to establish when beliefs are not uniformly bounded away from each-others, for example when more agents learn the correct probabilities but with different speed of convergence. See also Sandroni (2000), Blume and Easley (2006), and Massari (2014).

Non Log-optimal Portfolios When an agent i portfolio rule is not log-optimal, portfolio choices do not correspond to beliefs. It is still convenient to evaluate agents' portfolios through the lenses of the log-optimal agent and growth premia can be rewritten as

$$\mu_t^i = \left[I_P(Q^0) - I_P(Q^i)\right] + \left[I_P(Q^i) - I_P(\alpha_t^i)\right] .$$

The first part is the return of the log-optimal agent with same beliefs Q^i, and it is thus related to the accuracy of beliefs. The second part of the growth premium measures instead weather the agent is better-off or worse-off, in terms of expected log-returns, by using a non-log optimal rules α_t rather than the log-optimal rule derived using his beliefs. Equivalently it measures the difference in accuracy of beliefs Q^i and effective beliefs α_t^i. If agent i beliefs are correct, this second part is negative since i would have been better off by using a log-optimal portfolio. Effective beliefs are less accurate. However, when agent i beliefs are not correct, effective beliefs could be more accurate than beliefs. Equivalently, the portfolio decision could be 'closer' to the optimal growth portfolio than the log-optimal portfolio under incorrect beliefs Q^i, and thus give a positive contribution. I name this component the Non-Log-Optimality (NLO) compensation and denote it with ν_t^i. The difference of growth premia of i and j can thus be re-written in terms of the relative accuracy of beliefs and of the difference of NLO contributions.

$$\mu_t^{i,j} = I_P(Q^j) - I_P(Q^i) + \nu_t^{i,j} . \tag{20}$$

All results in the paper are essentially due to the role of these NLO terms. Here, since each agent takes the same saving decisions, (16-20) imply that the difference of survival indexes does not only depend on relative beliefs accuracy but also on the difference of NLO compensations.

The sign of (20) determines agents' relative performance. Although I assume that beliefs are exogenous, NLO terms depend on equilibrium prices. It turns out that it is sufficient to characterize the relative portfolio performance at the prices set by each agent to establish if on agent survives, vanishes, or dominates. In the limit of agent i having all the consumption, from (10) and (17) one finds

$$\nu^{i,j}|_i = \frac{1-\gamma^j}{\gamma^j}\left(I_P(Q^j) - I_P(Q^i)\right) + \Delta^{i,j}|_i \tag{21}$$

where

$$\Delta^{i,j}|_i = \log \frac{\sum_{s\in\mathcal{S}} (\mathrm{Q}_s^j\, \hat{g}_s^{1-\gamma^j})^{\frac{1}{\gamma^j}} (\mathrm{Q}_s^i\, \hat{g}_s^{1-\gamma^i})^{1-\frac{1}{\gamma^j}}}{\sum_{s\in\mathcal{S}} \mathrm{Q}_s^i\, \hat{g}_s^{1-\gamma^i}} . \tag{22}$$

Adding the contribution due to beliefs relative accuracy, the difference of generalized survival indexes when agent i dominates is can be written as

$$k^i|_i - k^j|_i = \mu^{i,j}|_i = \frac{1}{\gamma^j}\left(I_P(Q^j) - I_P(Q^i)\right) + \Delta^{i,j}|_i .$$

Even if agent i has correct beliefs, $\mathrm{Q}^i = \mathrm{P}$, $\Delta^{i,j}|_i$ could still be so negative to imply a higher portfolio expected log-return for agent j at the prices determined by agent i. The same holds for $\mu^{i,j}|_j$ which can be found by interchanging the role of agent i and j above and using that $\mu^{i,j}|_j = -\mu^{j,i}|_j$. Not only $\mu^{i,j}|_j$ can still be negative even if i has correct beliefs but, also, the signs of $\mu^{i,j}|_i$ and $\mu^{i,j}|_j$ can be different. The following proposition show that it is sufficient to characterize both signs to determine the outcome of the long-run consumption dynamics. I state the result by comparing the relative accuracy of beliefs with $\Delta^{i,j}|_i$ and $\Delta^{i,j}|_j$.

Proposition 4.1. *Under the Assumptions 2.1, 2.2, A.1, consider the equilibrium paths of an economy with two agents, i and j, maximizing an Epstein-Zin utility with $\rho^i = \rho^j = 1$ and $\beta^i = \beta^j = \beta$.*

i) If

$$\gamma^i \Delta^{j,i}|_j < I_P(Q^j) - I_P(Q^i) < -\gamma^j \Delta^{i,j}|_i \,,$$

then $k^i|_j - k^j|_j > 0$, $k^i|_i - k^j|_i < 0$, and both agents survive P*-almost surely.*

ii) If

$$-\gamma^j \Delta^{i,j}|_i < I_P(Q^j) - I_P(Q^i) < \gamma^i \Delta^{j,i}|_j \,,$$

then $k^i|_j - k^j|_j < 0$, $k^i|_i - k^j|_i > 0$, and there exists two sets Γ^+ and Γ^- with $\mathrm{P}(\Gamma^+ \cup \Gamma^-) = 1$ such that agent i dominates on σ when $\sigma \in \Gamma^+$ and agent j dominates on σ when $\sigma \in \Gamma^-$.

iii) If

$$I_P(Q^j) - I_P(Q^i) > \gamma^i \Delta^{j,i}|_j \quad \textit{and} \quad I_P(Q^j) - I_P(Q^i) > -\gamma^j \Delta^{i,j}|_i \,,$$

then $k^i|_j - k^j|_j > 0$, $k^i|_i - k^j|_i > 0$, and agent i dominates P*-almost surely. Likewise, if both reversed inequalities hold, so that $k^i|_j - k^j|_j < 0$ and $k^i|_i - k^j|_i < 0$, then agent j dominates* P*-almost surely.*

For a given relative beliefs accuracy, the endogenous component of the NLO term can be such that both agents survive, meaning that disagreement is persistent; both dominate, but on different path; or only one agent dominates, but not necessarily the one with most accurate beliefs. Assume trader i has less accurate beliefs than trader j. The reason behind the survival, or even dominance, of a trader i is that his NLO compensation can be larger than the corresponding compensation of agent j. Stated in different terms, the accuracy of effective beliefs, which depend also on preferences and equilibrium prices, can overturn the component of the growth premium given by the accuracy of beliefs. Overall, agent i could hold a portfolio closer to the growth optimal portfolio than the portfolio held by

agent j. It is enough to check for the above to happen at the prices set by either agent consuming all the endowment in one period to say whether i dominates on almost all path, survives on almost all path, or dominates on a set of paths with positive measure. I provide a graphical representation of all possible outcomes in Section 5.

A restriction on the possible long-run dynamics occurs when agents have homogeneous risk preferences γ. The following corollary relies on the fact that under no-aggregate risk, or with aggregate risk and $S = 2$, the difference of NLO compensations can be ordered.[22] When $\gamma > 1$, agent i cannot have a higher NLO term at the prices determined by j than he has at his prices, thus excluding that both agents dominate on different path. When $\gamma < 1$, agent i cannot have a higher NLO term at the prices he determines than he has at the prices determined by j, thus excluding that both agents survive and that beliefs disagreement is persistent. The following corollary proves the statement.

Corollary 4.1. *Under the assumption of Proposition 4.1, assume $\gamma^i = \gamma^j = \gamma$, no aggregate risk, or aggregate risk but $S = 2$. If $\gamma > 1$, then only cases i) and iii) are possible. If instead $\gamma < 1$, then only cases ii) and iii) are possible. If otherwise $\gamma = 1$, and $I_P(Q^i) \neq I_P(Q^j)$, only case iii) is possible.*

In homogeneous risk aversion economies, for all γ, either agent could dominate almost surely. However, survival of both agents can only occur when they have less risky portfolio than log-optimal ones. On the contrary, path dependency can only occur when agents hold more risky portfolio than log-optimal ones. More risk averse portfolios with incorrect beliefs tend to be close to log-optimal portfolios with correct beliefs at the prices set by the other agent, thus leading to accurate effective beliefs. On the contrary, less risk averse portfolios with incorrect beliefs tend to do the opposite and be very far from the log-optimal portfolio with correct beliefs at the prices set by the other agent, thus leading to less accurate effective beliefs. Beliefs heterogeneity is persistent in the first case and transient in the second.

Finally, the next corollary is another application of Proposition 4.1 that addresses the fate of an agent with correct beliefs.

Corollary 4.2. *Under the assumption of Proposition 4.1, assume that agent i has correct beliefs, $\mathrm{Q}^i = \mathrm{P}$.*

i) If agent i has $\gamma^i = 1$, then he dominates P*-almost surely*

ii) If the economy has no aggregate risk, then either i dominates P*-almost surely or case ii) of Proposition 4.1 can occur.*

[22] The case of $S = 2$, a binomial tree economy, is the one exploited in the examples of Section 5.

iii) Otherwise, any of the cases of Proposition 4.1 can occur.

Since the log-optimal portfolio derived under correct beliefs guarantees the highest growth-premium for all prices, an agent who use this portfolio dominates almost surely.[23] When instead the agent with correct beliefs does not use the log-optimal rule, anything can happen. Not only can he vanish, but there are also cases where he is not the only survivor and beliefs heterogeneity is persistent. However, agents' co-existence can never occur when there is no-aggregate risk. The reason is that in such an economy, if agent i has correct beliefs the equilibrium portfolio he holds in the limit of having all the consumption is also log-optimal (both imply fair pricing under no aggreate risk), leading to correct effective beliefs $\alpha^i|_i = \mathrm{P}$. As a result $\mu^{i,j}|_i$ is always positive and by having a higher growth premium at the returns he sets, both long-run heterogeneity and almost sure vanishing never occur.[24]

4.2 Selection in CRRA Economies

I turn to analyze the outcome of selection when agents not only hold different portfolios but also differ in how they transfer consumption intertemporally. I start with CRRA economies, for which it is known that only one agent dominates generically, see Sandroni (2000) and Blume and Easley (2006) for bounded economies and Yan (2008) for unbounded economies. I illustrate how all their results emerge in terms of substitution and portfolio decisions, and how they can be generalized.

Throughout this section I assume that for all $i \in \mathcal{I}$ the RRA coefficient γ^i and the IES coefficient ρ^i coincide, leading to substitution and portfolio decisions that are optimal for a CRRA agent with RRA coefficient γ^i.

When agent i is the representative agent (homogeneous preferences and beliefs), the joint solution of (7) and (9-10) leads to

$$\begin{aligned} \mathrm{Q}^0_{s,t}|_i &= \frac{\mathrm{Q}^i_s\, \hat{g}_s^{-\gamma^i}}{\sum_{s'\in\mathcal{S}} \mathrm{Q}^i_{s'}\, \hat{g}_{s'}^{-\gamma^i}}\,, \quad \text{for all } s\,, \\ \delta_t|_i &= \beta^i e^{-\gamma^i \mathrm{E}_{\mathrm{P}}[\log g]} \sum_{s\in\mathcal{S}} \mathrm{Q}^i_s\, \hat{g}_s^{-\gamma^i} = \frac{\beta e^{-\gamma^i \mathrm{E}_{\mathrm{P}}[\log g]}}{\sum_{s\in\mathcal{S}} \mathrm{Q}^0_{s,t}\, \hat{g}_s^{\gamma^i}}\,. \end{aligned} \tag{23}$$

[23] The result is well known at least since Kelly (1956). It was first extended to economies where prices are set in equilibrium by Theorem 5.1 of Blume and Easley (1992). See also the discussion after Proposition 1 in Sandroni (2000).

[24] When $\rho = 1$ the economy is equivalent to one where saving is exogenously fixed to β and portfolio are chosen myopically. With this respect the possibility that an agent with correct beliefs vanishes P-almost surely in economies with aggregate risk is equivalent to Theorem 5.4 of Blume and Easley (1992). All other MSH failures are new.

Agent i equilibrium saving and portfolio decisions are

$$\begin{aligned} \delta_t^i|_i &= \beta^i e^{(1-\gamma^i)\,\mathrm{E_P}[\log g]} \sum_{s\in\mathcal{S}} \mathrm{Q}_s^i\, \hat{g}_s^{1-\gamma^i}\,, \\ \alpha_{s,t}^i|_i &= \frac{\mathrm{Q}_s^i\, \hat{g}_s^{1-\gamma^i}}{\sum_{s'\in\mathcal{S}} \mathrm{Q}_{s'}^i\, \hat{g}_{s'}^{1-\gamma^i}}\,, \quad \text{for all } s\,. \end{aligned} \tag{24}$$

The situation is different when agents have heterogeneous beliefs. I consider economies with homogeneous preferences first.

