A contest success function for networks

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12th June 2015

Abstract

This paper models conflict within a network of friendships and enmities between players. We assume that each player is either in a friendly or in an antagonistic relation with every other player and players compete for a fixed prize by exerting costly efforts. We axiomatically characterize a success function which determines the share of each player given the efforts and the network of relations. This framework allows for the study of strategic incentives and friendship formation under conflict as well as the application of stability concepts of network theory to contests.

Keywords: contest, conflict, success function, networks, alliances, pairwise stability
JEL classification: C70, D72, D74, D85.

1 Introduction

In many conflictual situations, we often observe that competing parties join forces to fight together against others or refrain from fighting with each other. For instance, lobby groups may cooperate in supporting the same legislation when their interests coincide; belligerent states may form alliances for joint action if they face a common threat; competing firms may collude to increase their market share and so on. These parties do not necessarily act in a perfectly coordinated way, especially when their relation is an occasional opportunistic cooperation rather than a long term commitment, and these relations mostly rely on informal bilateral agreements which may form a complex network.

This paper introduces networks of relations to contest models, which are widely used to represent conflict over scarce resources. We consider players who compete for shares of some fixed resources by exerting costly efforts. The novelty of our setting is that each pair of players is either in a friendly or in an antagonistic relation, and the relations between all pairs define a network. We axiomatically characterize a success function which determines the share of resources of each player given all efforts and the network of relations. To our knowledge, this paper is the first to characterize a unified framework for contests defined on any type of networks, including those where *a friend of a friend is an enemy.* So far, the axiomatic work in the contest literature has exclusively focused on conflict between groups or between individuals. In the former, players are divided into mutually exclusive groups and groups compete with each other, while in the latter players compete all against

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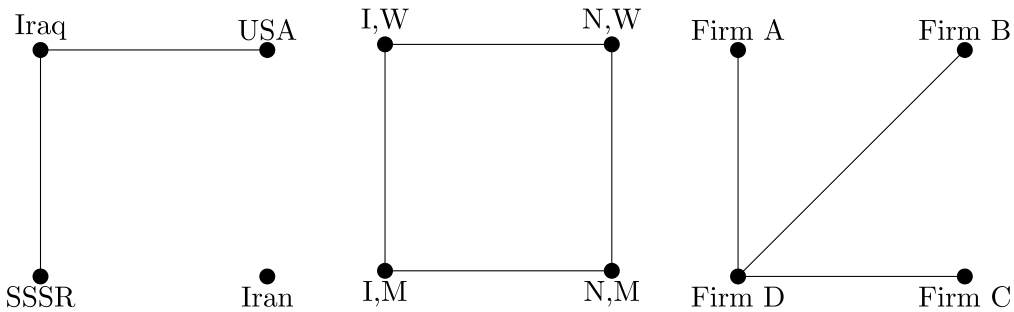
all. Yet, many competitive situations lead to networks which are different than these. We now describe three examples in international relations, political lobbying and R&D races respectively.

(a) In international relations, an alliance can be a rather informal agreement to join forces against common threats or to temporarily refrain from fighting against each other. These alliances are typically opportunistic; they do not mean perfect co-ordination or long term commitments and the friend of a friend can be an enemy at times. For instance, the United States and the Soviet Union were well-known to be enemies in the 1980s. However, they both supported Iraq in the Iran-Iraq War as they both perceived post-revolutionary Iran as a threat. We illustrate the corresponding network in Figure 1(a).

(b) Interest groups in political lobbying can have multiple characteristics. Let us describe a very simplified example where interest groups are based on origin and gender. We consider four interest groups defined by all combinations of native/immigrant and woman/man. There is a fixed public budget and each group lobbies for a larger share. The group of immigrant women lobbies for transfers to all immigrants and all women, while indirectly helping immigrant men and native women in their lobbying activities. This implies an informal alliance between groups that share one attribute. On the other hand, native men always lobby in the opposite direction to immigrant women. See Figure 1(b) for the network that represents these relations.

(c) Finally, we describe an example of R&D race of firms competing for market shares. Consider four firms, A, B, C and D in the electronics industry. Suppose firms A, B and C share the exact product portfolio of hardware products, while firm D additionally produces some software. Generally, the more a firm invests in R&D, the higher its market share is. This means in general, firms harm each other by increasing their R&D activities. However, the R&D efforts of firm D have also positive externalities on firms A, B and C and vice versa. So, the nature of the competition between, say A and D, is different than the one of A and B. This leads to the network given in Figure 1(c).



(a) International relations (b) Political lobbying (c) R&D race

Figure 1: Representations of networks in the examples above. A link between two players represents friendship. For the case on political lobbying, N,W denotes the group "native woman" and other groups are analogously labeled.

If we restrict our attention to all against all contests, our success function is equivalent to the well-established success function characterized by Skaperdas (1996). For contests between groups, our success function does not immediately link to the success function axiomatized in Münster (2009) as it determines shares of individual players rather than groups. However, the function obtained by summing up the shares of group members belongs to the class axiomatized in Münster (2009). In a broader sense, our success function is related to the probabilistic choice literature largely inspired by the seminal contribution of Luce (1959).

We characterize our success function by six axioms. *Exhaustivity* and *anonymity* are straightforward extensions of well-known axioms in the literature. The former requires that the sum of shares of all players is equal to the total of the resources while the latter implies that the shares do not depend on identities of players. We then have two monotonicity axioms, specifying how shares change in response to increasing efforts and to new friendships respectively. *Monotonicity of efforts* requires that the share of a player strictly increases in its own effort. *Monotonicity of relations* demands that the share of a player increases whenever the player becomes friend with a stronger player. We have two independence axioms. *Independence of efforts of commons* requires that the relative share of any two players is independent of the efforts of their common friends and common enemies. Our final axiom, *independence of relations of others*, is concerned with how relative shares change in response to a new friendship. The axiom imposes this change to be the same across all pairs of networks which differ only by the new friendship.

