Forward Guidance: Stabilization and Stabilizability under Endogenous Policy Rules at the Zero lower bound

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Abstract: Forward guidance has recently been implemented in a number of countries, including the EMU. We apply the theory of economic policy to a world with rational expectations and strategic policy announcements. We show that shocks of any size can always be stabilized by a policy design that includes forward guidance, contingent policy rules and an appropriate, but not restrictive, choice of the policy rule parameters. This provides enhanced controllability and stabilizability – especially where such properties have not been available before. There are some differences between the bounds on the permissible policy parameters, depending on whether we are tied to the zero lower bound or not, but stabilizing policy choices are *always* feasible and available.

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1. Introduction

While forward guidance, together with Quantitative Easing, has been at the forefront of practical (and largely successful) innovations in recent monetary policy, the existing literature on forward guidance has been, at least so far, limited, mostly restricted to a few specific problems or circumstances, and mainly of an empirical content.[2]

Part of the problem is that forward guidance has been seen in terms of ***controllability*** (the ability to reach or regain certain target values) rather than ***stabilizability*** (its ability to return an economy to a steady state growth path, following an arbitrary shock and at a chosen speed). This paper removes that constraint by making the policy rules themselves forward looking. In the jargon of the literature, this is just a question of using Odyssean rather than Delphic forward guidance.[3]

At the theoretical level, some attention has been given to the issues of the limits to stabilizability when the horizon of forward guidance is not long enough to let private sector expectations adjust, or where there are bounds to legitimate policy action (Evans, 2012, Gavin et al, 2013, Woodford, 2013). We deal with both issues by noting that, under stabilization, the policy horizon is large (effectively infinite) and repeatable. This implies that stabilizability is unconditional. That in turn means, as we show, that time inconsistency is not an issue for stabilizability since there is always time to achieve the desired goals (in expectation) – unless there are *further* shocks to respond to (a shift in information) or a change in the desired speed of absorbing them

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[2] See, e.g., Kool and Thornton (2012), Del Negro et al (2013), Filardo and Hofmann (2014).

[3] Delphic guidance is guidance defined in terms of intended/expected outcomes without specific commitments: for example “low inflation”, “interest rates lower for longer”, “inflation of 2% or a little less”. Odyssean guidance is guidance offered as a commitment to specific policies or (contingent) policy rules defining how the authorities will react to certain conditions (Den Haan, 2013). See Hughes Hallett and Acocella (2015) for a formal statement of this distinction.

(a change in preferences). Neither of those two can be properly classified as time inconsistency. Thus, the two cases of apparent time inconsistency commonly considered in the literature – that outcomes better than expected will require policymakers to retain their previous policies in order to maintain (time consistent) credibility, and that worse outcomes will persuade them to speed up the return to their equilibrium path[4] – never arise in the stabilizability case.[5]

The contribution by Levin et al (2010) is of specific relevance here, since these authors warn that the economy's stability may be at risk if there are physical or formal bounds to policies that can be implemented. If that is the case, forward guidance can be used to stabilize the system only when shocks are moderately negative. When negative shocks are large (such as would appear in a big recession or in a serious deflation), they conjecture forward guidance will be unable to take them into account and ensure economic stability. This contrasts with the theory of dynamic controllability and stabilizability which, as demonstrated below, shows that stabilization is *always* possible when the private sector has forward looking expectations, even when coping with large shocks. The explanation is that Levin *et al* did not consider the possibility that their results might depend on the choice of insufficiently responsive policy rules, or on an inappropriate type of forward guidance. These results are of particular significance for the Euro zone as it contemplates *expanding* quantitative easing in the face of accelerating deflationary pressures and as the ECB has adopted some form of forward guidance (e.g., see Praet 2013); and also for the US and UK as they try to avoid inflationary pressures in their exit path from quantitative easing and unusually low interest rates. In this paper we show that, in the presence of rational expectations, stabilizability can always be ensured by forward guidance,

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[4] See Krugman (1998) or den Haan (2013); and Plosser (2013) or Buiter (2013) respectively.

[5] Praet (2013) underlines this point, by arguing that to exploit good fortune or to speed the return to stability is to reaffirm the policymakers' objectives, not suspend them.

independently of the size of shocks, if an appropriate policy rule is chosen, even if the economy is at or near the zero lower bound. Such rules are always feasible.

The paper is organized as follows. The next section presents our policy model with forward looking expectations. Section 3 deals with the concepts of stabilizability and dynamic controllability. Section 4 investigates the issue of stabilizability under all sizes of shock. Section 5 deals more specifically with the impact of zero lower bound constraints. Section 6 concludes.

2. A Generic Dynamic Model with Forward Expectations

Without loss of generality, we can write the standard linear RE model in its reduced form for a single policy authority, as follows:

(1) $y_t = Ay_{t-1} + Bu_t + Cy_{t+1|t} + v_t$ for $t = 1,\ldots,T$.

where $y_{t+1|t} = E\left[y_{t+1} \mid \Omega_t\right]$ denotes the mathematical expectation of y_{t+1} conditional on Ω_t, an information set common to all agents at t, and u_t is a vector of m control variables in the hands of the policymakers. Matrices A, C and B are constant and of order S, S, and $S \times m$, respectively, and have at least some elements which are nonzero. In this representation, y_0 is a known initial condition, and $y_{T+1|1}$ is a known, assumed or expected terminal condition (most likely one that describes the economic system's long run equilibrium state[6]). Both are part of the information set Ω_t. But the values in u_t are not part of Ω_t since they are determined by the policy makers. Finally v_t is a vector of exogenous shocks and other influences on y_t with a known mean but from an

[6]There is no indeterminacy here. The dynamic conditions which guarantee the existence of a solution are automatically satisfied, given any particular information set, ***if*** the inverse in (2) exists – which we show to be the case in Hughes Hallett et al (2012b). Given such an inverse, Hughes Hallett and Fisher (1988) show that the saddle point property (that the system has the correct number of stable and unstable roots to ensure a solution; Blanchard and Khan, 1980) is satisfied. It then no longer matters what the value of the terminal condition is, or if none is specified, if the policy horizon is far enough away ($T \rightarrow \infty$). But indeterminacy may follow if the $y_{(T+1|1)}$ values cannot be specified.

unspecified probability distribution. Notice that the policy authority may have only $q \leq S$ explicit targets, but the m instruments are assumed to be linearly independent.