4.2.1 Homogeneous γ

When agents have heterogeneous beliefs and discount factors, but have the same RRA coefficient, the difference of agent i and agent j NLO terms computed at the generic set of prices Q_t^0 is

$$\nu_t^{i,j} = \left(\frac{1-\gamma}{\gamma}\right)\big(I_P(Q^j) - I_P(Q^i)\big) + \log \frac{\sum_{s\in\mathcal{S}} (\mathrm{Q}_s^j)^{\frac{1}{\gamma}} (\mathrm{Q}_{s,t}^0)^{1-\frac{1}{\gamma}}}{\sum_{s\in\mathcal{S}} (\mathrm{Q}_s^i)^{\frac{1}{\gamma}} (\mathrm{Q}_{s,t}^0)^{1-\frac{1}{\gamma}}}\,.$$

As we have shown in the previous section, for a given relative beliefs accuracy there could be prices for which agent i has higher growth premium and prices where the opposite occurs. Preferences play a role. (The result applies also here since, for a given γ, CRRA portfolio decisions and Epstein-Zin portfolio decisions coincide.)

Given the difference of generalized survival indexes in (15), in order to establish long-run outcomes we should complement the analysis of portfolio decisions with the analysis of saving. Give the CRRA intertemporal substitution rules in (9), the log-ratio of agent i to agent j substitution rate is

$$\log \frac{\delta_t^i}{\delta_t^i} = \frac{1}{\gamma} \log \frac{\beta^i}{\beta^j} + \log \frac{\sum_{s\in\mathcal{S}} (\mathrm{Q}_s^i)^{\frac{1}{\gamma}} (\mathrm{Q}_{s,t}^0)^{1-\frac{1}{\gamma}}}{\sum_{s\in\mathcal{S}} (\mathrm{Q}_s^j)^{\frac{1}{\gamma}} (\mathrm{Q}_{s,t}^0)^{1-\frac{1}{\gamma}}}\,.$$

Other than by the discount factor and IES coefficient $1/\gamma$, the comparison of agents' substitution rates ratio depends also on beliefs Q^i, Q^j and normalized state prices Q^0. This last terms reflect how agents adjust their saving in speculative markets. As a function of Q^0,

$$\sum_{s\in\mathcal{S}} (\mathrm{Q}_s^i)^{\frac{1}{\gamma}} (\mathrm{Q}_{s,t}^0)^{1-\frac{1}{\gamma}}$$

has a maximum of 1 in Q^i when $\gamma > 1$ and a minimum of 1 in Q^i when $\gamma < 1$. When $\gamma < 1$ an agent postpones consumption from one period to the next whenever normalized state prices do not coincides with his beliefs. The opposite occurs when $\gamma > 1$.

Importantly, the comparison of the saving under uncertainty terms depends on normalized state prices and individual beliefs through a term that matches exactly the price dependent part of the NLO term. Given that the two terms off-set each others, the difference of generalized survival indexes is determined, for all t and σ_t, only by (exogenously given) discount factors and beliefs:

$$k_t^i - k_t^j = \frac{1}{\gamma}\left(\log\frac{\beta^i}{\beta^j} + I_P(Q^j) - I_P(Q^i)\right).$$

CRRA economies with homogeneous preferences behave, as market selection is concerned, as log-economies: controlling for discount factors only beliefs accuracy matters.

It is explained why the comparison of generalized survival indexes can be given in terms of the survival index defined in Blume and Easley (2006),

$$k_{BE}^i = \log\beta^i - I_P(Q^i)\,.$$

In fact, although k_t and k_{BE} differ, it is the sign of $k_t^i - k_t^j$ that matters and, as we have just showed, this sign is equal to the sign of $k_{BE}^i - k_{BE}^j$ for all t and σ_t. Note however that the RRA coefficient still matters for the relative consumption dynamics in that it determines the speed of convergence.[25] We recover the following result:[26,27]

Proposition 4.2. *Under the Assumptions 2.1, 2.2, A.1, consider the equilibrium paths of a CRRA economy with I agents where $\gamma^i = \gamma$ for all $i \in \mathcal{I}$. If there exists an agent, say i, such that*

$$k_{BE}^i > k_{BE}^j \quad \textit{for all } j \neq i\,,$$

then for all t, σ_t, $k_t^i - k_t^j > 0$ for all $j \neq i$, and agent i dominates P*-almost surely. In particular, if agent i is the only agent with correct beliefs, $\mathrm{Q}^i = \mathrm{P}$, and if he has $\beta^i \geq \beta^j$ for all $j \neq i$, then he dominates* P*-almost surely.*

If the agent with most accurate beliefs has also the largest discount factors, then dominates. Note, however, that the dominance is due by the aggregate effect of saving and portfolio decision. Whereas in log-economies, the agent with the most

[25] Note that the speed of convergence has direct effects on long-run survival in large economies, see Massari (2014)

[26] See Section 3.1 of Blume and Easley (2006) for the same result in an economy with bounded aggregate endowment. The result holds also when the aggregate endowment is not bounded and the growth process is i.i.d., as shown by Yan (2008).

[27] When discount factors and beliefs are such that survival indexes are equal results are more subtle, see Blume and Easley (2009). These cases are however non-generic.

accurate beliefs dominates due to his portfolio, in non-log economies portfolios may not always reward the agent with most accurate beliefs. Whether growth premia favor the agent with the most correct beliefs depends on both beliefs and risk preferences of all the other agents, as we have pointed out in Section 4.1. Dominance still occur but the differentiated saving is crucial to the result. In other words, despite log-economies and CRRA non-log economies are equivalent in terms of the long-run outcome of the relative consumption process, economically how those long-run outcomes are achieved is rather different.[28]

Note also that the same result of Proposition 4.2 can be established also for more general aggregate endowment processes, such as unbounded economy with non i.i.d. growth. In fact, despite the growth process influences both equilibrium growth premia and log substitution rates, through normalized state prices Q^0, we have just shown that its impact drops out in their sum.

4.2.2 Heterogeneous γ

When preferences are heterogeneous the difference of log substitution rates and log-optimality premia depends on state prices and, thus, on agents' consumption distribution. However, it is still possible to characterize the long-run dynamics based only on (exogenous) agents' characteristics by computing the relative effect of saving and portfolio decisions in the limit of one agent having most of consumption in one period.

As in Section 4.1, I concentrate on two-agent economies, and denote agents with i and j. In the limit of agent i having all of the consumption the difference of portfolio log-optimality premia coincides with (21). The rate of intertemporal substitution is instead given by

$$\frac{\delta^i}{\delta^j}|_i = \left(\frac{\beta^i}{\beta^j} e^{(\gamma^j-\gamma^i)\,\mathrm{E_P}[\log g]}\right)^{\frac{1}{\gamma^j}} \frac{\sum_{s\in\mathcal{S}} \mathrm{Q}^i_s\, \hat{g}_s^{1-\gamma^i}}{\sum_{s\in\mathcal{S}} (\mathrm{Q}^j_s\, \hat{g}_s^{1-\gamma^j})^{\frac{1}{\gamma^j}} (\mathrm{Q}^i_s\, \hat{g}_s^{1-\gamma^i})^{1-\frac{1}{\gamma^j}}}\,. \tag{25}$$

As with homogeneous preferences, the price dependent component of the NLO terms in (22) and the saving component that incorporates beliefs heterogeneity cancel out. As a result

$$k^i|_i - k^j|_i = \frac{1}{\gamma_j}\left(\log\frac{\beta^i}{\beta^j} + I_P(Q^j) - I_P(Q^i) + (\gamma^j-\gamma^i)\,\mathrm{E_P}[\log g]\right)\,. \tag{26}$$

In particular, asymptotic drifts depend on beliefs onnly through their accuracy. In this case, however, the drift depends also on the expected log growth rate of the economy. Defining the modified survival index

$$k^i_Y = \log\beta^i - I_P(Q^i) - \gamma^i\,\mathrm{E_P}[\log g] \tag{27}$$

[28]The result could be useful when asset markets are incomplete.

one obtains

$$k^i|_i - k^j|_i = \frac{1}{\gamma^j}\left(k_Y^i - k_Y^j\right) \quad \text{and} \quad k^i|_j - k^j|_j = \frac{1}{\gamma^i}\left(k_Y^i - k_Y^j\right) .$$

The modified survival index, which is the equivalent to the one established by Yan (2008) for continuous-time economies, takes into account discount factors, beliefs, and IES/RRA coefficients γ. The latter matters for survival when $\mathrm{E_P}[\log g] \neq 0$ because it influences an agent's substitution rate, as it is evident from (25). When $\mathrm{E_P}[\log g]$ is positive, a large IES (low γ) denotes a high propensity to transfer consumption to future dates and it is thus advantageous for survival.

Similarly to Proposition 4.1 it is only the 'asymptotic' drift that matters, so that the relative size of the modified survival index k_Y determines if an agent dominates or vanishes.[29] Generically, no long-run heterogeneity of beliefs is possible.[30]

Proposition 4.3. *Under the Assumptions 2.1, 2.2, A.1, consider the equilibrium paths of a CRRA economy with two agents, i and j. If*

$$k_Y^i > k_Y^j ,$$

then $k^i|_j - k^j|_j > 0$, $k^i|_i - k^j|_i > 0$, and agent i dominates. In particular, if agent j has correct beliefs and agent i has non correct beliefs but

$$I_P(Q^i) < \log\frac{\beta^i}{\beta^j} + (\gamma^j - \gamma^i)\,\mathrm{E_P}[\log g] ,$$

then agent j vanishes.

As with homogeneous preferences economies, the derivation of the survival index clarifies the relative importance of portfolio and saving decisions. As with Proposition 4.2, the proposition can be generalized for economies where the growth process is not i.i.d.. In this case the survival index becomes time dependent and the relative importance of having a lower IES or having accurate beliefs changes over time. For example, provided discounts rate are equal, only one of the two factors matters for survival in the limit cases of $\mathrm{E_P}[\log g(t)]$ converging to zero or diverging (provided the equilibrium is still well defined).

I appendix B I discuss the applicability of these results to other expected time-separable utilities.

[29] Also in these cases however, preferences, in particular those of the negligible agent, still matter the speed of convergence.

[30] Previous works establish the result also in I-agent economies, see Sandroni (2000), Blume and Easley (2006), Yan (2008), or even in economies with a continuous of agents, see Massari (2014).

4.3 CRRA and (IES$=1$, RRA$\neq 1$) Economies

In an economy where not all agents have CRRA preferences, the effects of saving under uncertainty and NLO compensation terms should not cancel out. As a result all MSH failures $\mathbf{F1}-\mathbf{F3}$ should be possible. I show that this is indeed the case by studying an economy with a CRRA agent, agent i, and an Epstein-Zin agent with IES 1, agent j. For this purpose I shall exploit the results from the previous section and compare agent j to a CRRA investors with same RRA coefficient and beliefs, an investor that would hold the same one period portfolio α, but who uses a different substitution rule. The portfolio analysis is thus the same as for CRRA economies, implying that $\nu^{i,j}|_i$ is as in (21), and we can focus the attention on changes in substitution rates.

For this purpose one needs to distinguish the case of agent i, the CRRA agent, dominating from the case of agent j, the Epstein-Zin agent, dominating. When agent i dominates, he sets the market discount rate. Agent j differs from a corresponding CRRA agent with $IES=1/\gamma^j$ in that his substitution rate is $\delta^j|_i=\beta^j$ instead of

$$\delta^{(j,CRRA)}|_i=(\beta^j)^{\frac{1}{\gamma^j}}(\beta^i)^{1-\frac{1}{\gamma^j}}\sum_{s\in\mathcal{S}}(\mathrm{Q}_s^j)^{\frac{1}{\gamma^j}}(\mathrm{Q}_s^i)^{1-\frac{1}{\gamma^j}}g_s^{\frac{\gamma^i}{\gamma^j}(1-\gamma^j)} \tag{28}$$

which is obtained from (9) with $\rho^j=\gamma^j$ and market discount factor δ set by agent i. As a consequence

$$\frac{\delta^i}{\delta^j}|_i=\delta^{i,(j,CRRA)}|_i\,\frac{\delta^{(j,CRRA)}|_i}{\beta^j}\,.$$

Expliciting $\delta^{(j,CRRA)}|_i$ and using the difference of survival indexes found for the CRRA case one finds

$$k^i|_i-k^j|_i=\frac{1}{\gamma^j}(k_Y^i-k_Y^j)+\left(1-\frac{1}{\gamma^j}\right)\log\frac{\beta^i}{\beta^j}+\log\sum_{s\in\mathcal{S}}(\mathrm{Q}_s^j)^{\frac{1}{\gamma^j}}(\mathrm{Q}_s^i)^{1-\frac{1}{\gamma^j}}g_s^{\frac{\gamma^i}{\gamma^j}(1-\gamma^j)}\,. \tag{29}$$

The relative performance of agent i and j is not only governed by the differences of k_Y survival indexes, as in the CRRA case. Provided $\gamma^j\neq 1$, otherwise we would be back in a CRRA economy, there is an extra term, due to saving, that matters for the survival of agent j. Even when agent i has a higher modified survival index than agent j, the ordering of generalized survival indexes might be different. In fact agent j can still survive provided he postpone consumption more than he would have done as a CRRA agent with $IES=1/\gamma^j$. The above equation gives the precise trade-off.