Our framework is useful in connecting two major fields, namely contests and network theory. Our model can in fact be used to study a specific network formation problem where the total value of the network is fixed and the success function works as the allocation rule for given efforts of players. We state this problem and show that under symmetric efforts the unique pairwise stable network is the peace network, where all players are friends. In the context of international relations, this result simply means that if countries had equal power, the only stable outcome would be peace. Alternatively, our model can be used to study strategic choice of efforts given a network of relations, with Nash equilibrium as the solution concept. This would be the standard setting in contest theory if the network was all against all. We provide an example where we consider all networks with equal number of friends for each player, and show that aggregate equilibrium efforts decrease in the number of friendships.

Section 2 reviews the literature. We introduce our model formally in Section 3. In Section 4, we present our axioms and state our main characterization result. We discuss possible applications of our framework in Section 5. Section 6 concludes. Proofs are in Appendix.

2 Related literature

The contest is the workhorse model for representing conflict and competition over scarce resources. It appears that Haavelmo (1954) was the first in formulating a contest model. Since then, contest models have been applied in a variety of areas of economics and social sciences, such as rent seeking (e.g., Tullock, 1975, Nitzan, 1994), industrial organization (e.g., Schmalensee, 1978, Fullerton and McAfee, 1999), incentives within organizations (e.g., Rosen, 1986, Müller and Wärneryd, 2001) and armed conflict (e.g., Skaperdas, 1992, Hirshleifer, 1995). See Konrad (2009) for an introduction to contest theory and its applications.

The axiomatic approach in contest theory started with the seminal work by Skaperdas (1996), who defines a success function for all against all contests. This characterization has been extended in several directions by relaxing some of the axioms (e.g., Clark and Riis, 1998; Blavatskyy, 2010; see Jia et al., 2013, for a review), by generalizing to multi-dimensional efforts of players (e.g., Rai and Sarin, 2009; Arbatskaya and Mialon, 2010), or by allowing rankings as the outcome of a contest instead of a single winner (Vesperoni, 2013; Lu and Wang, 2014). While these contributions are exclusively on all against all contests, Münster (2009) axiomatically characterizes a success function for contests where the competition takes place between mutually exclusive groups of players, i.e., coalitions. Our paper follows the axiomatic approach in contests, allowing players to compete in every possible network of friendships and enmities. A recent paper by König et al. (2014) focuses on networks where each pair of players can be in a friendly, neutral or antagonistic relation. Like us, they propose a function to determine each player's share of the prize. Their approach is not axiomatic; instead they show that the equilibrium effort of a player is related to an index of its centrality in the network under some restrictions; and they perform an empirical analysis using data from the Second Congo War, which involves many groups in a complex network of alliances and enmities.

Several papers in the broad subject of coalition formation in conflicts study contests between groups. Following the seminal contribution of Olson (1965), these works focus on the collective action problem in groups, showing that the power of a group may not increase in its size due to free-riding in the provision of collective effort (e.g., Esteban and Ray, 2001; Esteban and Sákovics, 2003; Konrad and Kovenock, 2009). See Bloch (2010) for a review on endogenous formation of groups in conflict. Although not strictly related, there are contributions on the broader subject of conflict within networks. Franke and Öztürk (2009) define a model where players are embedded in a network of bilateral conflicts and each pair can choose to fight in each conflict by spending efforts or refrain from fighting. In this setting, they characterize equilibrium efforts given specific types of conflict networks. Hiller (2011) analyzes a model where there are as many local conflicts as pairs of players and the win probabilities for each pair are determined by the number of their friends. In this model, there is no endogenous choice of efforts and payoffs are fully determined by the network of relations. Jackson and Nei (2014) define and analyze a new solution concept, called war-stability, for networks where each player is in a friendly or antagonistic relation with every other player. A necessary condition for war-stability is that no coalition of players can successfully attack another coalition. Unlike us, they associate a fixed effort to each player and their success function is deterministic. Goyal and Vigier (2014) consider a two-player game where a designer chooses a network and allocates specific efforts to defend each node, while an adversary allocates specific efforts to attack each node after observing these. They find the optimal network structure for the designer when the pair of efforts determines the probability of destruction of each node via Tullock (1975) success function.

3 Modeling networks in conflict

We consider a set of players $N = \{1, \ldots, n\}$, where $n \geq 3$. Players compete in a contest for increasing their shares of a given prize whose value is normalized to 1. A player $i \in N$ is either in a *friendly* relation or in an *antagonistic* relation with every other player in N. We write $F_i \subseteq N$ for the set of friends of i including i itself. Relations between friends (or enemies) are mutual, hence for any pair of players $i, j \in N$, we have $i \in F_j$ if and only if $j \in F_i$. We define a *network* as the profile of sets of friends $F := (F_1, ..., F_n)$ and we denote by $\mathcal{F}$ the set of all networks.[1] Each player $i \in N$ is associated with an effort $x_i > 0$. We write $x = (x_1, ..., x_n) \in X \subseteq \mathbb{R}^n_{++}$ for the profile of efforts. For each player $i \in N$, we define a *success function* as a mapping $s_i : X \times \mathcal{F} \to (0, 1)$, which maps any effort profile and network pair (x, F) into player i's share $s_i(x, F)$.[2]

In a standard contest model, it is generally assumed that each player fights alone; so the resulting network is the all against all network, where each player's only friend is itself. Hence, a player exerts effort to increase its share of the prize, and efforts of all other players – of the enemies – work in the opposite direction. We need to go beyond standard contest models to incorporate possible friendships in a conflict. Our model proposes a simple environment to do so. As we discuss in Section 1, conflictual parties may form links with each other whenever they find it beneficial. This does not necessarily mean that they are no longer in competition; they may still pursue their self-interest in competition with each other but their friendship may impose negative externalities on their enemies. In the next section, we introduce some properties on success functions that are in line with the idea of conflict and networks of friendships.