This model can now be solved from the perspective of any particular period, say $t = 1$, by putting it into final form conditional on the information set available in that period:

$$(2) \quad \begin{pmatrix} y_{1|1} \\ : \\ : \\ : \\ y_{T|1} \end{pmatrix} = \begin{bmatrix} I & -C & 0 & . & 0 \\ -A & I & & . & . \\ 0 & & . & & 0 \\ . & . & & . & -C \\ 0 & . & 0 & -A & I \end{bmatrix}^{-1} \left\{ \begin{bmatrix} B & 0 & . & . & 0 \\ 0 & . & & & . \\ . & & . & & . \\ . & & & . & 0 \\ 0 & . & . & 0 & B \end{bmatrix} \begin{pmatrix} u_{1|1} \\ . \\ . \\ . \\ u_{T|1} \end{pmatrix} + \begin{pmatrix} v_{1|1} \\ . \\ . \\ . \\ v_{T|1} \end{pmatrix} + \begin{pmatrix} Ay_0 \\ 0 \\ . \\ . \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ . \\ . \\ 0 \\ Cy_{T+1|1} \end{pmatrix} \right\}$$

Although equation (1) has been solved from the point of view of Ω_1, it must be understood that it could have been solved for each Ω_t, $t = 1,...,T$ in turn, where $y_{j|t} = E_t(y_j)$ if $j \geq t$, but $y_{j|t} = y_j$ if $j < t$; similarly for u and v.

The structural equation to which (2) is the solution makes it clear that *neither* policymakers, *nor* the private sector are required to make expectation errors for the policies to work as planned. In fact, those expectations are exactly matched, aligned and consistent with what the private sector/policymakers expect/intend the outcomes to be (given the model and information set in use). It then only remains to determine if it is really possible to shift expectations in such a way that the economy's outcomes reach certain specified target values at certain points of time.

It is easy to show that this final form model always exists since the inverse matrix in (2) is well defined provided the matrix product AC does not contain a unit root (see Hughes Hallett et al, 2012a,b for an explicit proof).[7]

3. Controllability and Stabilizability

Controllability

[7]The no unit root condition is equivalent to specifying that the feed-forward and feedback dynamic elements do not cancel each other out.

In previous papers (Hughes Hallett et al, 2012b, Hughes Hallett and Acocella, 2015) we were able to show that the model discussed in Section 2, containing not only backward-looking but also forward-looking effects, has interesting controllability properties. Given that the inverse in (2) above exists, T_T^{-1}, we can write (2) as $y = Ru + b$, where R is defined as $T_T^{-1}(I \otimes B)$ and contains both the conventional policy multipliers in the sub-matrices $R_{t,j}$ where $t \leq j$ and $j = 1 \ldots T$ for each t; ***and*** multipliers that define the anticipatory effects of announced or anticipated policy changes $u_{j|1}$ to be made at various points in the future in sub-matrices $R_{t,j}$ where $t > j$. Finally the external information required is contained in $b = T_T^{-1}\{E(v|\Omega_1 + (A':0)'y_0 + (0:C')'y_{T+1|1}\}$.

Multi-period static controllability defines the set of conditions which must hold if an arbitrary set of target values is to be achieved for the endogenous targets in each period. Under REs, as in any conventional backwards looking model, a single policy maker will achieve this type of controllability when he possesses as many independent policy instruments as target variables in each time period.

A model is dynamically controllable, however, if a sequence of instrument values $u_1,\ldots,u_t$ exists that will reach arbitrary values, $\bar{y}_t$, for the target variables in period *t* (in expectation) given any starting point y_0. The economy represented by (1) is dynamically controllable over the interval (1, *t*), when $T \geq S$ and $t < T$, if in the *t*-th row block of *R*, $r[R_{t,1}\ldots.R_{t,S}]=S$. This is a sufficient condition for dynamic controllability. Necessary conditions may involve smaller subsets of $R_{t,j}$ having full rank depending on how many policy instruments are available. The conditions for dynamic controllability stated here contain an important generalization over the traditional case. In this case, policymakers can use policy announcements, in addition to instrument interventions, to guide the course of the economy. This is true even at *t = 1*: the targets are controllable from the first period, even if there are insufficient instruments, provided that the conditions in the theorem are satisfied and credible announcements are made about anticipated future actions.

It is important to note that time inconsistency will not appear here, so credibility is no restriction if the no unit root condition (eq. (1) possesses a solution) and the rank condition apply at $t<T$. Policymakers are of course free to set $u_{t|t} \neq u_{t|1}$. But they would never do so because $\overline{y}_t$ is the first best value, and is reachable given no further information changes or unforeseen shocks. Hence, to assert time inconsistency is to claim that rational policymakers would choose to make themselves worse off than they need to be: a contradiction.

For example, suppose low interest rates in a recession had been promised for a period of time but the economy starts to recover in that period. Policymakers clearly have an incentive to break their promise and raise interest rates early. Knowing that policymakers can reach their ultimate objectives, the private sector can see this rise coming and will factor in revised outcomes with the result that an early rise in interest rates would have no additional effect. Knowing that, policy makers won't make the change.