The other case of interest is when the Epstein-Zin agent, j, dominates. With respect to a CRRA economy there are two differences. As when i dominates,

agent j saves at a different rate then if he had a CRRA substitution rule with $IES = 1/\gamma^j$. Now, however, also agent i substitutes at a different rate because the equilibrium interest rate imposed by agent j is not as in the corresponding CRRA economy. As a result:

$$\frac{\delta^i}{\delta^j}|_j = \delta^{i,(j,CRRA)}|_{(j,CRRA)} \frac{\delta^i|_j}{\delta^i|_{(j,CRRA)}} \frac{\delta^{(j,CRRA)}|_{(j,CRRA)}}{\delta^j|_j},$$

After some simplifications,

$$\frac{\delta^i}{\delta^j}|_j = \delta^{i,(j,CRRA)}|_{(j,CRRA)} \left(\sum_{s \in \mathcal{S}} \mathrm{Q}_s^j \, g_s^{1-\gamma^j} \right)^{\frac{1}{\gamma^i}}.$$

Exploiting what we know for CRRA economies the difference of generalized survival indexes becomes

$$k^i|_j - k^j|_j = \frac{1}{\gamma^i}(k_Y^i - k_Y^j) + \frac{1}{\gamma^i} \log \sum_{s \in \mathcal{S}} \mathrm{Q}_s^j \, g_s^{1-\gamma^j} \tag{30}$$

The ordering of survival indexes k_Y is not enough to determine long-run outcomes. There is an extra term due to the fact that the Epstein-Zin agent has not $IES = 1/\gamma^j$, so that both the amount he saves and the discount rate he imposes differ from the corresponding CRRA economy.

Based on the sign of (29) and (30) it is still possible to characterize long run outcomes along the lines of Proposition 4.1. I particular I shall concentrate on the case when both agents survive, so that beliefs heterogeneity is persistent and consumption keeps fluctuating. For simplicity, I assume that the economy has a constant aggregate endowment, $g_s = 1$ for all $s \in \mathcal{S}$. When agent j dominates, he sets as the same equilibrium discount factor $\delta_t = \beta^j$ as a CRRA agent with $IES = 1/\gamma^j$, and also saves at the same rate β^j. As a result agents' relative performance can be given in terms of survival indexes as in a CRRA economy and (30) becomes

$$k^i|_j - k^j|_j = \frac{1}{\gamma^i}(k_{BE}^i - k_{BE}^j).$$

However, when agent i dominates, agent j substitutes differently than he would have done under CRRA preferences with $IES = 1/\gamma^j$, as can be seen from (29) with $g_s = 1$ for all s. Survival of both agents is established in the next corollary.

Corollary 4.3. *Under the Assumptions 2.1, 2.2, A.1, consider the equilibrium paths of a two-agent exchange economy, with a CRRA agent, i, and an Epstein-Zin agent with $IES = 1$, j, and assume that the aggregate endowment is constant. If*

$$0 < k_{BE}^i - k_{BE}^j < (\gamma^j - 1) \log \frac{\beta^j}{\beta^i} - \gamma^j \log \sum_{s \in \mathcal{S}} (\mathrm{Q}_s^j)^{\frac{1}{\gamma^j}} (\mathrm{Q}_s^i)^{1 - \frac{1}{\gamma^j}},$$

then $k^i|_j - k^j|_j > 0$, $k^i|_i - k^j|_i < 0$, *and both agents survive* P*-almost surely.*

Provided beliefs are heterogeneous, the last term of the inequality is positive when $\beta^j \geq \beta^i$ and $\gamma^j > 1$, so that $IES = 1 > 1/\gamma^j$. The result confirms that it is the differentiated saving of the Epstein-Zin agent j with respect to the corresponding CRRA agent that, in not balancing exactly the term coming from the portfolio NLO term, might keep him alive even when his modified survival index is lower than that of agent i.

4.4 Selection of Intertemporal Substitution Rates

In Section 4.1, we have seen that speculation that results only in different portfolios could generate MSH failures. Here I address the opposite issue, that is, whether saving under uncertainty could lead to market selection failures when growth premia do not play a role.

In order to answer this question, I investigate the outcome of market selection when all agents hold the same portfolio in equilibrium (the market portfolio) so that only saving matters. It turns out that the constraint imposed by the equal portfolio requirement, see Assumption 4.1 below, is such that the 'ordering' of substitution rates is stable. An intuitive result holds: the agent who fixes the highest market discount rate when alone in the market is also the one who saves the most for all possible equilibrium prices, and thus dominates in the long run.

Agents hold the same portfolio when they agree on normalized prices, or

$$\mathrm{Q}^0_s = \frac{\mathrm{Q}^i_s \, g_s^{\gamma^i}}{\sum_{s' \in \mathcal{S}} \mathrm{Q}^i_{s'} \, g_{s'}^{\gamma^i}} \quad \text{for all } s \in \mathcal{S}\,, \tag{31}$$

for all $i \in \mathcal{I}$. The condition can always be met in the sense that, given a set of normalized state prices Q^0, for any RRA coefficient γ there exists beliefs Q^i, namely

$$\mathrm{Q}^i_s = \frac{\mathrm{Q}^0_s \, g_s^{-\gamma^i}}{\sum_{s' \in \mathcal{S}} \mathrm{Q}^0_s \, g_s^{-\gamma^i}} \quad \text{for all } s \in \mathcal{S}\,,$$

such that (31) holds. We can thus assume

Assumption 4.1. *There exists a vector* $\mathrm{Q} \in \Delta^S_+$ *such that for all* $i \in \mathcal{I}$ *beliefs* Q^i *and RRA coefficients* γ^i *satisfy*

$$\frac{\mathrm{Q}^i_s \, g_s^{\gamma^i}}{\sum_{s' \in \mathcal{S}} \mathrm{Q}^i_{s'} \, g_{s'}^{\gamma^i}} = \mathrm{Q}_s \quad \textit{for all } s \in \mathcal{S}\,. \tag{32}$$

If the aggregate endowment were not risky, or if it were risky but all agents had the same risk preferences γ, each agent holding the market portfolio would only occur under homogeneous beliefs. However the combination of a risky aggregate endowment and heterogeneous risk preferences is such that agents could still hold the same portfolio in equilibrium even when they have heterogeneous beliefs. The case is non-generic, perturbing the belief of an agent would break (32), but it serves the purpose of analyzing selection of substitution rates in stochastic economies.

When all agents hold the same portfolio long-run outcomes are only determined by the comparison of their substitution rates in (9). When the initial allocation is such that each agent i starts with a fraction ϕ^i of the aggregate endowment, agents exchange claims on the aggregate endowment to transfer their consumption across dates. Agents with a long position are saving more than agents with a short position and are thus gaining consumption in relative terms. No other asset is traded.

The next proposition establishes that whether an agent has a long or short position can be established by comparing the discount rate that they would set when alone in the market. Whether an agent dominates or vanishes thus depends on the comparison of these single-agent economy rates. At this purpose I derive $\delta|_i$, the equilibrium rate when i is the representative agent. Simple computations lead to

$$\delta|_i = \beta^i e^{-\rho^i \operatorname{E_P}[\log g]} \left(\sum_{s \in \mathcal{S}} \mathrm{Q}_s^i \, \hat{g}_s^{-\gamma^i} \right) \left(\sum_{s \in \mathcal{S}} \mathrm{Q}_s^i \, \hat{g}_s^{1-\gamma^i} \right)^{\frac{\gamma^i - \rho^i}{1-\gamma^i}} . \tag{33}$$

The role of the IES coefficient ρ^i in setting discount rates stands out.

Proposition 4.4. *Under the Assumptions 2.2, 4.1, A.1, consider the equilibrium paths of an economy with I agents maximizing Epstein-Zin preferences.*

i) if for all $j \neq i$

$$\delta|_i > \delta|_j ,$$

then for all t, σ_t, and P, $k_t^i - k_t^j > 0$ for all $j \neq i$, and i dominates on all $\sigma \in \Sigma$;

ii) if there exist a $j \neq i$ such that

$$\delta|_i < \delta|_j ,$$

then for all t, σ_t, and P, $k_t^i - k_t^j < 0$, and i vanishes on all $\sigma \in \Sigma$.

The proof relies on showing that for each pair of agents i and j, the ordering of market discount rates $\delta|_i$ and $\delta|_j$ implies a stable ordering of substitution rates δ_t^i and δ_t^i for all equilibrium discount factors δ_t.

It is important to note that although beliefs do matter, in that through the saving under uncertainty channel they induce higher or lower substitution rates, the truth does not matter in these economies. Agents transfer consumption only across time and not across states. The relative consumption dynamics is deterministic and sufficient conditions i) and ii) imply a stable order of survival indexes for all measures P.[31] Dominance and vanishing hold on every path σ.

Proposition 4.4 establishes only sufficient conditions in that it could happen that two agents define the same maximal discount rate in equilibrium, yet save differently. These situations are even less generic than Assumption 4.1.

Proposition 4.4 can be combined with Proposition 4.3 in the case of CRRA preferences, $\gamma^i = \rho^i$ for all $i \in \mathcal{I}$. Under Assumption 4.1, modified survival indexes k_Y reflect only a differentiated saving behavior and dominance occurs universally, the truth has no role. Indeed, although the survival index k_Y seems to depend on P, the constraint imposed on beliefs by (32) is such that survival indexes computed under different P are all equal. The agent with the highest survival index dominates on all path $\sigma \in \Sigma$.

Proposition 4.5. *Under the Assumptions 2.1, 2.2, 4.1, A.1, consider the equilibrium paths of a CRRA economy. Survival indexes* $\{k_Y^i,\, i \in \mathcal{I}\}$ *do not depend on the truth* P. *The agent with the highest survival index* k_Y *dominates on all* $\sigma \in \Sigma$.

The comparison of rates $\delta|_i$ is also particularly simple in an Epstein-Zin economy without aggregate risk leading to the following corollary.

Corollary 4.4. *Under the assumptions of Proposition 4.4, assume further that there is no aggregate risk,* $g_s = g$ *for all* $s \in \mathcal{S}$. *If for all* $j \neq i$

$$\beta^i g^{-\rho^i} > \beta^j g^{-\rho^j},$$

then i *dominates on all* $\sigma \in \Sigma$.

Under no aggregate risk, the saving economy 'survival index' $\delta|_i$ can be simplified to $\beta^i g^{-\rho^i}$, which expresses the trade off between discount factors and IES coefficient. As in deterministic economies, controlling for discount factors, when $g > 1$ the agent with highest IES (lowest ρ) dominates. The opposite result holds when $g < 1$.

Under aggregate risk, although the comparison of equilibrium discount factors can still be simplified due to Assumption 4.1, its implication for preferences, discount factors, and beliefs is not straightforward. The following Corollary analyzes 'growing' 2-agent economies.

[31] As long as agents' beliefs are i.i.d. and all hold the market portfolio, i.e. Assumptions 2.2 and 4.1 respectively, Assumption 2.1 can be relaxed and the statement holds for any P on $(\Sigma, \{\mathcal{F}\})$.

Corollary 4.5. *Under the assumptions of Proposition 4.4, consider only two agents, i and j, with $\beta^i = \beta^j$ and $\rho^i < 1 < \rho^j$. If $\log \mathrm{E}_{\mathrm{Q}^0}[\hat{g}] + \mathrm{E}_P[\log g] \geq 0$, then agent i dominates surely.*

Controlling for discount factors, in an growing economies where risk neutral market beliefs are not too pessimistic, having a IES larger than 1 is sufficient for dominating against an agent with IES lower than 1, irrespectively of risk preferences.

5 Examples

In this section I shall consider simple illustrative examples for a two-state economy, $S = 2$. The advantage of working with only two states is that equilibrium substitution rates, portfolios, and state prices have a convenient graphical representation in a 2 dimensional plot. Due to the normalizations only the first component of normalized state prices needs to be tracked, the same holds for portfolios. Dropping time indexes to simplify the notation, Q^0 is the normalized price of state 1, Q^i the probability assigned from agent i to the realization of state 1, $\delta^i(\delta, \mathrm{Q}^0)$ the intertemporla substitution rule and $\alpha^i(\mathrm{Q}^0)$ the portfolio that allocates next period consumption.

