Determinacy and Identification with Optimal Policy*

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Abstract

Using Matthes (2015) estimation of Ramsey optimal policy under commitment (OPUC) in the new-Keynesian four-equations model, we interpret Volcker's Fed structural break of credibility as a "*Taylor principle saddlenode bifurcation" with a strictly negative optimal output gap rule parameter* and a shift of inflation rule parameter to a value larger than one (Taylor principle). Negative intertemporal elasticity of substitution, as in limited asset market participation (Bilbiie and Straub (2013)), and a positive marginal effect of current output on future inflation in the New-Keynesian Phillips curve (Mavroeidis et al. (2014), Taylor (1999)) would lead to an optimal positive output gap parameter. In the general linear quadratic case, OPUC admits a representation of the optimal rule as a non-inertial negative feedback function of current private sector's variables, with a unique initial anchor of forward variables. OPUC negative feedback optimal policy rule parameters never belong to the same set than positive feedback "optimal simple" rule parameters, which implies opposite policy recommendations. OPUC stabilizes the private sector's saddlepoint equilibrium in a *stable converging sink with determinacy* of optimal initial values of private sector's forward variables.

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"*The same equations have the same solutions*", Feynman, Leighton and Sands (1964, 12.1)

"*In new-Keynesian models, [the inflation Taylor rule parameter] $F_{\pi} > 1$ is the condition for a "dynamically unstable" model. New-Keynesian models want **unstable** dynamics in order to rule out multiple equilibria [sunspots] and force forward-looking solutions. In Taylor's model, $F_{\pi} > 1$ is the condition for **stable** dynamics, eigenvalues less than one, in which we solve for endogenous variables (including inflation) by backward looking solution. The conditions $F_{\pi} > 1$ sounds superficially similar, but in fact, its operation is diametrically the opposite. Taylor is worried about "spirals", not about determinacy... New-Keynesian models and results are often described with old-Keynesian intuition. This is a mistake.*" Cochrane (2011, p.602-604).

1 Introduction

Can the theory of Ramsey (1927) optimal policy under commitment of a decentralized economy rise on its ability to organize and interpret facts? Using data on inflation, output and interest rate including the Clarida, Gali and Gertler's (2000) period and assuming the private sector's four-equations new-Keynesian model, Matthes (2015, p.3) estimates the probability that *"only in 1980 were policy actions of the Volcker Federal Reserve able to significantly move the private sector's beliefs towards a central bank that acts under commitment and prefers lower inflation."*

We use Matthes (2015) estimates for computing the unique representation of optimal policy rule using the private sector's variables (more details below). Optimal policy under commitment confirms that U.S. inflation was conquered in the early 1980's by a change from a "passive" policy in which interest did not respond sufficiently to inflation to an "active" policy in which they do so. The inflation rule parameter is 2.76 in Matthes (2015), close to 2.15 in Clarida, Gali, Gertler (2000) and larger than one, according to the Taylor principle.

But a first surprise comes from that interest did respond *negatively* to output gap (We compute Matthes (2015) output gap rule parameter: -2.61) with optimal policy instead of positively 0.83 in the Clarida, Gali and Gertler

(2000) "simple" rule story. This sign difference reflects the opposite operation of monetary policy: negative feedback versus positive feedback. A second surprise is that this seemingly countercyclical optimal Taylor rule with respect to output is in fact *procyclical* in the new-Keynesian model! Conversely, the "*seemingly counter-cyclical*" positive output gap rule parameter is *key* in Clarida, Gali, Gertler (2000) to obtain a *procyclical positive feedback rule* in order to increase the number of unstable dimensions the private sector's dynamics. The sign of negative feedback rule parameters depends on the sign restriction of the monetary policy transmission mechanism. In the new-Keynesian model, current policy rate *increases* next period output gap (via the intertemporal substitution effect), Hence, a *negative* feedback rule parameter with respect to output gap should be *negative*. Then, one-time-step output gap *decreases* two-time-step inflation, so that a negative feedback rule parameter for the response of policy rate to inflation should necessarily be positive.