4 Characterization

In this section we present an axiomatic characterization of a particular success function through six axioms. The first three axioms are direct extensions of classical axioms in contest theory and they have similar justifications in our model. The latter three axioms incorporate the concept of friendships in conflict. The first condition, exhaustivity, requires that players always share the total of the prize.

Exhaustivity: For any $F \in \mathcal{F}$ and $x \in X$, $\sum_{i \in N} s_i(x, F) = 1$.

Anonymity states that shares are determined by efforts and networks, but not by players' identities. In short, it requires the contest to be a priori fair.

Anonymity: Let α be any permutation of N. For any $F \in \mathcal{F}$ and any $x \in X$, let $\alpha(F) = (F_{\alpha(1)}, ..., F_{\alpha(n)})$ and $\alpha(x) = (x_{\alpha(1)}, ..., x_{\alpha(n)})$. Then, $s_i(x, F) = s_{\alpha(i)}(\alpha(x), \alpha(F))$ for each $i \in N$.

[1] In standard network theory, a network or a *graph* g is defined as a list of unordered pairs of players $\{i, j\}$ which are linked. Here instead, we define a network as a profile of sets of friends for convenience. We can easily link the two definitions in the following way: for each $F \in \mathcal{J}$ we define g such that $\{i, j\} \in g$ if and only if $i \in F_j$. Note that this definition leads to non-directed graphs.

[2] We exclude the cases where efforts are zero or shares take value 0 or 1. These are marginal cases, but they lead to severe complications in our characterization which can only be dealt with ad hoc axioms. In applications, whenever necessary, it seems natural to assume that $s_i(x, F) = 0$ if $x_i = 0$ and $x_j > 0$ for some $j \neq i$, so that resources are exclusively shared by players which actively participate.

We now impose two monotonicity axioms; namely, monotonicity of efforts and monotonicity of relations. Monotonicity of efforts imposes that the share of a player is strictly increasing in its effort. Monotonicity of relations implies that being friends with a stronger player increases the share.

Monotonicity of efforts: Let $F \in \mathcal{F}$ be any network with $F_i \neq N$ for some player $i \in N$ and $x \in X$ be any effort profile. Then, $s_i(x', F) > s_i(x, F)$ for any $x' \in X$ with $x'_i > x_i$ and $x'_k = x_k$ for all $k \neq i$.

Monotonicity of relations: Let $F \in \mathcal{F}$ be any network and $x \in X$ be any effort profile such that there is a pair $i, j \in N$ with $i \notin F_j$. Consider $F' \in \mathcal{F}$ such that $i \in F'_j$ and $F'_h = F_h$ for all $h \notin \{i, j\}$. Then, $s_i(x, F') > s_i(x, F)$ if $x_j > x_i$.

We finally introduce two axioms of independence. The first one, independence of efforts of commons, imposes that the ratio of the shares of two players (their *relative* share) is independent of the efforts of their common friends and common enemies. The intuition is that the efforts of common friends and common enemies should similarly affect both players' shares; hence, they should not affect the relative share.

Independence of efforts of commons (IEC): Take a network $F \in \mathcal{F}$ and two effort profiles $x, x' \in X$. For any pair of players $i, j \in N$, $\frac{s_i(x,F)}{s_j(x,F)} = \frac{s_i(x',F)}{s_j(x',F)}$ if $x_k = x'_k$ for all $k \in (F_i \cup F_j) \setminus (F_i \cap F_j)$.

All axioms above except monotonicity of relations impose conditions on a success function for a given network. How should we expect shares to change when networks change, i.e., when new friendships/enmities are made, besides an increase in the share upon making a stronger friend? The second independence axiom, independence of relations of others, focuses on the relative share of two players and identifies how the relative share should change when one of these players makes a new friend or enemy. More specifically, the axiom requires the rate of change of this relative share to remain the same across all pairs of networks which differ only by the new friendship. So, it can also be seen as a consistency requirement imposing the change that results from befriending a player to be consistent across networks.

Independence of relations of others (IRO): Let $F \in \mathcal{F}$ be a network such that there are two players $i, j \in N$ with $i \in F_j$, and $x \in X$ be any effort profile. Let $F' \in \mathcal{F}$ be the network such that $i \notin F'_j$ and $F'_h = F_h$ for all $h \notin \{i, j\}$. Then, $\left(\frac{s_i(x,F')}{s_h(x,F')}\right) / \left(\frac{s_i(x,F)}{s_h(x,F)}\right) = \left(\frac{s_i(x,G')}{s_h(x,G')}\right) / \left(\frac{s_i(x,G)}{s_h(x,G)}\right)$ for all $h \in N \setminus \{i, j\}$ and $G, G' \in \mathcal{F}$ with $i \in G_j$, $i \notin G'_j$ and $G_k = G'_k$ for all $k \notin \{i, j\}$.