We concentrate on portfolio rules first. Figure 1 illustrates two examples of portfolio rules, showing how normalized equilibrium prices are determined by their aggregation. In the left panel, there is no aggregate risk. In the right panel, there is aggregate risk. In both cases, according to the market equilibrium equation (7), the equilibrium price Q^0 is found at the intersection of the convex combination of rules $\alpha^i(\mathrm{Q}^0)$, $i = 1, 2$, with the first component of the normalized supply, $l_1(\mathrm{Q}^0)$. In a dynamic economy weights $(\varphi, 1-\varphi)$ are given both by relative consumption and substitution rates, as in (7).

On the same plot one can also visualize the stability conditions. For this purpose we shall assume that the two states are equally likely, $\mathrm{P} = (1/2, 1/2)$. The advantage is that the relative entropy $I_P(Q)$ becomes symmetric around its minimum $\mathrm{Q} = 1/2$, $I_P(Q) = I_P(1-Q)$. As a result portfolio premia can be evaluated using the euclidean distance of their first components: given $\alpha^i, \alpha^j \in (0,1)$, $I_P(\alpha) \gtreqless I_P(\alpha)$ if and only if $|\alpha^i - 1/2| \gtreqless |\alpha^j - 1/2|$.[32]

In Figure 2, I add a graphical representation of growth premia and of their decomposition:

$$\mu^i = I_P(\mathrm{Q}^0) - I_P(\alpha^i(\mathrm{Q}^0)) = \left[I_P(\mathrm{Q}^0) - I_P(Q)\right] + \nu^i \,.$$

[32]If $\mathrm{P} \neq (1/2, 1/2)$ one should simply re-scale the vertical axis to compare 'left' and 'right' portfolio deviations from the truth.

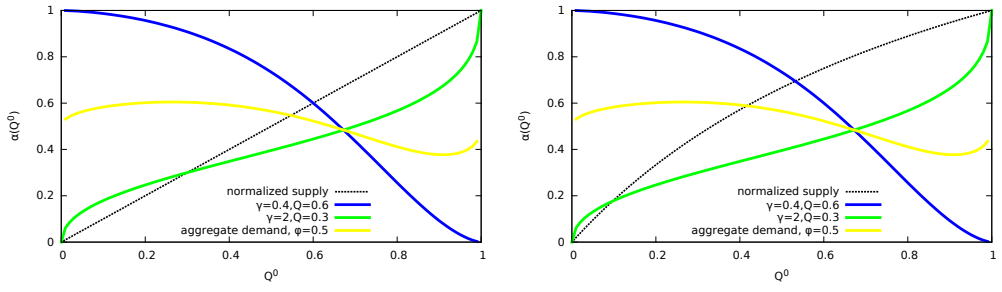

Figure 1: Market equilibrium with CRRA rules. In homogeneous (heterogeneous) economies clearing prices are determined by the intersection of a rule (a convex combination of rules) with the normalized supply. Left panel: no-aggregate risk. Right panel: aggregate risk, $g_1 = 2g_2$.

In the panel the solution of $\alpha^i(\mathrm{Q}^0) = \mathrm{Q}^0$ is equal to agent i belief Q^i. The horizontal line P represent the true probability that state 1 is realized. The vertical euclidean distance between the two horizontal lines P and Q is proportional to $I_P(Q)$. The full expected return μ^i computed at a price Q^0 can thus be visualized as the difference of the distances of $q = 0.3$ and $\alpha^i(q)$ from P. When, as in the plot, $\alpha^i(q)$ is further than q, the growth premium is negative. Although beliefs are more accurate than prices, $I_P(q) - I_P(Q) > 0$, by non using log-optimal portfolio agent i has a negative NLO term, the difference of the distances of Q^i and $\alpha^i(q)$ from P. Effective beliefs are less accurate than beliefs. As we shall see also the opposite might occur.

This graphical analysis can be used to illustrate the finding of Corollary 4.1. The left panel of Figure 3 shows an example where agents are more risk adverse than the log agent, $\gamma^i = \gamma^j > 1$, and $S = 2$. Assume that agent i beliefs are more accurate than agent j beliefs. If both agents had log-optimal portfolios then agent j would have positive expected log-returns in every period. However by using less risky portfolios each agent receives a particularly high NLO term when state prices coincide with the other agent belief. As a result, notwithstanding that agent i has less accurate beliefs, there exist state prices where agent i has a higher expected log-return, and the other-way round. The right panel of Figure 3 shows the same example where $\gamma^i = \gamma^j < 1$. In this case each agent receives a particularly low NLO compensation term when prices are close to the beliefs of the other agent. As a result there exist state prices where agent i has a lower expected log-return.

In Epstein-Zin economies where agents have the same $\rho = 1$ and the same β, only growth premia matter for survival. Portfolios as in the left panels are associated to long-run heterogeneity whereas portfolios as in the right panel are associated to dominance depending on the initial conditions, case i) and ii) of

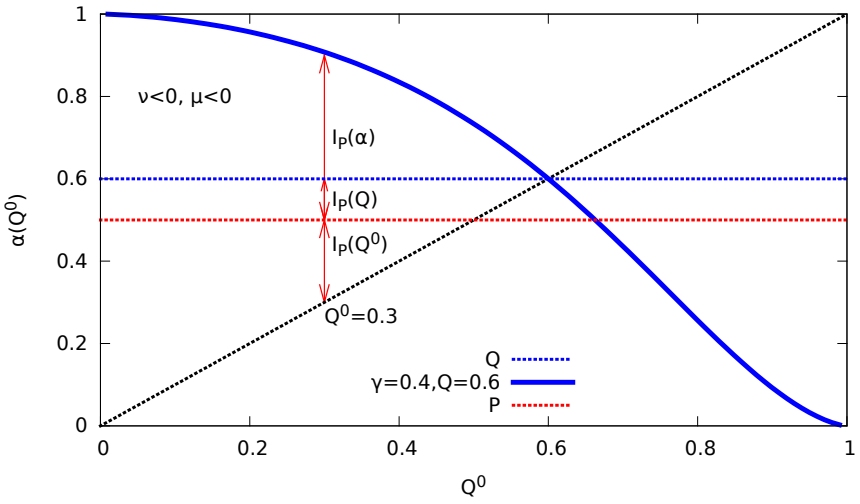

Figure 2: Expected log-returns decomposition when $q_1 = 0.3$.

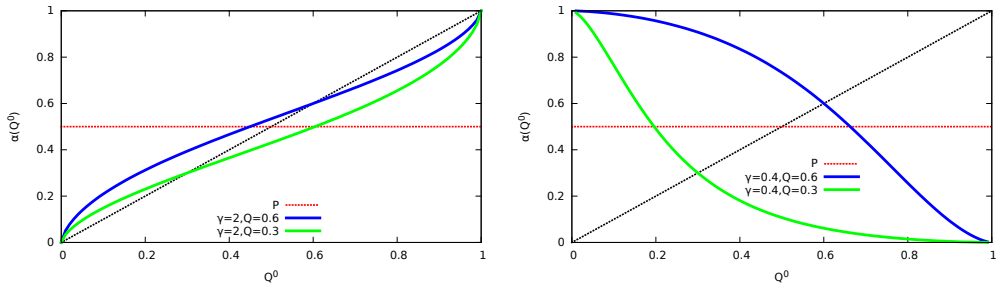

Figure 3: Comparison of growth premia in a market without aggregate risk and two agents with the same RRA coefficient γ and heterogeneous beliefs. Left panel: $\gamma > 1$. Right panel: $\gamma < 1$.

Proposition 4.1 respectively. The figure clarifies that both cases are generic: by locally perturbing preferences or beliefs long-run outcomes do not change.

In Figure 4, I illustrate the finding of Corollary 4.2 on the three possible failures of the MSH. In the left panel $\gamma > 1$, in the right panel $\gamma < 1$. In both examples there exists one agent with correct beliefs, and both no-aggregate and aggregate risk cases can be visualized. Only a couple of agents, the one who knows the truth and one among i, j, and k, should be considered at a time.

When $\gamma > 1$, knowing the truth leads to dominance under no aggregate risk, but might lead to dominance (against k), survival of both (against j), or even

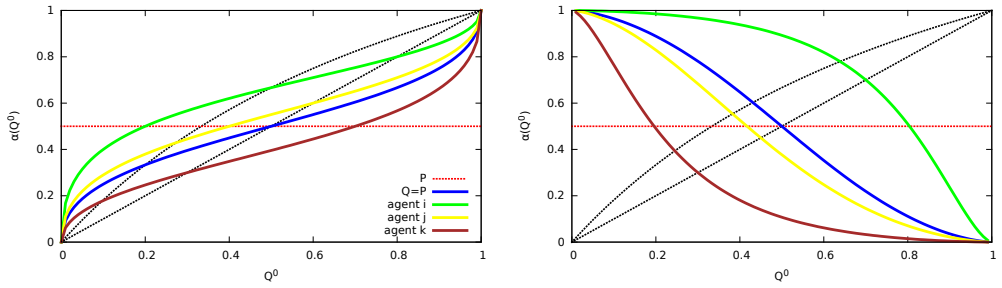

Figure 4: Comparison of growth premia in a market with and without aggregate risk when one agent has correct beliefs. Only two agents at a time, i.e. two portfolio rules, should be considered. Left panel: $\gamma > 1$. Right panel: $\gamma < 1$.

vanishing (against i), under aggregate risk. The plot confirms that results found in DeLong et al. (1991) are valid also in a general equilibrium model. When agents' RRA is higher than $\gamma = 1$, a noise trader with optimistic beliefs might have a portfolio that is closer to the log-optimal portfolio derived under the truth than the portfolio derived using correct beliefs. When optimism is mild (as for agent j) the latter observation holds for all possible equilibrium prices and the optimistic trader dominates. When the optimism is strong (as for agent i) there are prices where it is the agent with correct beliefs that has a higher growth premium. Since each agent has a higher growth premium at the prices set by the other agent, both survive. Trading never settles and state prices keep fluctuating between the evaluation of the rational trade and that of the noise trader. An equilibrium path of normalized state prices and of relative consumption shares is shown in Figure 5.

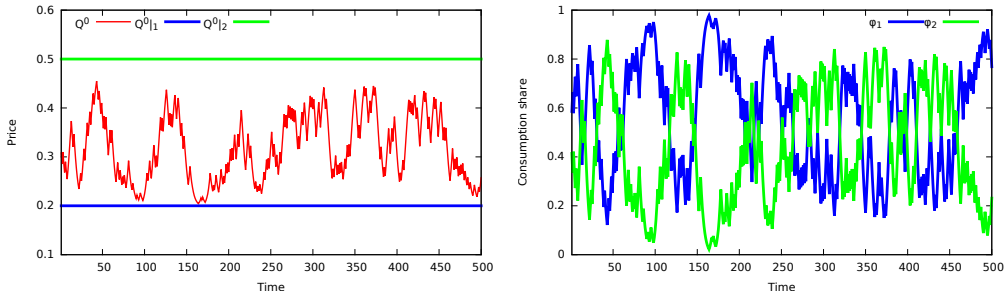

Figure 5: Equilibrium path of state $s = 1$ normalized prices (left panel) and agents' relative consumption shares (right panel) when $g_1 = 2g_2$, $\rho^1 = \rho^2 = 1$, $\beta^1 = \beta^2 = \beta \in (0,1)$, $\gamma^1 = \gamma^2 = 2$, $\mathrm{Q}^1 = \mathrm{P} = (1/2, 1/2)$, $\mathrm{Q}^2 = (4/5, 1/5)$.

When $\gamma < 1$ long-run heterogeneity never occurs. Under no-aggregate risk

either the rational trader dominates almost surely (against i) or he dominates only for some initial conditions but vanishes for others (against j and k). Under aggregate risk, both outcomes are still possible (against i and k respectively) but the rational agent might even vanish (against j). The result extends Theorem 5.4 of Blume and Easley (1992) in that long-run heterogeneity is possible and, even under no aggregate risk, the rational trader might vanish depending on the initial conditions.

As explained in Section 4.2, despite in CRRA economies the relative size of portfolios growth premia is exactly as just discussed for these Epstein-Zin economies, the component of intertemporal substitution rates that incorporates beliefs heterogeneity compensates for the inaccuracy of effective beliefs. In the left panel of Figure 6, I plot the ratio between the intertemporal rate of substitution of the agent with correct beliefs and the intertemporal rate of substitution of agent i as in the left panel of Figure 4. In the right panel of the same figure I consider agent k of the right panel of Figure 4 instead of agent i. In both cases the ratio is plotted for the interval of state 1 normalized prices that are possible in the corresponding economy. In the left panel both agents have RRA higher than 1. The agent with correct beliefs saves always more than the agent with incorrect beliefs, especially more at the prices he sets, $Q^0 = 0.2$, thus counter-balancing the bad performance of his portfolio there. The same happens when both agents have RRA lower than 1. In this case however the agent with correct beliefs saves more than the other agents at the prices set by the other agent, $Q^0 \approx 0.23$, counter-balancing the bad performance of his portfolio in this price range. In both cases the combined effect of saving and portfolio is such that the agent with accurate beliefs dominates. In both cases saving is crucial. Only the decomposition of the generalized survival index in terms of saving and portfolio contributions can shed light on this point.