Stabilizability

The controllability properties of an economy with REs through forward guidance are important as they rule out one of the more frequent obstacles against use of this type of policy, time inconsistency, if there are enough instruments or sufficient time periods (that is, if the rank condition above is satisfied). Another obstacle however can be occurrence of shocks. These could be such as to lead the economy towards an unstable path, thus making it uncontrollable. We now show that this is not the case, since any economy hit by a shock of any size can be stabilized to an arbitrary degree under rational, forward looking expectations if it is also dynamically controllable. An arbitrary degree of stabilization means that rules can be found to make the economy follow an arbitrarily stable path, based on an arbitrary set of eigenvalues, such that it returns to the original path following a shock. Theorem 1 is the Rational Expectations analogue of the standard stabilizability theorem for backward looking, physical systems.[8]

Theorem 1 (*stabilizability under REs*)**.** For any economy represented by (1), with arbitrary coefficient matrices *A*, *B* and *C*, we can always find a series of dynamic but forward-looking policy rules, $u_{t|1} = \sum_{j=1}^{T} y_{j-1|1} + k_{t|1}$,[9] such that the controlled economy is stabilizable up to an arbitrary set of eigenvalues, if that economy is dynamically controllable.

Proof. (see Hughes Hallett and Acocella, 2015; reproduced here for convenience):

Equation (1), with arbitrary coefficient matrices *A*, *B* and *C*, can be reduced to its final form (2). Substituting the policy rule $u_{t|1} = \sum_{j=1}^{T} K_{tj} y_{j-1|1} + k_{t|1}$ for each t = 1,…,T shows that the controlled economy will behave similarly to $y = Ru + b$,:

$$\text{(3)} \qquad \begin{pmatrix} y_{1|1} \\ . \\ . \\ . \\ y_{T|1} \end{pmatrix} = \begin{bmatrix} R_{11} & . & . & . & R_{1T} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ R_{T1} & . & . & . & R_{TT} \end{bmatrix} \begin{bmatrix} K_{1,1} & K_{1,2} & . & . & K_{1,T} \\ K_{2,1} & . & & & . \\ . & & . & & . \\ . & & & . & K_{T-1,T} \\ K_{T,1} & . & . & K_{T,T-1} & K_{T,T} \end{bmatrix} \begin{pmatrix} y_0 \\ y_{1|1} \\ . \\ . \\ y_{T-1|1} \end{pmatrix} + \begin{pmatrix} c_{1|1} \\ . \\ . \\ . \\ c_{T|1} \end{pmatrix}$$

where $y_{0|1} = y_0$ and $c_{t|1} = b_{t|1} + T_T^{-1} B k_{t|1}$. Rewriting (3), we now have

$$\text{(4)} \qquad \tilde{y}_t = RK\tilde{y}_{t-1} + c$$

where $\tilde{y}_t$ is the stacked vector on the left of (3).

The economy will now be stable (that is stabilized by the policy rule above for $u_{t|1}$) if the iteration matrix, *RK*, has its roots inside the unit circle (Wonham, 1974). But we can go further.

[8] See Wonham (1974).

[9] Note: this control rule, for use in period $t<T$, employs actions ***and*** anticipated actions up to the end of period *T*.

Any particular $y_{t|1}$ will follow an arbitrarily stable path if we can pick $K_{t1}.....K_{tT}$ to generate an arbitrary set of eigenvalues for that matrix for each *t*. Suppose we want to choose iteration matrix $D = Z\Lambda Z^{-1}$, where Λ is a diagonal matrix of the desired eigenvalues, and *Z* is a matrix of corresponding eigenvectors. Then, as long as $T>S$ and the matrix *R* has full rank *ST* (i.e., $m \geq S$, so that static controllability applies), we can calculate the required *K* from $K = R^{-1}Z\Lambda Z^{-1}$. But if $m<S$ and dynamic controllability applies (as in the theorem), then we can use a generalized left-inverse instead: $K = R^{+}Z\Lambda Z^{-1}$ with $R^{+} = \left(R'R\right)^{-1} R'$ as one obvious possibility. This generalized inverse always exists, given dynamic controllability, since $R'R$ has full rank with r[*R*]= *mT* by Sylvester's inequality. To see this, recall $R = T_T^{-1}\left(I \otimes B\right)$ where T_T^{-1} is a square *ST*x*ST* matrix of full rank and $\left(I \otimes B\right)$ is a block diagonal matrix with rank *mT*. Hence, by Sylvester's inequality, r[*R*]≥ *ST*+*mT*–*ST* = *mT*. But if $m<S$, then $\mathrm{r}\left[I \otimes B\right]$ cannot be greater than *mT* by definition. Hence r[*R*]=*mT*, which means that $\left(R'R\right)^{-1}$exists and this value of R^{+} is always available. ■

Corollary: Hence a RE model that is dynamically controllable at t = 1 is also stabilizable from t = 1. Theorem 1 generalizes Wonham's original theorem.

Intuition: The policy rules described in Theorem 1 are both forward and backward looking in that they react to expected future developments, including the effects of these rules applied in the future, *and* to feedback from past outcomes (past "failures") – in exactly the same way as the private agents in the economy have been assumed to do. The practical implication is that the constrained (policy) optimization problem no longer satisfies the Markov property with respect to time, which means that traditional optimization techniques like dynamic programming cannot be used[10] – at least, not without the risk of creating time inconsistent decisions as policymakers exploit their opportunity to re-optimize to improve successive outcomes.

The lesson here is that in models with forward looking behavior, the closed loop (as opposed to simple feedback) characteristics of our policy rules are of special importance. Closed loop means reacting to changes in expectations of future events as they appear, in addition to past outcomes as they deviate from plan. In backward looking models, future events are represented by (fixed) future exogenous variables. As a result, the distinction between closed loop and feedback rules will be ignored as if it were unimportant. But given forward looking markets, where current behavior and outcomes depend on expectations of the future, and expectations of the future depend on current outcomes, the distinction can be large (Hughes Hallett et al, 2012a).

4. Stabilisability and the size of shocks.

The clear implication of Theorem 1 is that an economy subject to forward looking expectations is ***always*** stabilizable if a suitably chosen policy rule is applied together with forward guidance - defined here as planned outcomes $y_{j|1}$ applied to u_t for $j \geq t+1$. This result has been challenged by Levin et al (2010), who assert that such models may be stabilizable if the shocks are small; but not if they are large, such as those in the 2008-12 recession. This section reconciles the results above with that assertion and explains the source of the difference.