Our axioms uniquely characterize a particular success function s^*_i. For each network $F \in \mathcal{F}$, effort profile $x \in X$ and player $i \in N$, this function is defined as

$$s^*_i(x, F) = \frac{\prod_{h \in F_i} f(x_h)}{\sum_{j \in N} \prod_{h \in F_j} f(x_h)}, \tag{1}$$

where $f : \mathbb{R}_{++} \to (1, +\infty)$ is a strictly increasing function.[3]

[3]When we restrict our attention to all against all network where each player is friend with only itself, this success function reduces to the one introduced by Skaperdas (1996), except that in his framework f is not necessarily greater than 1. Note that the restriction $f > 1$ guarantees that when a player becomes friend with a stronger player its share always increases.

We are ready for the characterization result.

Theorem 1 *A success function $s_i : X \times \mathcal{F} \to (0,1)$ satisfies exhaustivity, anonymity, monotonicity of efforts, monotonicity of relations, IEC and IRO if and only if $s_i(x,F) = s_i^*(x,F)$ for any $i \in N$, $F \in \mathcal{F}$ and $x \in X$.*

This characterization is tight. We demonstrate the tightness of the axioms by means of examples of success functions which satisfy all but one. Proofs are left to the reader.

1. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, we define the success function $s_i^1(x,F)$ as
$$s_i^1(x,F) = \frac{\prod_{h\in F_i} f(x_h)}{\sum_{j\in N}\prod_{h\in F_j} f(x_h) + 1},$$
where $f : \mathbb{R}_{++} \to (1,+\infty)$ is strictly increasing. This success function satisfies all our axioms except exhaustivity.

2. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, we define the success function $s_i^2(x,F)$ as
$$s_i^2(x,F) = \frac{\prod_{h\in F_i} f_h(x_h)}{\sum_{j\in N}\prod_{h\in F_j} f_h(x_h)},$$
where $f_i : \mathbb{R}_{++} \to (1,+\infty)$ is strictly increasing for each $i \in N$ and $f_h \neq f_j$ for some $h,j \in N$. This success function fulfills all our axioms except anonymity.

3. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, we define the success function $s_i^3(x,F)$ as
$$s_i^3(x,F) = \frac{\prod_{h\in F_i} f(x_h)}{\sum_{j\in N}\prod_{h\in F_j} f(x_h)},$$
where $f : \mathbb{R}_{++} \to (1,+\infty)$ is strictly decreasing. Then, s_i^3 satisfies all axioms except monotonicity of efforts.

4. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, we define the success function $s_i^4(x,F)$ as
$$s_i^4(x,F) = \frac{\prod_{h\in F_i} f(x_h)}{\sum_{j\in N}\prod_{h\in F_j} f(x_h)},$$
where $f : \mathbb{R}_{++} \to (0,1)$ is strictly increasing. Then, s_i^4 fulfills all axioms but monotonicity of relations.

5. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, consider the success function $s_i^5(x,F)$ which takes the form
$$s_i^5(x,F) = \frac{f(x_i)^2\prod_{h\in F_i\setminus\{i\}} f(x_h)}{\sum_{j\in N} f(x_j)^2\prod_{h\in F_j\setminus\{j\}} f(x_h)},$$
where $f : \mathbb{R}_{++} \to (1,+\infty)$ is a strictly increasing function. This success function violates IEC, while it satisfies all other axioms.

6. For all $F \in \mathcal{F}$, $x \in X$ and $i \in N$, let us define the success function $s_i^6(x,F)$ as
$$s_i^6(x,F) = \begin{cases} \dfrac{f(x_i)^2}{\sum_{j\in N} f(x_j)^2} & \text{if } F_k = \{k\} \text{ for all } k \in N, \\[2ex] \dfrac{\prod_{h\in F_i} f(x_h)}{\sum_{j\in N}\prod_{h\in F_j} f(x_h)} & \text{otherwise,} \end{cases}$$

where $f : \mathbb{R}_{++} \to (1, +\infty)$ is a strictly increasing function. Then, s_i^6 fulfills all axioms but IRO.

Among these six functions, it is easy to see that s_i^1 and s_i^2 belong to the class of success functions characterized by the five axioms they satisfy. On the other hand, s_i^3 and s_i^4 can easily be fully characterized by further imposing the 'opposite' of the monotonicity axiom that each fails to satisfy. Characterization of s_i^5 and s_i^6 requires introducing other axioms and is not straightforward or necessarily desirable.

5 Applications

Due to its novelty and generality, our framework has potential for many applications. In this section, we illustrate two examples. First, we consider games where players choose relations taking efforts as given. Then, we study games where players choose efforts taking relations as given.

5.1 Network formation problems

In these games, players take their efforts as given and simultaneously choose whether to become friend with each other player or not. This setting belongs to the literature on network formation. See Jackson (2005) for a review on network formation games. These games are suitable for applications in international relations. Each player can represent a country whose effort is a measure of its military capability. While military capability is a stock, relations can change at convenience due to their ephemeral nature.

Our framework can be seen as a specific network formation problem where all networks generate the same value which is 1, hence the value function is constant. The success function works as the allocation rule. A common solution concept in this literature is *pairwise stability.* To formally define pairwise stability, we introduce some notation. Given a network $F \in \mathcal{F}$ and a pair of players $i, j \in N$ with $i \in F_j$, let $F - ij$ be the network where players i and j become enemies while all other relations remain as in F. Similarly, given a network $F \in \mathcal{F}$ and a pair of players $i, j \in N$ with $i \notin F_j$, let $F + ij$ be the network where players i and j become friends while all other relations remain as in F.