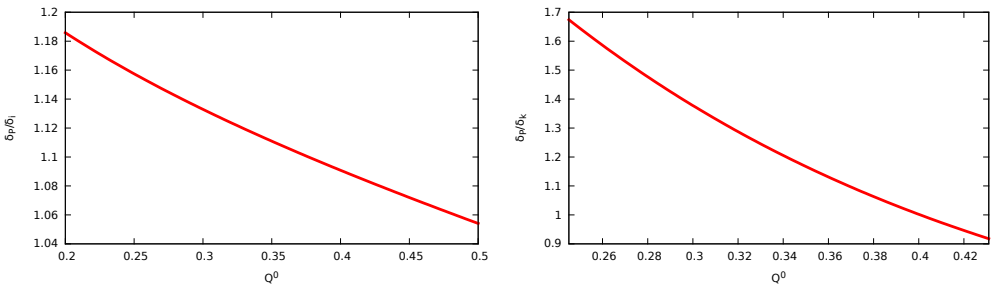

Figure 6: Ratio of intertemporal substitution rates of the agent with correct beliefs and agent i (left panel) or agent k (right panel) as a function of state 1 normalized prices. Left panel: both agents have $\gamma = \rho = 2$. Right panel: both agents have $\gamma = \rho = 0.4$.

5.1 Generalizations

All the above examples are in two-agent economies. When more than two agents are trading in the same market there is no conceptual difficulty, growth premia and substitution rates are still determining long-run outcomes. The limitation is technical in that growth premia need to be evaluated in the limit of one group of agents consuming all the aggregate endowment, and in these limits state prices depend on all remaining agents' consumption distribution. Sufficient conditions similar to those of Proposition 4.1 can still be established but are far from being tight.[33] In Figure 7, I present three examples of three-agent economies. In the left panel, under no-aggregate risk, agent i survives against the combination (j,k), because the growth premium of i is larger than the growth premium of (i,k) for all prices determined by (i,k). However, agent i does not dominate. Under aggregate risk, instead, agent i dominates. In the right panel, agent i vanishes.

The same weakening of the sufficient conditions applies also to non i.i.d. economies. When the growth rate g follows a generic process, or when beliefs are not i.i.d., equilibrium prices computed under the assumption that an agent, or a group of agents, consumes the aggregate endowment become a random variable. Growth premia should thus be compared for all the relevant possible histories of the process. Only when inequalities hold in all these cases, they are sufficient to characterize long-run outcomes.

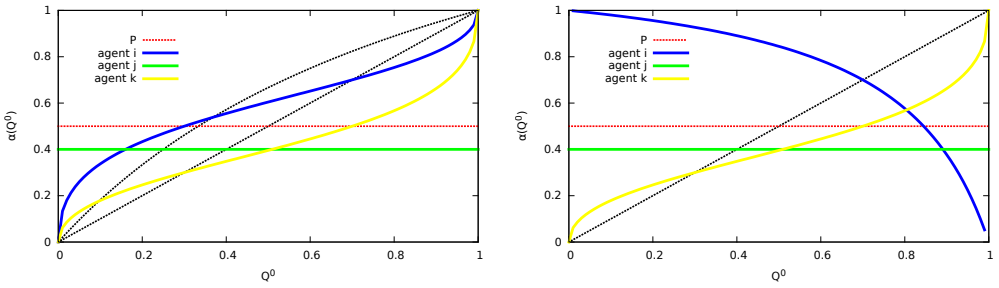

Figure 7: Three-agent economies. Left panel: agent i survives but does not dominate under no-aggregate risk, and dominates under aggregate risk. Right panel: agent i vanishes.

Finally note that the same approach used in this paper can be extended beyond Epstein-Zin economies whenever an equilibrium path of prices and consumption distribution can be shown to exist, and date t one-period portfolio and substitution

[33] I do not have an explicit proposition for I-agent economies, the interested reader can refer to Bottazzi et al. (2015).

rules depend on information up to t. The system of equations (3) and (5-7) can be used to characterize consumption and state prices in the long-run, and thus to address the MSH.

Consider for example a temporary equilibrium model of sequential trading where agents are not assumed to have rational expectations on future prices. Each agent decides how much wealth to save and how to allocate the saved wealth to the purchase of Arrow securities by using (adapted) rules $\bar{\delta}_t^i$ and $\bar{\alpha}_t^i$ that depend only on the information available till time t. If future prices are involved, the expectation should be computed given the information up to t. One period saving and portfolio rules δ_t^i and α_t^i, can be easily derived from the wealth dynamics as

$$\begin{array}{rcl} \delta_t^i & = & \frac{\bar{\delta}_t^i}{1-\bar{\delta}_t^i} \sum_{s\in\mathcal{S}} (1-\bar{\delta}_{t+1}^i(\sigma_t,s))\bar{\alpha}_{s,t}^i \\ \alpha_{s,t}^i & = & \frac{(1-\bar{\delta}_{t+1}^i(\sigma_t,s))\bar{\alpha}_{s,t}^i}{\sum_{s'\in\mathcal{S}}(1-\bar{\delta}_{t+1}^i(\sigma_t,s'))\bar{\alpha}_{s',t}^i}\,, \quad \text{for all } s\in\mathcal{S}\,. \end{array} \tag{34}$$

One period substitution and portfolio rules also depend only on the information up to period t. Two-agent economies where one agent maximizes a CRRA utility and the other agent uses a behavioral rule of this sort can be thus analyzed. Whether an agent vanishes, survives, or dominates is determined the by comparison of generalized survival indexes given by log substitution rates and growth premia.

6 Conclusion

This paper shows that in dynamic stochastic exchange economies where agents have heterogeneous beliefs and maximize Epstein-Zin preferences, speculation does not support the Market Selection Hypothesis. The result is established by characterizing long-run outcomes of agents' relative consumption process in terms of the comparison of agents' log substitution rates and portfolio growth premia.

In the special case of log-economies, provided discount factors are equal, portfolio growth premia depend only on beliefs accuracy. Portfolio speculative returns favor the agent with correct beliefs. However, outside the log-utility framework, the growth premium depends both on beliefs accuracy and on the comparison of an agent's portfolio choice with the corresponding log-optimal portfolio. This last term, named the Non-Log-Optimality (NLO) contribution, leads to generic failures of Market Selection Hypothesis. In an economy where all agents use the same intertemporal substitution rates, three types of failures are identified: multiple agents survive a.s., leading to heterogeneity of beliefs also in the long run; the agent with accurate beliefs vanishes on some paths and dominates on others; the agent with accurate beliefs vanishes a.s.. The failures are shown to be robust to cases where agents use different intertemporal substitution rates. CRRA economies are instead special because, due to interdependence of intertemporal and

risk preferences, the response to beliefs heterogeneity incorprated in intertemporal substitution rates and NLO terms compensate each-others. The only long-run outcome is the dominance of a unique agent, so that the only possible MSH failure is the vanishing of the agent with most accurate beliefs. However, also in CRRA economies, the relative importance for long/run survival of saving and portfolio decisions depends on all agents' preferences and beliefs.

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A General Equilibrium with Epstein-Zin Agents

A.1 Saving and portfolio decisions

Saving and portfolio decisions, $\bar{\delta}_t^i$ and $\bar{\alpha}_t^i$ respectively, can be computed starting from intertemporal substitution rates and one-period portfolio decisions, δ_t^i and α_t^i respectively. Agent i wealth after history σ_t is

$$w_t^i = c_t^i + \sum_{T>0,\, \sigma_{t+T}} q_{\sigma_{t+T},t}\, c_{t+T}^i \,,$$

where σ_{t+T} takes values in $\Sigma_{t+T}(\sigma_t)$, the subset of Σ_{t+T} whose elements have a common history σ_t. Iterating the consumption dynamics (3) in the expression for w_t^i above one finds

$$w_t^i(\sigma_t) = c_t^i(\sigma_t) \left(1 + \delta_t^i(\sigma_t) + \delta_t^i(\sigma_t) \sum_{T>0,\, \sigma_{t+T}} \prod_{\tau=1}^{T} \delta_{t+\tau}^i(\sigma_{t+\tau}) \alpha^i_{s_{t+\tau},t+\tau-1}(\sigma_{t+\tau-1}) \right) .$$

Agent i date t saving decision, $\bar{\delta}_t^i$ such that $c_t^i = w_t^i(1-\bar{\delta}_t^i)$, is thus

$$\bar{\delta}_t^i = \delta_t^i \frac{1 + \sum_{T>0,\, \sigma_{t+T}} \prod_{\tau=1}^{T} \delta_{t+\tau}^i \alpha^i_{s_{t+\tau},t+\tau-1}}{1 + \delta_t^i + \delta_t^i \sum_{T>0,\, \sigma_{t+T}} \prod_{\tau=1}^{T} \delta_{t+\tau}^i \alpha^i_{s_{t+\tau},t+\tau-1}} \,. \tag{35}$$

Finiteness of total wealth and positiveness of consumption guarantee that $\bar{\delta}_t^i \in (0,1)$.

From the saving decision $\bar{\delta}_t^i$ and the consumption dynamics (3) it is possible to derive the wealth dynamics

$$w_{t+1}^i = \bar{\delta}_t^i \frac{\bar{\alpha}^i_{s,t}}{q_{s,t}} w_t^i \quad \text{on} \quad (\sigma_t, s) \,, \tag{36}$$

where the portfolio decision $\bar{\alpha}_t^i \in \Delta_+^S$ is

$$\bar{\alpha}^i_{s,t}(\sigma_t) = \alpha^i_{s,t}(\sigma_t) \frac{\delta_t^i(\sigma_t)}{\bar{\delta}_t^i(\sigma_t)} \frac{1-\bar{\delta}_t^i(\sigma_t)}{1-\bar{\delta}_{t+1}^i(\sigma_t, s)} \,. \tag{37}$$

Using the expression for $\bar{\delta}_t^i$ in (35) it is possible to explicit the portfolio decision $\bar{\alpha}^i$ in terms of one-period portfolios and substitution rates:

$$\bar{\alpha}^i_{s,t} = \alpha^i_{s,t} \frac{1 + \delta_{t+1}^i(\sigma_t, s) + \delta_{t+1}^i(\sigma_t, s) \sum_{T>0,\, \sigma_{t+1+T}} \prod_{\tau=1}^{T} \delta_{t+\tau}^i(\sigma_{t+\tau}) \alpha^i_{s_{t+\tau},t+\tau-1}(\sigma_{t+\tau-1})}{1 + \sum_{\tau>0,\, \sigma_{t+\tau}} \prod_{\tau=1}^{T} \delta_{t+\tau}^i(\sigma_{t+\tau}) \alpha^i_{s_{t+\tau},t+\tau-1}(\sigma_{t+\tau-1})} \,,$$

where, for every s, σ_{t+1+T} takes values in $\Sigma_{t+1+T}((\sigma_t, s)) \subset \Sigma_{t+1+T}$.

Although full and one-period decisions differ, there exists a limit under which they coincide. The following result characterizes when it is the case.

Lemma A.1. *If in equilibrium agent i intertemporal substitution rates are history-independent (deterministic),*

$$\delta_t^i(\sigma_t) = \delta_t^i(\sigma_t') \quad \textit{for all} \quad t, \sigma_t, \sigma_t'$$

then

$$\bar{\alpha}_t^i = \alpha_t^i$$

If, moreover, intertemporal substitution rates are time-independent,

$$\delta_t^i = \delta_{t+1}^i = \delta^i \quad \textit{for all} \quad t,$$

then

$$\bar{\delta}_t^i = \delta^i \,.$$

Proof. Provided intertemporal substitution rates are history-idependent, i.e. deterministic, the ratio

$$\frac{\delta_t^i(\sigma_t)}{\bar{\delta}_t^i(\sigma_t)} \frac{1 - \bar{\delta}_t^i(\sigma_t)}{1 - \bar{\delta}_{t+1}^i(\sigma_t, s)}$$

on the left hand side of (37) is one for all $s \in \mathcal{S}$, proving the first part of the Lemma. If substitution rates are also stationary, the sum of the geometric series of compounded rates can be computed leading to $\bar{\delta}_t^i = \delta^i$. □

A.2 Portfolio and saving decisions under Epstein-Zin preferences

In agent agent i maximizes a recursive utility of the type (8) one can use the first order conditions, see e.g. Epstein and Zin (1991), to characterize equilibrium allocation and prices. In terms of saving and portfolio decisions one finds

$$\frac{\mathrm{Q}_s^i}{q_{s,t}} (\beta^i)^{\frac{1-\gamma^i}{1-\rho^i}} \left(\frac{\delta_t^i \alpha_{s,t}^i}{q_{s,t}} \right)^{-\rho^i \frac{1-\gamma^i}{1-\rho^i}} \left(\frac{\bar{\alpha}_{s,t}^i}{q_{s,t}} \right)^{\frac{\rho^i - \gamma^i}{1-\rho^i}} = 1 \quad \text{for all} \quad s,\, t,\, \sigma_t \,. \tag{38}$$

Unless we are in the CRRA case, $\gamma^i = \rho^i$, the full portfolio $\bar{\alpha}_t^i$ enters in the first order condition. Since $\bar{\alpha}_t^i$ depends on all future one-period substitution and portfolio decisions, all first order conditions are coupled. However when Lemma A.1 applies, $\bar{\alpha}_t^i$ and α_t^i coincide so that (38) can be solved to find one-period optimal decisions in terms of beliefs, preferences, and market prices.