In this section we show that any economy remains stabilizable, irrespective of the size or type of shock, ***if*** the stabilization rule contains forward looking elements[11]; ***if*** the policy interventions are endogenized (specifically the forward looking elements, which means that there is an articulated

[10] Bellman (1961)

[11] To be fair, Levin et al (2010) conjecture that forward guidance might be a way to resolve the potential instabilities remaining in their model. But the two further properties noted in this sentence are not present in the Levin et al analysis. They turn out to be necessary components for achieving stability – as shown below.

exit strategy even if current policies are stuck at their zero lower bound); ***and*** if the parameters of the stabilization rule are chosen to be strong enough to make the policy interventions effective.

To demonstrate these results, we take the same model as in the Levin et al analysis. First, there is a standard forward looking New Keynesian Philips curve to explain inflation π_t:

(5) $$\pi_t = \beta E_t(\pi_{t+1}) + kx_t$$

where $\beta \in [0,1]$ measures the sensitivity of current inflation to expectations of future inflation; and $k > 0$ is the corresponding sensitivity (elasticity) of inflation to changes in the existing output gap x_t. Second, a forward looking IS curve to explain the output gap (unused spare capacity) in the economy:

(6) $$x_t = E_t(x_{t+1}) - \sigma E_t(i_t - \pi_{t+1} - r_t^n)$$

where i_t is the short term interest rate (policy rate), r_t^n is the long run equilibrium interest rate, $\sigma > 0$ is the interest rate elasticity of aggregate demand (output), and $E_t(.) = E_t(.|\Omega_t)$.

The general solution to this model, eliminating x_t from (5), is:

(7) $$\binom{\pi_t}{x_t} = \begin{bmatrix} k\sigma + \beta & k \\ \sigma & 1 \end{bmatrix} \binom{\pi_{t+1|t}}{x_{t+1|t}} + \sigma \binom{k}{1} (r_t^n - E_t i_t).$$

If we are stuck at the zero lower bound for the interest rate, then $i_t = 0$; otherwise we operate with equation (7) as it is. Notice, if policy is not constrained to its zero lower bound, then the natural effects of expectations and dynamics can be offset by policy or guidance: $E_t i_t \neq 0$. This confirms our earlier stabilization results, whatever the shocks may be. The question is, can it still be done if $E_t i_t$ has to be nonnegative?

The eigenvalues controlling the dynamics of (7) can be obtained from the iteration matrix:

(8) $$\lambda_{1,2} = \tfrac{1}{2}\left\{\beta + k\sigma + 1 \pm \sqrt{(\beta + k\sigma + 1)^2 - 4\beta}\right\}.$$

Stability here requires both eigenvalues ***outside*** the unit circle because (7) is an exclusively forward looking dynamic equation.[12] The problem with (8) is that the positive root lies outside the unit circle, but the negative root lies inside. That means we are condemned to suffer some form of instability unless β and either k or σ are equal to zero.[13] To see this, we first check that $\lambda_{1,2}$ are real. That requires

(9) $(\beta + k\sigma + 1)^2 > 4\beta$ or $(\beta - 1)^2 + 2k\sigma(1 + \beta) > 0$

which is evidently true since β, k and σ are all nonnegative. This then implies the negative root is inside the unit circle, $|\lambda_2| < 1$, but the positive root is outside, $\lambda_1 > 1$, as demonstrated in Appendix A at the end of the paper.

The clear implication of these results is that the model at (7) is inherently unstable, whether or not monetary policy is stuck at its zero lower bound, ***unless*** β = 0 and either k or σ are zero; or the stabilizing interventions follow a policy rule of the type used in Theorem 1. These results are the first extension of the Levin et al analysis.

Next, this tendency to instability gets worse the larger is output sensitivity with respect to interest rates, σ; the larger is the sensitivity of inflation to output, k; and the greater is the impact of inflation expectations on current inflation: β. One could expect these parameters to be rather small near the zero lower bound if there is excess capacity in the economy. But otherwise, stabilizing policies will have to work harder to achieve their aims; or the forward guidance announcements will have to have a lot of credibility. Conversely, if those three parameters are small (σ because of the liquidity trap effect; k because there is excess capacity; and β because

[12] Renormalising on variables indexed by $t+1$ gives the conventional result; stability follows if the eigenvalues of the *inverse* of the iteration matrix in (7) are inside the unit circle, meaning $\lambda_{1,2}$ have to be outside. Having one root inside and one outside is not the usual saddle point property, in the sense of Blanchard and Khan (1980), since in this case there are no lag terms in the model.

[13] This is the particular case that Levin et al were actually concerned with.

there is a fear of deflation, and no fear of inflation), forward guidance and policy interventions will not have to work so hard. That is the second extension of the Levin et al results.

The parameters of the policy rule therefore matter: offsetting large negative shocks requires less effort near the zero lower bound; but elsewhere (i.e. before the recession has fully taken hold), it will require considerably more effort (a stronger rule or stronger forward guidance).

5. Stabilisability at the Zero Lower Bound.

The next step is to determine if stabilizability and policy is still possible when the zero lower bound constraint is taken into account. Theorem 1 states that any economy such as (1), with (11) as a special case, is in general stabilizable; but not that it is stabilizable without cost [in the sense that the policy instruments might need to take values that are undesirable, if not infeasible]. We need to check if stabilization is still possible when interest rates are required to respect their zero lower bound: $\mathrm{i_t} \geq 0$ but $E_t(i_{t+1}) > 0$, representing a possible zero lower bound restriction and a requirement to announce an exit strategy, respectively, in a two period problem.