Definition 1 *For a given effort profile $x \in X$, a network $F \in \mathcal{F}$ is pairwise stable if*

(i) *for all $i, j \in N$ with $i \in F_j$, $s_i(x, F) > s_i(x, F - ij)$ and $s_j(x, F) > s_j(x, F - ij)$ and*

(ii) *for all $i, j \in N$ with $i \notin F_j$, if $s_i(x, F) \leq s_i(x, F + ij)$ then $s_j(x, F) \geq s_j(x, F + ij)$.*

In a pairwise stable network no pair of friends can both be better off by breaking their friendship and no pair of enemies can both be better off by becoming friends. We now provide a result which characterizes the unique pairwise stable network when efforts are symmetric.

Proposition 1 *For any symmetric effort profile $x \in X$, the unique pairwise stable network is the peace network where all players are friends.*

In the context of international relations, this setting allows countries to choose their relations taking their military capabilities as given. Our result then simply means that if all countries had equal military capabilities, peace for all would be the only pairwise stable network.

5.2 Contest games

In these games, players take the network of relations as given and simultaneously choose their efforts. This is the standard setting in contest theory if we restrict our attention to all against all network. This setting is suitable for applications to political lobbying among others. For instance, in the political lobbying example illustrated in the introduction, relations between interest groups are based on origin and gender, so are unlikely or impossible to change.

Following common assumptions in the literature, for any network $F \in \mathcal{F}$ and effort profile $x \in X$, we define the payoff of player $i \in N$ as $\pi_i(x, F) = s_i^*(x, F) - cx_i$, where $c > 0$ is the marginal cost of effort. Each player $i \in N$ chooses x_i to maximize $\pi_i(x, F)$ and we solve for Nash equilibrium of the corresponding game. We state a simple result for a network where all players have equal number of friends.

Proposition 2 *Let $f(x_i) = 1 + x_i$. For any network $F \in \mathcal{F}$ where all players have the same number of friends $\varphi < n$, there exists an equilibrium $x^* \in X$ such that for all $i \in N$ $x_i^* = (n - \varphi)/(n^2 c) - 1$ if $c < 1/n^2$.*

This simple result shows that equilibrium efforts decrease in the number of friends φ. Let us illustrate this proposition by the political lobbying example given in the introduction. Consider the network given in Figure 1(b), where there are 4 players: an immigrant woman, an immigrant man, a native woman, a native man. We take each player as an individual rather than a group for simplicity. Every player has the same number of friends which is 3 including itself. If $f(x_i) = 1 + x_i$ and $c < 1/n^2$, equilibrium efforts are $1/(16c) - 1$ as $n = 4$ and $\varphi = 3$. We now compare this result with another one for the alternative setting where both women are native and both men are immigrants. This gives a different network, where each player has 2 friends including itself (native women are friends with each other and immigrant men are friends with each other). The corresponding equilibrium efforts are $1/(8c) - 1$ by Proposition 2. Since $1/(16c) - 1 < 1/(8c) - 1$ for any $c < 1/n^2$, our success function predicts more effort – more lobbying – in the second case, when ethnic origin and gender are perfectly correlated leading to a more polarized set of players compared to the first setting.

This is of course a very simple example but this analysis can be extended to more complex networks to study, for instance, how the intensity of political lobbying in a country is affected by the structure of the identities of citizens. This approach would be similar to Esteban and Ray (1999) and Esteban et al. (2012), which show the positive link between ethnic polarization and conflict from a theoretical and empirical view point respectively. In our setting the identity of a citizen can be defined by the intersection of multiple categories such as ethnicity, gender, age, education, wealth, etc. Then, Esteban et al. (2012) can be extended to this multivariate setting by measuring the degree of correlation across these categories via the multivariate inequality index in Tsui (1995, 1999) or the multivariate polarization index in Srisuma and Vesperoni (2015).

6 Conclusion

We define and axiomatically characterize a success function for contests where each pair of players can be in a friendly or antagonistic relation, and all these pairwise relations form a network. For all against all contests, our success function is equivalent to the function characterized in Skaperdas (1996). For contests between groups, the aggregate share of group members always belongs to the class axiomatized in Münster (2009). Our success function treats every friend equally, however, it can easily be extended to the setting where each friendship has a weight. Another possible extension is to consider networks where relations between players are not necessarily mutual or equally weighted. We leave these extensions for future research.

Our framework allows to study strategic interaction between parties in conflict who are connected by a complex network of relations. Among many other environments, we commonly see such complex networks in international relations between countries, in industrial organization between competing firms and in political lobbying between interest groups. We have illustrated two possible applications of our model. We show in Section 5.1 that under symmetric efforts the unique pairwise stable network is the peace network when our success function works as the allocation rule. A very natural question that arises is whether this result persists under asymmetric effort profiles. We know that this result persists only for some profiles, and we leave characterization of such profiles for future research. Similarly, it is interesting to extend the analysis in Section 5.2 to any type of networks, or to study an environment where players choose their efforts and friendships simultaneously.

Our success function has potential for empirical applications as well. For instance, our model can be used to test for network effects by maximum likelihood or related methods. We refer to Jia et al. (2013) for a review of empirical issues in the estimation of success functions. The success function can also be used as an index of power adjusted for the network of relations. In the context of international relations, a particularly suited collection of datasets is presented by The Correlates of War Project, which spans for about two centuries and provides material for case studies as well as econometric analysis. In this context, the effort of a country can be estimated by its National Material Capabilities (see Singer et al., 1972) while its set of friends can be estimated by its Formal Alliances (see Gibler, 2009).

Appendix

Proof of Theorem 1

We leave to the reader to verify that (1) satisfies the axioms given in the theorem. To show that the converse holds, for each player $i \in N$, we take a success function $s_i : X \times \mathcal{F} \to (0, 1)$ satisfying the axioms. We want to show that for any $i \in N$, $x \in X$ and $F \in \mathcal{F}$, $(*)$ $s_i(x, F) = s_i^*(x, F)$.