We have the following

Proposition A.1. *If in equilibrium agent i intertemporal substitution rates are history-independent, then for all t and σ_t, it holds*

$$\delta_t^i = \delta_t \left(\frac{\beta^i}{\delta_t}\right)^{\frac{1}{\rho^i}} \left(\sum_{s' \in S} (\mathrm{Q}_{s'}^i)^{\frac{1}{\gamma^i}} (\mathrm{Q}_{s',t}^0)^{1-\frac{1}{\gamma^i}}\right)^{\frac{\gamma^i}{\rho^i}\frac{1-\rho^i}{1-\gamma^i}}, \tag{39}$$

$$\alpha_{s,t}^i = \frac{(\mathrm{Q}_s^i)^{\frac{1}{\gamma^i}} (\mathrm{Q}_{s,t}^0)^{1-\frac{1}{\gamma^i}}}{\sum_{s' \in S} (\mathrm{Q}_{s'}^i)^{\frac{1}{\gamma^i}} (\mathrm{Q}_{s',t}^0)^{1-\frac{1}{\gamma^i}}}, \quad s = 1, \ldots, S. \tag{40}$$

Moreover, $\bar{\alpha}_t^i = \alpha_t^i$.

Proof. The result follows from the application of Lemma A.1, which allows to use α_t^i in place of $\bar{\alpha}_t^i$, and from the direct solution of (38) in terms of α_t^i and δ_t^i. □

When substitution rates are history-independent both substitution and portfolio decisions depend only on contemporaneous market prices and rates, as in an expected utility framework. The functions $\delta^i(\cdot,\cdot)$ and $\alpha^i(\cdot)$ such that $\delta_t^i = \delta^i(\delta_t, \mathrm{Q}_t^0)$ and $\alpha_t^i = \alpha^i(\mathrm{Q}_t^0)$ are, respectively, the intertemporal substitution rule and one-period portfolio rule of agent i.

Although the proposition does not say when equilibria are such that the one-period substitution decision coming from these rules is history, or time, independent, one can judge directly from the functional form in (39). This is the content of the next two corollaries. The first illustrates the well-known case of simple saving rules when the IES parameter is 1.

Corollary A.1. *If agent i has $\rho^i = 1$, then for all t and all σ_t $\bar{\delta}_t^i = \delta_t^i = \beta^i$ and $\bar{\alpha}_t^i = \alpha_t^i = \alpha^i(\mathrm{Q}_t^0)$.*

Proof. Other than from direct substitution of $\rho = 1$ in (39), the result can be established starting from the Euler equation of the recursive formulation limit, see Epstein and Zin (1991). □

The corollary applies also when growth rates are not i.i.d.. Instead the next result applies only when the economy is i.i.d., both in beliefs and growth rates, see Assumptions 2.1-2.2.

Corollary A.2. *In an economy where Assumptions 2.1-2.2 hold, if all agents hold the market portfolio, then for all t and σ_t agent i one-period substitution decisions are as in* (39) *with*

$$\mathrm{Q}_{s,t}^0 = \left(\frac{\mathrm{Q}_s^i}{g_s^{\gamma^i}}\right) \left(\sum_{s' \in \mathbb{S}} \frac{\mathrm{Q}_{s'}^i}{g_{s'}^{\gamma^i}}\right)^{-1}, \quad s = 1, \ldots, S. \tag{41}$$

Proof. Under Assumptions 2.1-2.2 beliefs and growth rates are i.i.d. so that, when all agents hold the market portfolio, state prices as well as one-period portfolio decisions do not depend on time and states. One-period substitution rules still depend on time, through the market discount rate, but do no depend on partial histories as the dynamics of discount rates is deterministic. As a result Lemma A.1 applies and Proposition A.1 holds. □

A.3 General equilibrium

Given an economy with a set $\mathcal{I}$ of Epstein-Zin agents, consumption paths $\{c_t^i\}$ for all i, normalized states prices $\{\mathrm{Q}_t^0\}$, and market discount rates $\{\delta_t\}$ generated by (3-7) with rules as in (39-40) are an equilibrium of the exchange economy for a given initial allocation $\{c_0^i \text{ for all } i\}$ provided that: i) an interior equilibrium is shown to exist, otherwise the system (3-7) might have no solutions; ii) agents value function are finite in equilibrium, so that recursive preferences are well defined and Euler equations are sufficient, see also Epstein and Zin (1989) and Ma (1993).

Regarding i), under time-0 trading the existence of an equilibrium follows from Peleg and Yaari (1970), provided the recursive formulation of utility gives a well define utility over consumption streams and provided strict desirability holds. Both require finiteness of the value functions, that is ii). Since Epstein-Zin preferences are dynamically consistent, as long as markets are (dynamically) complete and an equilibrium exists, time-0 trading and sequential trading achieve the same equilibrium allocations. Depending on the chosen asset structure, different assumptions on the budget constraint are necessary to guarantee the existence of an equilibrium: under date $t = 0$ trading no bankruptcy is allowed, under sequential trading no bankruptcy and no Ponzi schemes are allowed, see also Araujo and Sandroni (1999). When an equilibrium exists, it must be interior: agents consumption is positive on all paths σ due to the fact that for every t consumption in t and expected value of date $t+1$ utility are evaluated via a CES aggregator with a finite elasticity of substitution equal to $1/\rho$. As a result, it is never optimal to have zero consumption. Regarding ii) a sufficient condition is that each agent value function is finite when he consumes all the aggregate endowment along the paths of maximal and minimal growth, that is, assuming that there is no uncertainty in the economy. To see why, name s^+ the state of maximal growth and s^- the state of minimal growth. Agent i utility on the path $\{e_t^s\} = \{e_0, g_s e_0, g_s^2 e_0, \ldots\}$, with s either s^+ or s^-, can be easily computed from (8) as

$$U_0^i = \left((1-\beta) \sum_{t=0}^{\infty} e_0 \left(\beta^i g_s^{1-\rho^i} \right)^t \right)^{\frac{1}{1-\rho^i}} .$$

The time zero utility is finite provided that

$$\beta^i g_s^{1-\rho^i} < 1 \quad s = s^+, s^- \tag{42}$$

Note also that since the path of maximum and minimum growth are certain, and no agent consumes all the aggregate endowment, then

$$\{e_t^{s^-}\} \leq \{c_t^i\} \leq \{e_t^{s^+}\}$$

(inequalities for sequences are valid component by component). Adding that preferences are monotone, the latter implies

$$U_0^i(\{c_t^i\}) \in (\infty, \infty)$$

for all the feasible allocations $\{c_t^i\}$, provided that agent i preferences satisfy the bound (42).

The argument is concluded by assuming that for each agent $i \in \mathcal{I}$ discount factors β^i and IES coefficients ρ^i are such that both inequalities (42) hold. We have the following.

Assumption A.1. *For every agents $i \in \mathcal{I}$, the discount factor β^i and the IES parameter ρ^i are such that* (42) *in both the maximum and minimum growth state.*

Finally, I have not excluded the possibility that multiple equilibria exist. As long as each equilibrium obeys (3-7), the market selection results derived from growth premia and intertemporal rates of substitutions apply.

B Time-separability beyond the CRRA case

Sandroni (2000) and Blume and Easley (2006) show that discount factors and beliefs determine long-run survival for all economies where preferences are represented by an expected time-separable utility with Bernoulli utility $u(c)$ satisfying $u'(c) > 0$, $u''(c) < 0$, and $\lim_{c\to 0} u'(c) = +\infty$, provided that the aggregate endowment is bounded from above and from below. Does the the same trade-off between portfolio log-returns and log substitution ratios hold also when u is not of the CRRA type? Under the same assumptions on u, the marginal utility $f_i(c^i) = du^i(c^i)/dc^i$ is a strictly decreasing positive function unbounded from above with well defined inverse $f_i^{-1}(\cdot)$. Solving the Euler equations leads to the following portfolio and substitution rule:

$$\begin{aligned} \delta_t^i &= \textstyle\sum_{s' \in \mathcal{S}} f_i^{-1}\left(\frac{\mathrm{Q}^0_{s',t}}{\beta^i \mathrm{Q}^i_{s',t}} f_i\left(c_t^i\right)\right) \frac{\mathrm{Q}^0_{s',t}}{c_t^i}, \\ \alpha_{s,t}^i &= \frac{f_i^{-1}\left(\frac{\mathrm{Q}^0_{s,t}}{\beta^i \mathrm{Q}^i_{s,t}} f_i\left(c_t^i\right)\right) \frac{\mathrm{Q}^0_{s,t}}{c_t^i}}{\delta_t^i}. \end{aligned}$$

Although it is difficult to use the former to characterize long-run consumption, it is evident that the two are related. Moreover, given the decomposition of a portfolio growth premium, f_i determines only the NLO term. As with CRRA preference, in order for the asymptotic relative 'ranking' not to depend on normalized state prices, NLO terms and differences of log substitution rates should compensate each-other.

C Proofs of Section 4

Proof of Proposition 4.1 Given a filtered probability space $(\mathrm{P}, \Sigma, \Im)$ and a real process x_t defined on $(\Sigma, \Im)$, adapted to the filtration $\{\Im_t\}$, Bottazzi and Dindo (2015) prove the following theorems, which rely on the Martingale Convergence Theorem and owe to Lamperti (1960).

Theorem C.1. *Consider a finite increments process x_t with $|x_{t+1} - x_t| < B$ P-a.s.. If there exist $M > B$ and $\epsilon > 0$ such that,* P*-a.s.,* $\mathrm{E}[x_{t+1}|x_t = x, \Im_t] < x - \epsilon$ *for all $x > M$ and* $\mathrm{E}[x_{t+1}|x_t = x, \Im_t] > x + \epsilon$ *for all $x < -M$, then there exists a real interval $L = (a, b)$ such that for any t it is* $\mathrm{Prob}\{x_{t'} \in L \text{ for some } t' > t\} = 1$.

Theorem C.2. *Consider a finite increments process x_t with $|x_{t+1} - x_t| < B$ P-a.s.. and such that for all t* $\mathrm{Prob}\{x_{t+1} - x_t > \gamma | \Im_t\} > \gamma$ *for some $\gamma > 0$. If there exist $M > B$ and $\epsilon > 0$ such that,* P*-a.s.,* $\mathrm{E}[x_{t+1}|x_t = x, \Im_t] > x + \epsilon$ *for all $x > M$ and* $\mathrm{E}[x_{t+1}|x_t = y, \Im_t] > x + \epsilon$ *for all $x < -M$, then* $\mathrm{Prob}\{\lim_{t\to\infty} x_t = +\infty\} = 1$.