The model: to examine such a case, consider an economy represented by the following model

(10) $$y_t = \rho y_{t-1} + \alpha(\pi_t - E_t\pi_{t+1}) - \beta(i_t - E_t\pi_{t+1}) + \varepsilon_t$$

(11) $$i_t = c_0 + c_1(\pi_t - \pi^*) + c_2 y_t + \nu_t$$

Equation (10) is an elaboration of the standard model which has been part of the theory of monetary policy since the Barro-Gordon model. It consists of a short run Phillips curve with persistence ($\rho \neq 0$), set within a standard forward looking Lucas supply function (a vertical long run Phillips curve) and extended to include the effect of real interest rate changes on output. It can therefore be interpreted as either a dynamic open economy Phillips curve; or as a forward looking New Keynesian IS curve with dynamics. In that case, y_t is the output gap; π_t is the rate of inflation; $E_t\pi_{t+1}$ the private sector's current expectation for inflation; π^* is the policymaker's target for inflation; and i_t is the nominal rate of interest. We define ε_t as a supply shock and ν_t a monetary shock, both with mean zero and constant variance. All parameters are positive.

The only policy instrument in this example is i_t. Policy therefore follows a Taylor rule: c_0 is an exogenous term incorporating the equilibrium rate of interest r_t^n; ν_t are possible control errors; and determinacy (the Taylor principle) suggests $c_1 > 1$.

To obtain a reduced form for (10)-(11), we renormalize (11) on π_t, set $u_t = \pi^* - c_1^{-1}(c_0 - v_t)$, and then solve for y_t and π_t. This transforms our system to:

$$(12) \quad \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \frac{\rho}{\Delta}\begin{bmatrix} 1 & 0 \\ -c_2c_1^{-1} & 0 \end{bmatrix}\begin{bmatrix} y_{t-1} \\ \pi_{t-1} \end{bmatrix} + \frac{\beta-\alpha}{\Delta}\begin{bmatrix} 0 & 1 \\ 0 & -c_2c_1^{-1} \end{bmatrix}\begin{bmatrix} y_{t+1/1} \\ \pi_{t+1/1} \end{bmatrix} + + \frac{1}{c_1\Delta}\begin{bmatrix} \alpha-\beta c_1 \\ \beta c_2+1 \end{bmatrix} i_t + \frac{1}{\Delta}\begin{bmatrix} 1 & \alpha \\ -c_2c_1^{-1} & 1 \end{bmatrix}\begin{bmatrix} \varepsilon_t \\ u_t \end{bmatrix}$$

where $\Delta = (1 + \alpha c_1^{-1} c_0)$. This form of model does not permit static controllability since the coefficient matrix multiplying i_t is singular. However, the two-period policy problem is:

$$(13) \quad \begin{bmatrix} y_t \\ \pi_t \\ y_{t+1} \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} I & -B \\ -A & I \end{bmatrix}^{-1}\left(\begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} i_t \\ E_t i_{t+1} \end{bmatrix} + \begin{bmatrix} A\begin{bmatrix} y_0 \\ \pi_0 \end{bmatrix} \\ B\begin{bmatrix} E_t y_{t+2} \\ E_t \pi_{t+2} \end{bmatrix} \end{bmatrix} + \frac{1}{\Delta}\begin{bmatrix} \varepsilon_t + \alpha u_t \\ u_t - c_2c_1^{-1}\varepsilon_t \\ \varepsilon_{t+1} + \alpha u_{t+1} \\ u_{t+1} - c_2c_1^{-1}\varepsilon_{t+1} \end{bmatrix}\right)$$

where *A*, *B*, *C* are the first, second and third coefficient matrices in (12), and correspond to equation (1). The policy multiplier matrix for this model is then:

$$(14) \qquad \frac{1}{\Phi}\begin{bmatrix} (\alpha-\beta c_1)\Delta & (\beta-\alpha)(1+\beta c_2) \\ \dfrac{\rho(\beta-\alpha)(\alpha-\beta c_1)c_1^{-1}c_2^{2}+(1+\beta c_2)\Phi}{c_1\Delta} & -c_2c_1^{-1}(\beta-\alpha)(1+\beta c_2) \\ \rho(\alpha-\beta c_1) & \dfrac{\rho(\beta-\alpha)(1+\beta c_2)c_1+(\alpha-\beta c_1)\Phi}{c_1\Delta} \\ -c_2c_1^{-1}\rho(\alpha-\beta c_1) & (1+\beta c_2)\Delta \end{bmatrix}\begin{bmatrix} i_t \\ E_t i_{t+1}\end{bmatrix}$$

where $\Phi = \Delta^2 c_1 + \rho(\beta-\alpha)c_2$. This model therefore tells us that (y_{t+1}, π_{t+1}) are both controllable using current policies and announcements or projections of future actions, because the lower 2x2 partition of the multiplier matrix in (14) is non-singular.[14] Policies to achieve stabilized values of $(0,\pi^*)$ for (y_{t+1}, π_{t+1}), defined for some point in the future, would therefore be:

$$(15) \quad \begin{bmatrix} i_t \\ E_t i_{t+1}\end{bmatrix} = \frac{\Phi}{D}.\begin{bmatrix} 1+\beta c_2 & -\rho(\beta-\alpha)(1+\beta c_2)c_1 + (\alpha-\beta c_1)\Phi/c_1\Delta \\ c_2c_1^{-1}\rho(\alpha-\beta c_1) & \rho(\alpha-\beta c_1) \end{bmatrix}\begin{bmatrix} 0 \\ \pi^* \end{bmatrix}$$

where $D = \rho(\alpha-\beta c_1)\{(1+\beta c_2)\Delta + c_2c_1^{-1}[\rho(\beta-\alpha)(1+\beta c_2) + (\alpha-\beta c_1)\Phi]/\Delta\}$.