Let $F \in \mathcal{F}$ be the network for which $F_i = N$ for each player $i \in N$. Take any effort profile $x \in X$. To show that $(*)$ is true, it suffices to show that $(**)$ $s_i(x, F) = 1/n$. Take any pair $i, j \in N$. Take a permutation α such that $\alpha(i) = j$, $\alpha(j) = i$, $\alpha(k) = k$ for all $k \notin \{i, j\}$.

By anonymity, $s_i(x,F) = s_j(\alpha(x),F)$ and $s_j(x,F) = s_i(\alpha(x),F)$. Note that $\alpha(F) = F$. Moreover, by IEC $\frac{s_i(x,F)}{s_j(x,F)} = \frac{s_i(\alpha(x),F)}{s_j(\alpha(x),F)}$ which then is equal to $\frac{s_j(x,F)}{s_i(x,F)}$. This implies that $s_i(x,F) = s_j(x,F)$. Together with exhaustivity this implies (∗∗).

To proceed in the proof, we now define a class of success functions slightly broader than (1). As an intermediate result we will argue that the axioms require the success function to belong to this class. For any $F \in \mathcal{F}$, $x \in X$ and $i \in N$, we define

$$\hat{s}_i(x,F) = \frac{\prod_{h \in F_i} f(x_h)}{\sum_{j \in N} \prod_{h \in F_j} f(x_h)} \tag{2}$$

where $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$. Note that (2) differs from (1) only by f not necessarily being strictly increasing and strictly greater than 1. Moreover it is easy to verify that (2) satisfies all our axioms except monotonicity of efforts and monotonicity of relations. Hence, (2) defines a class of success functions which includes (1).

To prove our intermediate result, we proceed by induction. First, let $F \in \mathcal{F}$ be a network with at least one pair of players which are enemies, i.e., $F_i \neq N$ for at least one player i. Let $x \in X$ be an effort profile. It is easy to show that there exists a sequence of networks $F^0, \ldots, F^m$ with $m \geq 1$ such that for $t \in \{0, \ldots, m-1\}$ (i) there is a pair of players $i, j \in N$ such that $i \in F_j^t$ and $i \notin F_j^{t+1}$, (ii) for all $k \notin \{i,j\}$, $F_k^t = F_k^{t+1}$ and (iii) $F_i^0 = N$ for all $i \in N$ and $F^m = F$. Note that there are $(m-1)!$ such sequences. We take any such sequence and we want to prove the following claim:

Claim: For each $i \in N$, $s_i(x,F^t) = \hat{s}_i(x,F^t) \implies s_i(x,F^{t+1}) = \hat{s}_i(x,F^{t+1})$.

Let $s_k(x,F^t) = \hat{s}_k(x,F^t)$ for all $k \in N$. Without loss of generality, we take two players $i, j \in N$ with $i \in F_j^t$ and we define the network F^{t+1} as the network where $i \notin F_j^{t+1}$ and $F_h^{t+1} = F_h^t$ for all $h \notin \{i,j\}$. Let h be any player in $N \backslash \{i,j\}$. By IEC, $\frac{s_i(x,F^t)}{s_h(x,F^t)}$ depends only on the efforts of the uncommon friends of i and h in F^t. Similarly, $\frac{s_i(x,F^{t+1})}{s_h(x,F^{t+1})}$ depends only on the efforts of their uncommon friends in F^{t+1}. Hence, there exists a real valued function $\gamma_{i,j,h}^{F^t}$ such that

$$\gamma_{i,j,h}^{F^t}(x_U) = \left(\frac{s_i(x,F^{t+1})}{s_h(x,F^{t+1})}\right) / \left(\frac{s_i(x,F^t)}{s_h(x,F^t)}\right) \tag{3}$$

where x_U is the profile of efforts of players $l \in U := [(F_i^t \cup F_h^t) \setminus (F_i^t \cap F_h^t)] \cup [(F_i^{t+1} \cup F_h^{t+1}) \setminus (F_i^{t+1} \cap F_h^{t+1})]$. By IRO, for any pair of networks $G, G' \in \mathcal{F}$ such that $i \in G_j$ and $i \notin G'_j$ and $G'_k = G_k$ for all $k \notin \{i,j\}$,

$$\gamma_{i,j,h}^{F^t}(x_U) = \left(\frac{s_i(x,G')}{s_h(x,G')}\right) / \left(\frac{s_i(x,G)}{s_h(x,G)}\right). \tag{4}$$

By IEC the right hand side of (4) is exclusively a function of $x_{U'}$, where $U' := [(G_i \cup G_h) \setminus (G_i \cap G_h)] \cup [(G'_i \cup G'_h) \setminus (G'_i \cap G'_h)]$. Note that $j \in U$ and $j \in U'$ by construction. As there is no restriction on G except that $i \in G_j$, the function $\gamma_{i,j,h}^{F^t}$ does not depend on the whole network F^t but only on the relation between i, j. Moreover, as (4) must hold for G such that $U' = \{j\}$, the function $\gamma_{i,j,h}^{F^t}$ is constant in all efforts except x_j. A similar set of expressions can be written for the relative share of players j and h in networks F^t and

F^{t+1}. Then, we can define the functions $g_{i,j,h} : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and $g_{j,i,h} : \mathbb{R}_{++} \to \mathbb{R}_{++}$ so that

$$g_{i,j,h}(x_j) = \left(\frac{s_i(x, F^{t+1})}{s_h(x, F^{t+1})}\right) / \left(\frac{s_i(x, F^t)}{s_h(x, F^t)}\right), \tag{5}$$

$$g_{j,i,h}(x_i) = \left(\frac{s_j(x, F^{t+1})}{s_h(x, F^{t+1})}\right) / \left(\frac{s_j(x, F^t)}{s_h(x, F^t)}\right). \tag{6}$$