Theorem C.3. *Consider a finite increments process x_t with $|x_{t+1} - x_t| < B$* P*-a.s.. and such that for all t* $\mathrm{Prob}\{x_{t+1} - x_t > \gamma | \Im_t\} > \gamma$ *and* $\mathrm{Prob}\{x_{t+1} - x_t < -\gamma | \Im_t\} > \gamma$ *for some $\gamma > 0$. If there exist $M > B$ and $\epsilon > 0$ such that,* P*-a.s.,* $\mathrm{E}[x_{t+1}|x_t = x, \Im_t] > x + \epsilon$ *for all $x > M$ and* $\mathrm{E}[x_{t+1}|x_t = y, \Im_t] < x - \epsilon$ *for all $x < -M$, then there exists two sets of initial conditions, Γ^+ and Γ^- with $\Gamma^+ \cup \Gamma^- = \mathbb{R}$, such that $\lim_{t\to\infty} x_t = +\infty$ if $x_0 \in \Gamma^+$ and $\lim_{t\to\infty} x_t = -\infty$ if $x_0 \in \Gamma^-$.*

Statement *i*) of the Proposition follows from Theorem C.1, provided we prove that the log consumption ratio $x_t = z_t^{i,j}$ has finite increments B. In fact, by continuity of the conditional drift[34], there exists an $M > B$ such that the drift hypothesis of Theorem C.1 are satisfied when the limit of the drift is positive for $z \to -\infty$ and negative for $z \to +\infty$. The latter follows from by the assumed inequality on relative beliefs accuracy and NLO terms, as explained in the text above the proposition. Using the same argument statements *iii*) and *ii*) follow

[34]The continuity of the conditional drift follows from the continuity CRRA rules α^i and α^j seen as a function of z, which in turn follows from the existence of continuous maps $q_s(\phi^i)$ in the neighborhood of $\phi^i = 1$ and $\phi^i = 0$ (due to the local uniqueness of homogeneous economy equilibria).

from Theorem C.2 and Theorem C.3, respectively, provided we prove that the log consumption ratio $z_t^{i,j}$ has finite increments and a finite probability to jump of at least a given step. This is the content of the following lemma.

Lemma C.1. *Under the assumption of Proposition 4.1 the log relative consumption process $z_t^{i,j} = x_t$ has finite increments, that is, there exists a $B > 0$ such that*

$$|x_{t+1} - x_t| < B \quad \mathrm{P} - \textit{almost surely}\,.$$

Moreover if one of the sufficient conditions i) to iii) hold the process has a finite probability of jumping of at least a given step, that is, there exists a $\gamma > 0$ such that

$$\mathrm{Prob}\left\{x_{t+1} - x_t > \gamma | \Im_t\right\} > \gamma \quad \textit{and} \quad \mathrm{Prob}\left\{x_{t+1} - x_t < -\gamma | \Im_t\right\} > \gamma\,.$$

Proof. The process $z_t^{i,j} = x_t$ has innovation

$$\epsilon_{s,t+1}^{i,j} = \log \frac{\alpha_s^i(q_t)}{\alpha_s^j(q_t)} \quad \text{on}\,(\sigma_t, s)\,,$$

where α are as in (10) with $q_t = \mathrm{Q}_t^*$. For each s and t, equilibrium prices $q_{s,t}$ are in the interval $(\min\{q_s^j, q_s^i\}, \max\{q_s^j, q_s^i\})$, where q^i (q^j) is the vector of state prices when agent i (j) dominates. Name $\Omega = \times_{s\in\mathcal{S}}\left[\min\{q_s^j, q_s^i\}, \max\{q_s^j, q_s^i\}\right]$. Regarding the finite increment requirement, note that for each s there exists a maximum innovation given by

$$\epsilon_s = \max\left\{\left|\log \frac{\alpha_s^i(q_t)}{\alpha_s^j(q_t)}\right| \text{for}\, q \in \Omega\right\}\,.$$

Choosing $B > \max\{\epsilon_s\,, s \in \mathcal{S}\}$ suffices for the requirement. Turning to the existence of γ, such that jumps of at least γ occur with probability at least γ, note that if $q_s^i = q_s^j$ for all s then $q_t = q^i = q^j$ for all t and, from the market clearing equation (7) $\alpha^i(q_t) = \alpha^j(q_t)$ for all t. It follows that not only $\epsilon_t^{i,j} = 0$ for all t but also $\mu_t^{i,j} = 0$ for all t so that none of the drift conditions i) to iii) can be satisfied. As a result we exclude that $q^i \neq q^j$ and equilibrium prices q_t belong to the interior of Ω. To conclude the proof note that in equilibrium there are no arbitrages, as a result for all t and $q_t \in \Omega$ there exists at least an s and an s' such that

$$\epsilon_{s,t}^{i,j} > 0 \quad \text{and} \quad \epsilon_{s',t}^{i,j} < 0$$

(Otherwise the zero-price portfolio $\alpha^i(q_t) - \alpha^j(q_t)$, or $\alpha^j(q_t) - \alpha^i(q_t)$, would be an arbitrage). For every $q \in \Omega$ let $\epsilon^+(q)$ the maximum of such jumps (the upper envelope of $\epsilon_s(q)$ for all s) and $\epsilon^-(q)$ the lowest of such jumps. The two functions

are continuous in q and Ω is compact so they have a maximum and a minimum. Moreover, since by the non arbitrage argument $\epsilon^+(q) > 0$ and $\epsilon^-(q) < 0$, the minimum of $\epsilon^+(q)$ is positive, $\epsilon^{+-} > 0$, and the maximum of $\epsilon^-(q)$ is negative, $\epsilon^{-+} < 0$. Choosing

$$\gamma = \min\{\epsilon^{+-}, |\epsilon^{-+}|, \mathrm{P}_s \; s \in \mathcal{S}\}$$

finishes the proof. □

Proof of Corollary 4.1 Under no-aggregate risk, $e_{s,t} = e$ for all $s \in \mathcal{S}$ and $t \in \mathbb{N}_0$. Computing the difference $\nu^{i,j}|_i - \nu^{i,j}|_j$ with rules as in (10) , gives

$$\nu^{i,j}|_i - \nu^{i,j}|_j = \log \sum_{s\in\mathcal{S}} (\mathrm{Q}^j)^{\frac{1}{\gamma}} (\mathrm{Q}^i)^{1-\frac{1}{\gamma}} + \log \sum_{s\in\mathcal{S}} (\mathrm{Q}^i)^{\frac{1}{\gamma}} (\mathrm{Q}^j)^{1-\frac{1}{\gamma}} .$$

For x in the simplex Δ^S, the function

$$f(x; \mathrm{Q}) = \sum_{s\in\mathcal{S}} (x)^{\frac{1}{\gamma}} (\mathrm{Q})^{1-\frac{1}{\gamma}}$$

is convex with a minimum equal to 1 in $x = \mathrm{Q}$ when $\gamma \in (0, 1)$, it is concave with a maximum equal to 1 in $x = \mathrm{Q}$ when $\gamma \in (1, \infty)$. As a result, when $\gamma > 1$, $\mu^{i,j}|_j > \mu^{i,j}|_i$ so that

$$\mu^{i,j}|_i > 0 \Rightarrow \mu^{i,j}|_j > 0 .$$

When $\gamma \in (0, 1)$ $\mu^{i,j}|_j < \mu^{i,j}|_i$ so that

$$\mu^{i,j}|_i < 0 \Rightarrow \mu^{i,j}|_j < 0 .$$

The two sign implications together with Proposition 4.1 prove the statement.

Consider now the aggregate risk case with $S = 2$. Since both state prices and beliefs belong to the simplex, growth premia can be seen as a function of one variable only. Focusing on state $s = 1$, e.g. the state with highest aggregate endowment growth, name $q \in (0, 1)$ the state price and $\mathrm{Q}^i, \mathrm{Q}^j \in (0, 1)$ agents beliefs, w.l.o.g. $\mathrm{Q}^j < \mathrm{Q}^i$. A CRRA portfolio rule (10) is thus a real function $\alpha(q; \mathrm{Q}) : (0, 1) \times (0, 1) \to (0, 1)$. The function is increasing in q when $\gamma \in (1, \infty)$ and decreasing when $\gamma \in (0, 1)$. It is always increasing in Q. Denote q^i as the state price that clears the market when i is the representative agent. From (17) when $S = 2$

$$q^i = \frac{\mathrm{Q}^i}{\mathrm{Q}^i + (1 - \mathrm{Q}^i) \left(\frac{g_1}{g_2}\right)^\gamma}$$

and similarly for q^j. Since $g_1 > g_2$ and $\mathrm{Q}^i > \mathrm{Q}^j$, then $q^i > q^j$ for all $\gamma \in (0, \infty)$.

When $\gamma > 1$,

$$\mu^{i,j}|_i > 0 \Rightarrow \mu^{i,j}|_j > 0$$

proves the statement together with Proposition 4.1. Since $\alpha(q;\mathrm{Q})$ is increasing in Q

$$\alpha^j(q^j) < \alpha^i(q^j) \quad \text{and} \quad \alpha^j(q^i) < \alpha^i(q^i)\,.$$

Moreover since $\alpha(q;\mathrm{Q})$ is increasing also in q

$$\alpha^j(q^j) < \alpha^j(q^i) \quad \text{and} \quad \alpha^i(q^j) < \alpha^i(q^i)\,.$$

Growth premia depend of relative entropies of the form $I_P(\alpha)$. The function $I_P(x)$ is defined on $(0,1)$, is convex, and has a minimum equal to zero in $x = P$. Assume by absurd that $\mu^{i,j}|_i > 0$ and $\mu^{i,j}|_j < 0$, then in must hold that

$$\alpha^j(q^i) < P < \alpha^i(q^j)$$

as all the other cases would result in a different signs combinations. Since $P < \alpha^i(q^j)$ and $\alpha^i(q^j) < \alpha^i(q^i)$ then

$$I_P(\alpha^i(q^i)) > I_P(\alpha^i(q^j))\,.$$

$\mu^{i,j}|_i > 0$ instead implies

$$I_P(\alpha^j(q^i)) > I_P(\alpha^i(q^i))\,,$$

and $\mu^{i,j}|_j < 0$ implies

$$I_P(\alpha^i(q^j)) > I_P(\alpha^j(q^j))\,.$$

The last three inequalities imply

$$I_P(\alpha^j(q^i)) > I_P(\alpha^j(q^j))$$

which is absurd given the fact that $P > \alpha^j(q^i) > \alpha^j(q^j)$.

The proof is similar for $\gamma \in (0,1)$. Now it is

$$\mu^{i,j}|_i < 0 \Rightarrow \mu^{i,j}|_j < 0$$

that proves the statement together with Proposition 4.1. $\alpha(q;\mathrm{Q})$ is still increasing in Q but it is now decreasing in q

$$\alpha^j(q^j) > \alpha^j(q^i) \quad \text{and} \quad \alpha^i(q^j) > \alpha^i(q^i)\,.$$

Assume by absurd that $\mu^{i,j}|_i < 0$ and $\mu^{i,j}|_j > 0$, then it must hold that

$$\alpha^i(q^i) < P < \alpha^j(q^j)$$

Since $P > \alpha^i(q^i)$ and $\alpha^i(q^j) > \alpha^i(q^i)$ then

$$I_P(\alpha^i(q^j)) > I_P(\alpha^i(q^i))\,.$$

$\mu^{i,j}|_i < 0$ instead implies

$$I_P(\alpha^i(q^i)) > I_P(\alpha^j(q^i))\,,$$

and $\mu^{i,j}|_j > 0$ implies

$$I_P(\alpha^j(q^j)) > I_P(\alpha^i(q^j))\,.$$

The last three inequalities imply

$$I_P(\alpha^j(q^j)) > I_P(\alpha^j(q^i))$$

which is absurd given the fact that $P < \alpha^j(q^j) < \alpha^j(q^j)$.

Proof of Corollary 4.2 Since when agent i has $\gamma^i = 1$ and correct beliefs his growth premium is maximum for all equilibrium returns, then both $\mu^{i,j}|_i$ and $\mu^{i,j}|_j$ are positive and statement i) follows from case iii) of Proposition 4.1. Regarding ii) since under no aggregate risk normalized state prices are equal to beliefs in the limit of an agent consuming all the aggregate endowment, irrespectively from his risk preferences, then an agent with correct beliefs has a maximal growth premium when he dominates. It follows that if i has correct beliefs, then $\mu^{i,j}|_i > 0$. Applying Proposition 4.1 either case ii) or iii), with him dominating P-almost surely, are possible. Regarding the statement iii) of the corollary, examples of preferences and beliefs such an agent with correct beliefs dominates, vanishes, or survives are given in Section 5 Figure 4.

Proof of Proposition 4.2 Given two agents i and j, w.l.o.g. $k^i_{BE} > k^j_{BE}$, so that

$$\mathrm{E}[\epsilon^{i,j}_{t+1}|\mathcal{F}_t\, s.t.\, z^{i,j}_t = z] = \frac{1}{\gamma}(k^i_{BE} - k^j_{BE}) > 0$$

for every log consumption ratio $z \in (-\infty, +\infty)$. The implication for the limit of $z^{i,j}_t$ follows from the Law of Large Numbers for uncorrelated random variables, see also the Proposition 1 of Sandroni (2000) for a similar application. Since the process $\{Z_t\}$ with

$$Z_t = \epsilon^{i,j}_t - \mathrm{E}[\epsilon^{i,j}_t|\mathcal{F}_{t-1}]$$

is a uncorrelated martingale then

$$\lim_{T\to\infty} \frac{\sum_{t=1}^T Z_t}{T} = 0 \quad \text{P-almost surely}\,.$$

Together with the fact that

$$\lim_{T\to\infty} \frac{\sum_{t=1}^{T} \mathrm{E}[\epsilon_t^{i,j}|\mathcal{F}_{t-1}]}{T} = \frac{1}{\gamma}(k^i - k^j) > 0 \quad \text{P-almost surely}\,,$$

the latter implies

$$\lim_{T\to\infty} z_T^{i,j} = +\infty\,.$$

Note that equilibrium prices, and thus whether there are more than only agent i and j in the economy do not matter for the result. Finally if $k^i > k^j$ for all $j \neq i$, then each j vanishes against i. Then also all agents (but i) vanish against i,

$$\limsup_{t\to\infty} \frac{\sum_{j\neq i} \phi_t^j}{\phi_t^i} = \limsup_{t\to\infty} \frac{1-\phi_t^i}{\phi_t^i} = 0,$$

so that i dominates. When only i has correct beliefs, provided $\beta^i \geq \beta^j$ for all $j \neq i$, then also $k^i > k^j$ for all $j \neq i$ and the same result follows.