i) Policy Analysis: To simplify the discussion, suppose that $\beta > \alpha$: that is, interest rate policy has more impact on the economy than inflation surprises (but not necessarily more than inflation itself). This implies $\Phi > 0$. As a consequence we will also assume that $(\alpha\text{-}\beta c_1) < 0$ and hence that $D < 0$. Last but not least, recall that $\Delta > 0$. Equation (15) now implies we need to set

$$(16) \qquad i_t = -\frac{\pi^*\Phi}{D}.\frac{\rho(\beta-\alpha)(1+\beta c_2)+(\alpha-\beta c_1)\Phi}{c_1\Delta} \geq 0, \qquad \text{and}$$

$$(17) \qquad E_t i_{t+1} = \frac{\pi^*\Phi}{D}.\rho(\alpha-\beta c_1)$$

$$= \frac{\pi^*\Phi}{\{(1+\beta c_2)\Delta + c_2c_1^{-2}[\rho(\beta-\alpha)(1+\beta c_2)+(\alpha-\beta c_1)\Phi]/\Delta\}} > 0$$

[14] Unless the underlying parameters satisfy $\alpha = \beta c_1$ exactly. Since we are free to choose c_1, we may assume $\alpha \neq \beta c_1$.

in order to achieve our targets, and hence stability, without breaching the lower bound constraint.

ii) Responses to inflation. The weak inequality in (16) depends on our choice of c_1. It requires

(18) $$c_1 \leq \beta^{-1}[\alpha + \rho(\beta - \alpha)(1 + \beta c_2)/\Phi]$$

to hold. This upper bound exceeds α/β, unless $\alpha = \beta$ (in which case any $c_1 \leq 1$ value will do). Thus the current policy will be retained at its zero lower bound if c_1 is chosen equal to its upper bound in (18), but will exceed zero if c_1 is chosen smaller. However (18) is an approximation, chosen for the purposes of illustration only, since Φ depends on c_1. Substituting for Φ, the correct upper bound on c_1 is given by:

(19) $$c_1^2 - hc_1 - \frac{\rho(\beta-\alpha)(1+\beta c_2)}{\beta(1+\alpha^2 c_2^2)} \leq 0 \quad \text{where} \quad h = \frac{\alpha}{\beta} + \frac{\rho(\beta-\alpha)(1+\beta c_2)}{\beta[\rho\beta+(2-\rho)\alpha]}$$

which implies upper and lower bounds on c_1 (they represent, respectively, sufficient conditions for (16), and more accurate restrictions when $\beta - \alpha$ is small) of:

(20) $$0 \leq c_1 \leq h.$$

In other words, we need to choose the policy rule parameters correctly, but not too strong, if we wish to stabilize the economy. That remains true even when we are constrained to the zero lower bound. Hence the size and direction of the shocks have nothing to do with our ability to stabilize the economy. Equally the Taylor principle may or may not be desirable or possible (Appendix C resolves that question), but it plays no role so long as our assumptions on α, β (or their replacements if $\beta \leq \alpha$) remain valid.

iii) Responses to output: The forward guidance and exit strategy, as part of the stabilization package implicitly available to everyone (even if not preannounced), is bounded away from zero if the parameters of the policy rule are chosen right. Inequality (17) then holds for any parameter values if c_2 satisfies

$$(21) \qquad c_2 > \frac{-[(\alpha\text{-}\beta c_1) + \rho(\beta-\alpha)]}{\rho\beta(\beta-\alpha)}.$$

Inequality (21) is in fact a sufficient condition, the necessary condition being quadratic in c_2 and more complicated.[15] It implies any $c_2 \leq 0$ will do if $(\alpha\text{-}\beta c_1) + \rho(\beta-\alpha) \geq 0$; or that any positive c_2 satisfying (21) will do even if we are currently constrained to be at the zero lower bound. Again the size and direction of shocks have nothing to do with it. Instead, with c_2 values away from the bounds above, the trick is to choose the parameters in the policy rule correctly. Together with the constraints on c_1 in (20), this has implications for "austerity". In particular, it shows the relaxations of the Taylor principle which might be used to start a recovery, or to overcome the fear of deflation, or provide the ability to stabilize the economy. Such policies all require forward guidance to place greater emphasis on output stabilization and potential growth. In addition, the exit strategy would never have zero interest rates. These results are all further extensions to the Levin et al analysis.

iv) Are these constraints on the policy rule compatible with the Taylor principle?

Appendix B, part v), shows that the constraints on c_1 and c_2 above are mutually compatible for all admissible parameter values in the problem, whether or not policy is restricted to the zero lower bound in period *t*. That means we can pick any values of c_1 and c_2 that lie in the shaded area of Figure 1 (Appendix B). But do they also allow us to deploy the Taylor principle in the policy rule (11); and is anything lost if they do not? Appendix C shows that the Taylor principle requires the following to hold:

[15] See appendix B, part ii) below.

(22) $$\rho > c_1 + \alpha c_2 + (\alpha c_2)^2/c_1 = c_1 + \alpha c_1(1 + \alpha c_2/c_1).$$

This is certainly possible: especially if output inertia ρ is, or can be made to be relatively strong; or if the impact of inflation surprises, α, is weak (this is likely at or near the zero lower bound); or if c_2 can be chosen to be rather small. That makes good intuitive sense.

Thus, the stronger is output persistence, the stronger the reactions in the policy rule – and hence in the forward guidance or exit strategies – can and should be. On the other hand, (A20) also says that the Taylor principle can only be applied if $\rho > 1$ can be chosen. This might be possible temporarily; but appears to involve an unlikely combination of parameter values (given their other restrictions) and implies a careful coordination of fiscal and monetary policies will be needed in any event.

6. Conclusions

i) Given the rank condition described for dynamic controllability with REs, forward guidance is necessary to secure controllability and stabilizability: that is to secure the ability to reach, and then stabilize around, specified values for the target variables.

ii) Without forward guidance, which provides the private sector with information about the policymaker's future intentions, the economy will not in general be controllable with respect to any given set of target values; and it may not be stabilizable either.

iii) The capacity to stabilize the economy in fact depends not on the size or direction of shocks, but on the parameters of the policy rule not being chosen *in*appropriately (outside the bounds indicated). Offsetting large shocks needs a stronger rule and more carefully designed forward guidance. The existence of these bounds may prevent the Taylor principle being applied in some instances. But these are largely the cases close the zero lower bound where that principle is unlikely to be permissible anyway.

iv) A functioning stabilization policy rule is ***always*** possible, even at the zero lower bound.

v) These results highlight the tension between the needs of austerity vs. the needs of growth. It is clear that careful coordination between fiscal and monetary policies is always going to be necessary, not least in the forward guidance/exit strategy components, and that announcing the exact timing of the latter is (just about) everything.