For $n = 3$, we can immediately write $g_{i,j,h}(x_j) = g_{i,j}(x_j)$ and $g_{j,i,h}(x_i) = g_{j,i}(x_i)$ for the unique player $h \notin \{i, j\}$. Let $n \geq 4$, so that there are at least two players $h, k \in N \setminus \{i, j\}$. If we write (5) for $h, k \in N \setminus \{i, j\}$ and we take the ratio of the two expressions we obtain

$$\frac{g_{i,j,k}(x_j)}{g_{i,j,h}(x_j)} = \left(\frac{s_h(x, F^{t+1})}{s_k(x, F^{t+1})}\right) / \left(\frac{s_h(x, F^t)}{s_k(x, F^t)}\right).$$

Similarly if we write (6) for $h, k \in N \setminus \{i, j\}$ we obtain

$$\frac{g_{j,i,k}(x_i)}{g_{j,i,h}(x_i)} = \left(\frac{s_h(x, F^{t+1})}{s_k(x, F^{t+1})}\right) / \left(\frac{s_h(x, F^t)}{s_k(x, F^t)}\right).$$

Let $G, G' \in \mathcal{F}$ be such that $i \in G_j$ and $i \notin G'_j$ and $G'_l = G_l$ for all $l \notin \{i, j\}$. Moreover let $G_h = G_k$. Consider the permutation β such that $\beta(k) = h$, $\beta(h) = k$, $\beta(l) = l$ for all $l \notin \{h, k\}$. By anonymity $s_h(x, G) = s_k(\beta(x), \beta(G))$. Note that $\beta(G) = G$, therefore $s_h(x, G) = s_k(\beta(x), G)$. As $G_h = G_k$ implies $G'_h = G'_k$, by anonymity we must also have $s_h(x, G') = s_k(\beta(x), G')$. Moreover, by IEC $\frac{s_h(x,G)}{s_k(x,G)} = \frac{s_h(\beta(x),G)}{s_k(\beta(x),G)}$ which then is equal to $\frac{s_k(x,G)}{s_h(x,G)}$. This implies that $s_h(x, G) = s_k(x, G)$. Similarly, by IEC we also have $s_h(x, G') = s_k(x, G')$. It follows by IRO that

$$1 = \left(\frac{s_h(x, F^{t+1})}{s_k(x, F^{t+1})}\right) / \left(\frac{s_h(x, F^t)}{s_k(x, F^t)}\right), \tag{7}$$

hence $g_{i,j,h}(x_j)$ does not depend on the identity of h as long as $h \notin \{i, j\}$. Then, we can write $g_{i,j,h}(x_j) = g_{i,j}(x_j)$ and $g_{j,i,h}(x_i) = g_{j,i}(x_i)$ for each $h \in N \setminus \{i, j\}$ also when $n \geq 4$.

Now, let $G' \in \mathcal{F}$ be the network such that $G'_k = \{k\}$ for all $k \in N$. Consider the network $G \in \mathcal{F}$ which differs from G' only by i, j being friends, so $i \in G_j$ and $G_k = G'_k$ for all $k \notin \{i, j\}$. Consider the permutation α defined above. By anonymity $s_i(x, G) = s_j(\alpha(x), \alpha(G))$. Note that $\alpha(G) = G$, therefore $s_i(x, G) = s_j(\alpha(x), G)$. Then, as by IEC $\frac{s_i(x,G)}{s_j(x,G)}$ is constant in x, we must have $s_i(x, G) = s_j(x, G)$. Using IRO, we can write

$$\left(\frac{s_i(x, G')}{s_j(x, G')}\right) / \left(\frac{s_i(x, G)}{s_j(x, G)}\right) = \frac{s_i(x, G')}{s_j(x, G')} = \frac{g_{i,j}(x_j)}{g_{j,i}(x_i)}. \tag{8}$$

Since $\alpha(G') = G'$, by anonymity $g_{i,j} = g_{j,i}$. Note that any permutation of players besides α also leads to G'. Then, anonymity implies that $g_{i,j}$ and $g_{j,i}$ do not depend on the identities of i and j, hence we can write $g_{i,j} = g_{j,i} = g$ for all $i, j \in N$.

Let $s_i(x, F^t) = \hat{s}_i(x, F^t)$ for all $i \in N$, $x \in X$ and some $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$. By (5) we can write

$$g(x_j) = \left(\frac{s_i(x, F^{t+1})}{s_h(x, F^{t+1})}\right) / \left(\frac{\hat{s}_i(x, F^t)}{\hat{s}_h(x, F^t)}\right),$$

therefore

$$\frac{s_i(x,F^{t+1})}{s_h(x,F^{t+1})} = \frac{\prod_{l\in F_i^t} f(x_l)g(x_j)}{\prod_{l\in F_h^t} f(x_l)}. \tag{9}$$