Note at last that although the relative log-consumption dynamics depends, through market equilibrium prices, on the growth process g_t its drift does not. It follows that the same result holds for any growth process g_t (provide the economic equilibrium is well defined).

Proof of Proposition 4.3 The relative consumption process $\{z_t^{i,j}\}$ has innovation $\epsilon_{s,t+1}^{i,j}$. As shown in the main text the relative size of the survival indexes K_Y determines the sign of $\mathrm{E}[\epsilon_{t+1}^{i,j}|\mathcal{F}_t\, s.t.\, z_t^{i,j} = z]$ in the limit of $z \to \pm\infty$. The proof follows by applying Theorem C.2 along the same line of the proof of Proposition 4.1.

In particular, we have to show that i) when both limit conditional drifts are positive there exists a $\gamma > 0$ such that $\mathrm{Prob}\{\epsilon_{t+1}^{i,j} > \gamma\} > \gamma$ and, vice-versa, when limits conditional drifts are negative there exists a $\gamma > 0$ such that $\mathrm{Prob}\{\epsilon_{t+1}^{i,j} < -\gamma\} > \gamma$; ii) there exists a B such that $\mathrm{Prob}\{|\epsilon_{t+1}^{i,j}| < B\} = 1$.

To prove i), for every s define

$$f_s(q_s) = \epsilon_{s,t+1} = \log \frac{(\beta^i\, \mathrm{Q}_s^i)^{\frac{1}{\gamma^i}}}{(\beta^j\, \mathrm{Q}^j)^{\frac{1}{\gamma^j}}} q_s^{\frac{1}{\gamma^j}-\frac{1}{\gamma^i}}$$

as the innovation when both i and j use CRRA saving and portfolio rules. When i dominates $q_s = q_s^i = \beta^i\, \mathrm{Q}_s^i\, g_s^{-\gamma^i}$ and

$$f_s(q_s^i) = \frac{1}{\gamma^j} \log\left(\frac{\beta^i\, \mathrm{Q}_s^i}{\beta^j\, \mathrm{Q}^j}\right) g_s^{\gamma^j-\gamma^i}\,.$$

Likewise when j dominates

$$f_s(q_s^j) = \frac{1}{\gamma^i} \log \left(\frac{\beta^i \, \mathrm{Q}_s^i}{\beta^j \, \mathrm{Q}^j} \right) g_s^{\gamma^j - \gamma^i} .$$

Since $f_s(q_s^i)$ and $f_s(q_s^j)$ differ only for a constant of proportionality they are either both positive or both are negative. Note also that $f_s(q_s)$ is monotone, either increasing or decreasing depending on the relative size of γ^i and γ^j. All this together and the fact that $q_s \in (q_s^i, q^j)$ (w.l.o.g. $q_s^i < q_s^j$) implies that

$$f_s(q_s) > \min\{f_s(q_s^i), f_s(q_s^j)\}. \quad \text{for all} \quad q_s \in (q_s^i, q^j) . \tag{43}$$

When the process is such that

$$\lim_{z \to \pm\infty} \mathrm{E}[\epsilon_{t+1}^{i,j} | \mathcal{F}_t \, s.t. \, z_t^{i,j} = z] > 0$$

then there exists at least one s such that $f_s(q^j) > 0$ and $f_s(q^i) > 0$. By (43)

$$f_s(q_s) > \min\{f_s(q_s^i), f_s(q_s^j)\}.$$

for all $q_s \in (q_s^i, q^j)$. Naming $\gamma = \min\{\min\{f_s(q_s^i), f_s(q_s^j)\}, \mathrm{P}_s \; s \in \mathcal{S}\}$ proves that

$$\mathrm{Prob}\{\epsilon_{t+1}^{i,j} > \gamma\} > \gamma \, .$$

When the conditional drift is negative at the borders, the proof follows the same lines with $\gamma = \min\{|\max\{f_s(q_s^i), f_s(q_s^j)\}|, \mathrm{P}_s \; s \in \mathcal{S}\}$.

To prove ii) we need to show that there exists a B such that

$$|f_s(q_s)| < B$$

for every s. Given the properties of f_s exploited to prove point i) it also

$$|f_s(q_s)| < \max\{|f_s(q_s^i)|, |f_s(q_s^j)|\}.$$

Choosing $B > \max\{|f_s(q_s^i)|, |f_s(q_s^j)|, \, s \in \mathcal{S}\}$ proves the result.

Proof of Corollary 4.3 The corollary follows from the application of Theorem C.1. The theorem has two hypothesis: that the log relative consumption process has finite increments, and that the drift at $\pm\infty$ points to the center. This second hypothesis holds provided parameters are as specified, as it is proved in the main text immediately before the Corollary. I turn to show that also the finite increment hypothesis holds. Name q^i and $\delta|_i$ the normalized state price vector and market discount rate set by agent i when he consumes all the aggregate endowment. In a two-agent economy state prices are in the interior of the set

$\Omega_q = \times_{s\in\mathcal{S}} [\min\{q_s^j, q_s^i\}, \max\{q_s^j, q_s^i\}]$ and market discount rate in the interior of $\Omega_\delta = [\min\{\delta|_j, \delta|_i\}, \max\{\delta|_j, \delta|_i\}]$. Given the continuity of one-period substitution and portfolio rules of both agents, for each s there exists a maximum innovation given by

$$\epsilon_s = \max\left\{ \left| \log \frac{\delta^i(\delta,q)\alpha_s^i(q)}{\delta^j(\delta,q)\alpha_s^j(q)} \right| \text{for } q \in \Omega_q\,, \delta \in \Omega_\delta \right\}\,.$$

Choosing $B > \max\{\epsilon_s\,, s \in \mathcal{S}\}$ suffices for the requirement.

Proof of Proposition 4.4 Under Assumption 4.1 all agents $i \in \mathcal{I}$ hold the market portfolio so that $\alpha^i = \alpha^j$ for all i and j. As a result for any couple (i,j) the log relative consumption dynamics $z_t^{i,j}$ derived from (3) is deterministic and has innovation equal to

$$\epsilon_{t+1}^{i,j} = \log \frac{\delta_t^i}{\delta_t^j}\,.$$

Substitution rates are given by (9). Market rates δ_t are set by

$$\delta_t = \frac{\sum_{i\in\mathcal{I}} \delta_t^i \phi_t^i}{\sum_{s\in\mathcal{S}} \mathrm{Q}_s^0\, g_s}\,. \tag{44}$$

where Q^0 is the set of normalized state price that supports all agents holding the market portfolio. Defining for each agent i

$$k^i = \frac{(\beta^i)^{\frac{1}{\rho^i}}}{\sum_{s\in\mathcal{S}} \mathrm{Q}_s^0\, g_s} \left(\sum_{s\in S} (\mathrm{Q}_s^i)^{\frac{1}{\gamma^i}} (\mathrm{Q}_s^0)^{1-\frac{1}{\gamma^i}} \right)^{\frac{\gamma^i}{1-\gamma^i}\frac{1-\rho^i}{\rho^i}}\,,$$

which does not depend on time given that beliefs and market equilibrium prices are i.i.d., equation (44) becomes

$$1 = \sum_{i\in\mathcal{I}} \left(\frac{1}{\delta_t}\right)^{\frac{1}{\rho^i}} k^i \phi_t^i = \sum_{i\in\mathcal{I}} f^i(\delta_t)\phi_t^i\,,$$

where f^i is defined appropriately. Since $\rho^i > 0$ for all i, each function f^i is decreasing in δ_t. Moreover it holds $f^i(\delta|_i) = 1$, where $\delta|_i$ is the interest rate set by i when he has all the aggregate endowment:

$$\delta|_i = \frac{\beta^i}{\sum_{s\in\mathcal{S}} \mathrm{Q}_s^0\, g_s} \left(\sum_{s\in S} \mathrm{Q}_s^i\, g_s^{1-\gamma^i} \right)^{\frac{1-\rho^i}{1-\gamma^i}}\,.$$

It follows that for each i

$$f^i(\delta) \gtreqless 1 \;\Leftrightarrow\; \delta \lesseqgtr \delta|_i\,.$$

As a result, for each t

$$\delta_t \in (\min\{\delta|_i,\, i \in \mathcal{I}\}, \max\{\delta|_i,\, i \in \mathcal{I}\})\,.$$

Moreover for each t and j

$$f^j(q_t) < \max\{\delta|_i\, i \in \mathcal{I}\}$$

if $\delta|_j < \max\{\delta|_i\, i \in \mathcal{I}\}$, and

$$f^j(q_t) > \max\{\delta|_i\, i \in \mathcal{I}\}$$

if $\delta|_j = \max\{\delta|_i\, i \in \mathcal{I}\}$. Since by construction $\delta^i_t = \delta_t f^i(\delta_t)$ we have proved that if i defines the maximal rate $\delta|_i$, then his substitution rate is higher than that of all other agents and dominates. Dominance is sure because the dynamics of market discount rates and substitution rates is deterministic. As a result, the same relative consumption dynamics occurs for all path σ. In the same way, if i does not define the maximum rate $\delta|_i$, then he vanishes.

Proof of Proposition 4.5 The result follows by noticing that under Assumption 4.1

$$\frac{\mathrm{Q}^i_s\, g_s^{-\gamma^i}}{\mathrm{Q}^j_s\, g_s^{-\gamma^j}} = \frac{\sum_{s' \in \mathcal{S}} \mathrm{Q}^i_{s'}\, g_{s'}^{-\gamma^i}}{\sum_{s' \in \mathcal{S}} \mathrm{Q}^i_{s'}\, g_{s'}^{-\gamma^i}} \quad \text{for all} \quad s \in \mathcal{S}\,,$$

so that survival indexes k_Y do not depend on P, and by applying Proposition 4.4.

Proof of Corollary 4.4 Under no aggregate risk, Assumption 4.1 implies that for all $i, j \in \mathcal{I}$ $\mathrm{Q}^i = \mathrm{Q}^j = \mathrm{Q}^0$. From simple computation it holds

$$\delta|_i \lesseqgtr \delta|_j \;\Leftrightarrow\; \frac{\beta^i}{\beta^j}\, g^{\rho^j - \rho^i} \lesseqgtr 1\,.$$

The above and Proposition 4.4 prove the statement.

Proof of Corollary 4.5 Under Assumption 4.1, beliefs and normalized state prices are such that when $\beta^i = \beta^j$

$$\frac{\delta|_i}{\delta|_j} = (\sum_{s \in \mathcal{S}} \mathrm{Q}^0_s\, g_s)^{\rho^j - \rho^i} \frac{\left(\sum_{s \in S} (\mathrm{Q}^i_s)^{\frac{1}{\gamma^i}} (\mathrm{Q}^0_s)^{1 - \frac{1}{\gamma^i}}\right)^{\gamma^i \frac{1-\rho^i}{1-\gamma^i}}}{\left(\sum_{s \in S} (\mathrm{Q}^j_s)^{\frac{1}{\gamma^j}} (\mathrm{Q}^0_s)^{1 - \frac{1}{\gamma^j}}\right)^{\gamma^j \frac{1-\rho^j}{1-\gamma^j}}}\,.$$

As a function of beliefs Q

$$\left(\sum_{s\in S}(\mathrm{Q}_s)^{\frac{1}{\gamma}}(\mathrm{Q}_s^0)^{1-\frac{1}{\gamma}}\right)^{\gamma\frac{1-\rho}{1-\gamma}}$$

has a stationary point in $\mathrm{Q}=\mathrm{Q}^0$ where it is equal to one. Moreover it is convex when $\rho\in(0,1)$ and concave when $\rho>1$. It follows that, provided $\sum_{s\in\mathcal{S}}\mathrm{Q}_s^0\,g_s\geq 1$, $\rho^i<1<\rho^j$ implies that the ratio $\delta|_i/\delta|_j>1$. Applying Proposition 4.4 concludes the proof.