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Appendix A: Determinacy in the Levin model with endogenous policy rules

Equation (8) gave the roots of the Levin et al. model of the economy as

(A1) $$\lambda_{1,2} = ½\{\beta + k\sigma + 1 \pm \sqrt{(\beta + k\sigma + 1)^2 - 4\beta}\}.$$

Since both roots are real, (9), to demonstrate the negative root is inside the unit circle we must have:

(A2) $$-1 < ½\left\{\beta + k\sigma + 1 - \sqrt{(\beta + k\sigma + 1)^2 - 4\beta}\right\} < 1$$

The right hand inequality holds if $\beta + k\sigma - 1 < \sqrt{(\beta + k\sigma + 1)^2 - 4\beta}$, i.e. if (on simplifying)

(A3) $$-4k\sigma < 0;$$

which is true for all admissible values of k and σ. Similarly, the left hand inequality holds if

(A4) $$0 \leq \beta + k\sigma + 1 - \sqrt{(\beta + k\sigma + 1)^2 - 4\beta}$$

which is again true for all admissible values of β since $4\beta > 0$.

To show that the other, positive, root $\lambda_1 = ½\{\beta + k\sigma + 1 + \sqrt{(\beta + k\sigma + 1)^2 - 4\beta}\}$ lies outside the unit circle, note that $\lambda_1 = 1$ if $(\beta, k, \sigma) = 0$ - their lower bound values. Also that

$$\frac{\partial\lambda_1}{\partial k} = ½\sigma\{1 + [(\beta + k\sigma + 1)^2 - 4\beta]^{-½}(\beta + k\sigma + 1)\} > 0,$$

(A5) $$\frac{\partial\lambda_1}{\partial\sigma} = ½k\left\{1 + [(\beta + k\sigma + 1)^2 - 4\beta]^{-½}(\beta + k\sigma + 1)\right\} > 0,$$

$$\frac{\partial\lambda_1}{\partial\beta} = ½\{1 + [(\beta + k\sigma + 1)^2 - 4\beta]^{-½}(\beta + k\sigma + 1)\} > 0.$$

This means that λ_1 is increasing in β, k, σ, from $\lambda_1 = 1$, for all admissible values of β, k, σ. It lies outside the unit circle therefore.

Appendix B: is a policy rule of the form (11) always feasible?

i) *Derivation of (18)*: given $\Phi > 0$, $\Delta > 0$, $c_1 > 0$ and $D < 0$, then $i_1 \geq 0$ in (16) requires

(A6) $$[\rho(\beta - \alpha)(1 + \beta c_2) + (\alpha - \beta c_1)\Phi] \geq 0.$$

Solving for admissible values of c_1 yields (18):

(A7) $$c_1 \leq \beta^{-1}\left[\alpha + \frac{\rho(\beta-\alpha)(1+\beta c_2)}{\Phi}\right]$$

which exceeds $\alpha/\beta > 0$ unless $\alpha = \beta$, in which the bound is $c_1 \leq 1$.

ii) *Derivation of (21) under the same conditions*: $E_t i_{t+1} > 0$ requires the following inequality

(A8) $$(1 + \beta c_2) + \frac{c_2[\rho(\beta-\alpha)(1+\beta c_2)+(\alpha-\beta c_1)\Phi]}{(c_1\Delta)^2} > 0$$

to hold. This expression is quadratic in c_2:

(A9) $$\left(\frac{\beta\rho(\beta-\alpha)}{(c_1\Delta)^2}\right)c_2^2 + \left(\beta + \frac{\rho(\beta-\alpha)+(\alpha-\beta c_1)\Phi}{(c_1\Delta)^2}\right)c_2 + 1 > 0.$$

Taking the positive root so that $c_2 > 0$, the necessary and sufficient condition for $E_t i_{t+1} > 0$ is:

(A10) $$c_2 > \frac{-\left[\beta+\frac{\rho(\beta-\alpha)+(\alpha-\beta c_1)\Phi}{(c_1\Delta)^2}\right]+\sqrt{\left(\beta+\frac{\rho(\beta-\alpha)+(\alpha-\beta c_1)\Phi}{(c_1\Delta)^2}\right)^2 - 4\frac{\beta\rho(\beta-\alpha)}{(c_1\Delta)^2}}}{2\beta(\rho\beta-\alpha)/(c_1\Delta)^2}.$$

That could be tested in any particular case. But an easier sufficient condition can be obtained from (A8):

(A11) $$\rho(\beta - \alpha)(1 + \beta c_2) > -(\alpha - \beta c_1)\Phi,$$

which ensures that the denominator, and thus all of (17), remains positive. This yields:

(A12) $$c_2 > \frac{-[\rho(\beta-\alpha)+(\alpha-\beta c_1)\Phi]}{\beta\rho(\beta-\alpha)}.$$

Hence any $c_2 > 0$ will do, so long as $\rho(\beta-\alpha)+(\alpha-\beta c_1)\Phi$ remains positive. Otherwise c_2 may have to be restricted from below.

iii) *Are the limits on c_2 still valid in the general case*; i.e. when i_t is not held at its lower bound? In that case, the full denominator of (17) needs to remain positive:

(A13) $(1+\beta c_2)\Delta^2 + c_2 c_1^{-1}[\rho(\beta-\alpha)(1+\beta c_2)+(\alpha-\beta c_1)\Phi] > 0$. This is quadratic in c_2:

(A14) $$\left(\frac{\beta\rho(\beta\text{-}\alpha)}{(c_1\Delta)^2}\right)c_2^2 + \left(\beta + \frac{\rho(\beta-\alpha)+(\alpha-\beta c_1)\Phi}{(c_1\Delta)^2}\right)c_2 + 1 > 0.$$