To show that the Claim holds, we have to show that $g = 1/f$. Suppose for a contradiction, this is not the case. Consider the specific case where $\hat{F}^t \in \mathcal{F}$ be the network where $i \in \hat{F}_j^t$ and $\hat{F}_k^t = \{k\}$ for all $k \in N \setminus \{i,j\}$. Let $\hat{F}^{t+1}$ be defined accordingly. If we rewrite (9) for this pair of networks, we obtain

$$\frac{s_i(x,\hat{F}^{t+1})}{s_h(x,\hat{F}^{t+1})} = \frac{f(x_i)f(x_j)g(x_j)}{f(x_h)}, \tag{10}$$

which equals $\frac{g(x_h)}{g(x_i)}$ by (8). Then, we can write

$$\frac{f(x_i)f(x_j)g(x_j)}{f(x_h)} = \frac{g(x_h)}{g(x_i)}. \tag{11}$$

As the right hand side is constant in x_j, we must have $f(x_j)g(x_j) = c$ where c is a strictly positive constant. Plugging c/f for g in (11) and rearranging, we have

$$\frac{f(x_i)f(x_j)}{f(x_h)} = \frac{c/f(x_h)}{(c/f(x_i))(c/f(x_j))}, \tag{12}$$

which implies that $c = 1$, hence a contradiction. Therefore $g = 1/f$ and we can rewrite (9) as

$$\frac{s_i(x,F^{t+1})}{s_h(x,F^{t+1})} = \frac{\prod_{l\in F_i^{t+1}} f(x_l)}{\prod_{l\in F_h^{t+1}} f(x_l)}. \tag{13}$$

By (8) and (13), for any $i,j \in N$ we have

$$\frac{s_i(x,F^{t+1})}{s_j(x,F^{t+1})} = \frac{\prod_{l\in F_i^{t+1}} f(x_l)}{\prod_{l\in F_j^{t+1}} f(x_l)},$$

hence all relative shares are determined. For any $j \in N$, by exhaustivity,

$$\frac{1}{s_j(x,F^{t+1})} = \sum_{i\in N}\frac{s_i(x,F^{t+1})}{s_j(x,F^{t+1})} = \sum_{i\in N}\left(\frac{\prod_{l\in F_i^{t+1}} f(x_l)}{\prod_{l\in F_j^{t+1}} f(x_l)}\right) = \frac{\sum_{i\in N}\prod_{l\in F_i^{t+1}} f(x_l)}{\prod_{l\in F_j^{t+1}} f(x_l)}$$

therefore $s_j(x,F^{t+1}) = \hat{s}_j(x,F^{t+1})$. Then, we have shown that the Claim holds.

As (2) satisfies all axioms except monotonicity of efforts and monotonicity of relations, it can be proven that $s_i(x,F^0) = \hat{s}_i(x,F^0)$ for any positive function f. Then, for any network $F \in \mathcal{F}$ with at least one pair of players which are enemies, any effort profile $x \in X$, $s_i(x,F) = \hat{s}_i(x,F)$ for any $i \in N$ by induction.

Given this, to prove $(*)$ for such networks and effort profiles it is sufficient to show that the function f must always be strictly increasing and strictly greater than 1. Consider a network $F \in \mathcal{F}$ such that there is a pair $i,j \in N$ with $i \notin F_j$. As $i \notin F_j$, we have $s_i(x,F) = \hat{s}_i(x,F)$ and $s_j(x,F) = \hat{s}_j(x,F)$ for any $x \in X$. Take any $x,x' \in X$ such that $x_i' > x_i$ and $x_h' = x_h$ for $h \neq i$. By monotonicity of efforts $s_i(x',F) > s_i(x,F)$. Then, it is easy to verify that f must be strictly increasing. To show that $f > 1$, consider the network F' with $i \notin F'_j$ and $F'_k = F_k$ for all $k \notin \{i, j\}$. Without loss of generality, let $x_j > x_i$. Then, $s_i(x, F) > s_i(x, F')$, which implies that $f > 1$. So, we achieve the desired result $(*)$.

□

Proof of Proposition 1

Take any pair of players $i, j \in N$ and a network $F \in \mathcal{F}$ where i, j are friends. Let F' be the network in $\mathcal{F}$ where i, j are enemies and all other relations are the same as in F. For any effort profile $x \in X$, one can show that we have $s_i(x, F) > s_i(x, F')$ if and only if

$$f(x_i) - f(x_j) < (f(x_j) - 1)\left(\sum_{h \in N \setminus \{i,j\}} \prod_{k \in F_h} f(x_k)\right) \Big/ \left(\prod_{h \in F_j} f(x_h)\right). \tag{14}$$

Suppose the effort profile x is symmetric, i.e., $x_1 = \ldots = x_n > 0$. The LHS of (14) is always 0. Conversely the RHS is always positive. Then, by (14), any network F' where some players $i, j \in N$ are enemies is not pairwise stable, as they prefer to be friends. It follows that the only pairwise stable network is the one where all players are friends.

□

Proof of Proposition 2

Take any $F \in \mathcal{F}$ such that $\varphi < n$. It is easy to verify that, if $f(x_i) = 1 + x_i$, the second derivative of $\pi_i(x, F)$ with respect to x_i is negative for any $x \in X$ and $i \in N$. Then, a solution x^* of the system of equations given by the n first order conditions

$$s_i(x, F) \sum_{j \notin F_i} s_j(x, F) = cf(x_i)/f'(x_i)$$

of all players $i \in N$ defines a Nash equilibrium. Suppose that a solution satisfies $x_i^* = x_j^*$ for any $i, j \in N$. As $|F_i| = \varphi$ for any $i \in N$, we have $s_i(x^*, F) = 1/n$ for all $i \in N$ and each first order condition becomes $(n - \varphi)/n^2 = cf(x_i^*)/f'(x_i^*)$. As $f(x_i) = 1 + x_i$, we have $x_i^* = (n - \varphi)/(n^2 c) - 1$. Note that $x_i^* > 0$ for all $\varphi < n$ if and only if $c \in (0, 1/n^2)$. Then, x^* is an equilibrium if and only if $c \in (0, 1/n^2)$.

□

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