Simple sufficient conditions to guarantee that (A14) holds are:

(A15) $c_1 < \frac{\rho(\beta-\alpha)}{\beta\Phi} + \frac{\alpha}{\beta}$ from the middle term of (A14); or $c_2 < \frac{(c_1\Delta)^2}{(\beta c_1-\alpha)\Phi}$

from the last two terms of (A14). It is always possible to satisfy these two inequalities, although the upper bound of the former may fall short of satisfying the Taylor principle in some cases.

iv) *Are these bounds on c_2 still valid* if policy is already constrained to its zero lower bound in period *t*? Inserting the equality for c_1 from (A7) into (17) to get i_t=0, we find:

(A16) $$E_t i_{t+1} = \frac{\pi^*\Phi}{\Delta(1+\beta c_2)}\ .$$

This is always positive for all admissible c_2 values, so any $c_2 \geq 0$ will do. Moreover, $E_t i_{t+1}$ is never zero in this case: policymakers will *always* need to deploy forward guidance **and** an exit strategy after i_t has been at or near its zero lower bound.

v) *Are the upper and lower bounds on c_1 and c_2 always mutually compatible?* The slope of the constraint on c_1 in (c_1, c_2) space, taken from (16), is $\rho(\beta-\alpha)/\Phi$. But that on c_2 from (17) after renormalization to fit in the same parameter space, is identical. The two boundaries are therefore parallel: one an upper bound and one a lower bound. Their intercept terms however are different: $k_1 = \beta^{-1}[\alpha+\rho(\beta-\alpha)]$ for the upper bound on c_1; and $k_2 = -\beta^{-1}\left[1 + {}^{\alpha\Phi}/_{\rho(\beta-\alpha)}\right]$ for the lower bound on c_2.

< Place fig. 1 about here >

Hence there is *always* an admissible parameter space between the two bounds, meaning that compatible choices of c_1 and c_2 are always available in the shaded area in figure 1.

Appendix C: Can the Taylor Principle still be applied?

In the general case, will the Taylor principle still be available as part of the monetary policy rule? Given availability, whether a value of c_1>1 would be useful is quite another matter. Many would argue not in periods of deflation or serious recession when interest rates are already at their lower bound and cannot be reduced further. But in periods of deflation before the lower bound, or in normal times of positive inflation, $c_1 > 1$ might well be desirable – particularly as part of forward guidance and exit strategy announcements.

For the Taylor principle to be admissible, the upper bound on c_1 in (18) must exceed 1; that is, (A17) $$\beta^{-1}[\alpha + \rho(\beta - \alpha)(1 + \beta c_2)/\Phi] > 1$$

has to hold. Recasting (A17) in terms of c_2 and then substituting for Φ, yields

(A18) $$c_2 > \frac{\Phi-\rho}{\beta\rho} = \frac{\Delta^2 c_1-\rho+\rho(\beta-\alpha)c_2}{\beta\rho}, \quad \text{and thus} \quad c_2 > \frac{\Delta^2 c_1-\rho}{\alpha\rho}$$

as the necessary condition for the Taylor principle to be feasible. Substituting again for Δ reduces that condition to the following inequalities:

(A19) $$c_2>[c_1 + 2\alpha c_2 + (\alpha c_2)^2/c_1 - \rho], \quad \text{or}$$

(A20) $$\rho > c_1+\alpha c_2 + (\alpha c_2)^2/c_1 = c_1 + \alpha c_1(1 + \alpha c_2/c_1).$$

It is certainly possible to attain these conditions, especially if: i) output inertia ρ is, or can be made to be, relatively strong; ii) if the impact of inflation surprises, α, is weak; this is likely at or near the zero lower bound; or iii) if a suitably small value of c_2 is chosen. Hence the applicability of the Taylor principle depends on the choice of the policy parameters relative to the persistence in output and its sensitivity to inflation expectations. It does ***not*** depend on the impact of expected real interest rates or activity levels.

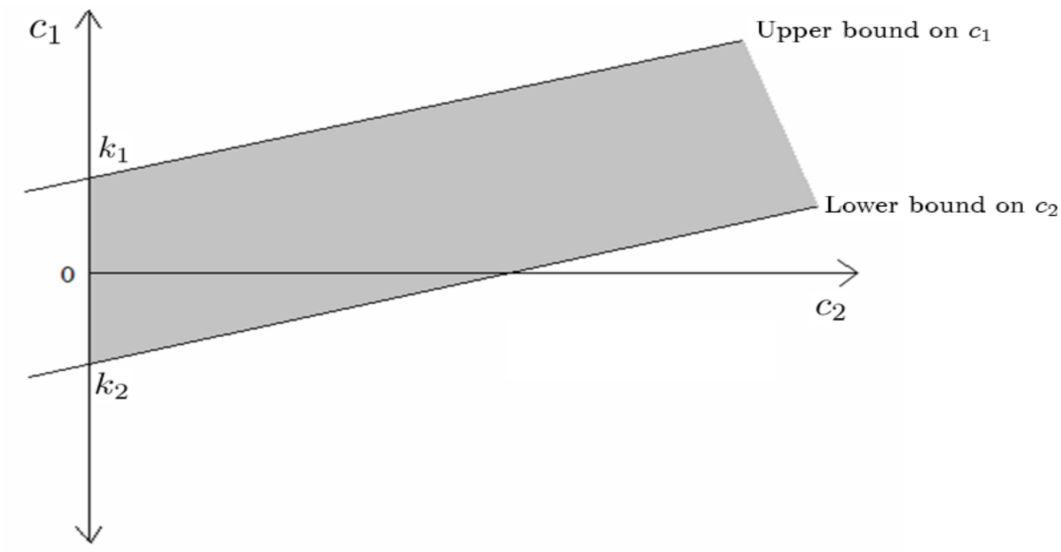

Figure 1: Compatible choices for policy rule parameters c_1 and c_2.