First, DSGE modellers shifting from the hypothesis of non-optimal policy maker's, "simple" rule, determinacy (with positive feedback rule) to optimal policy under commitment (with negative feedback rule) implies a dramatic change for the estimation of the model because forward variables turn to be necessarily included in the stationary recursive dynamics of the economy. In technical terms, changing determinacy assumptions is a *Hopf bifurcation* (Barnett and Duzhak (2008).Second, with optimal policy under commitment, one may interpret the shift from active to passive monetary policy in 1973 and from passive to active monetary policy in 1981 as back and forth *Taylor principle saddlenode bifurcations*. Following a mismearement of the output gap, and being uncertain on the structural break on the trend of output in 1973, Fed policy may have turned to give an excessive weight on output gap with respect to inflation. Because of a new-Keynesian monetary transmission mechanism where the policy rate modifies one-time-step output gap without a direct one-time-step effect on inflation, this opens the possibility to reduce the volatility of output, but to choose inflation rule parameter close to one and let inflation close to a unit root. Such a policy is risky, with a lack of robustness of stabilization to measurement errors of structural parameters. Diverging inflation occured with inflation spiral and indeterminacy.

In 1981, Volcker's created a structural break on inflation and output gap expectations, having built the credibility of Fed preferences with relatively large weight on inflation volatility, leading to a larger than one inflation rule parameter. Because of the new-Keynesian monetary transmission mechanism, to stabilize two-time-step inflation, one needs to stabilize first the intermediate target (output gap) at one time step with the policy rate. Then, both inflation and output gap turned to be stabilized, with a relatively safe distance from bifurcation borders and more robustness to measurement errors on structural parameters.

Policy makers may still worry on the negative sign of optimal policy output rule parameters, which contradicts countercyclical monetary policy Fed statements. Identical saddlenode bifurcations occurs with a negative feedback output gap parameter which is positive if one asssumes opposite sign restrictions than the current ones in the new-Keynesian model: a *negative* intertemporal elasticity of substitution and a *positive* sensitivity of future inflation to current output gap). Matthes (2015) Bayesian estimates depend heavily on the priors with positive sign restrictions on the intertemporal elasticity of substitution (prior 0.5) and on the negative sign restriction of the effect on future inflation of current output gap (prior 0.3). Bayesian estimation is indeed more likely to find negative posterior estimates with negative priors. An alternative theory and its estimations, such as limited asset market participation (Bilbiie (2007), Bilbiie and Straub (2013)), finds *negative* aggregate intertemporal elasticity of substitution. This estimates are also found in several countries by Havranek et al. (2015) meta-analysis despite a massive publication bias for positive IES estimated by Havranek (2013). Mavroeidis et al. (2014, figure 5) found nearly 40% *positive* correlation between future inflation and current output gap in the new-Keynesian Phillips curve. Alternative sign restriction as the key difference in Taylor's (1999) model with respect to the new-Keynesian four equations model. Assuming inflation is forward instead of backward in this model, optimal policy under commitment leads also to a Taylor principe with an intertemporal elasticity of substitution equal to zero and a positive correlation between future inflation and current output gap in the new-Keynesian Phillips curve. Uncertainty upon the sign (including zero) of structural parameters driving the monetary policy transmission is the worst type of uncertainty for the design of policy rule, in both optimal and non-optimal determinacy hypothesis, and in the case of robust optimal control applied on optimal policy under commitment.

This paper's novelty is a contribution to the interpretation of Ramsey (1927) optimal policy under commitment of a decentralized dynamic economy (Miller and Salmon (1985), Levine and Currie (1987)). We interpret optimal policy under commitment as a theory of how the credibility of policy maker's commitment perceived by the private sector leads to an initial *structural break* on private sector's expectations reflected by a unique optimal anchor (which differs from old-Keynesian model where all variables are backward looking), followed by the stabilization of private sector's forward and backward variables saddlepoint equilibrium dynamics into converging sink described by a "large" stationary vector auto-regressive of order one (VAR(1)) of minimal number of dimensions equal to the number of predetermined *and forward* variables, *exactly* as old-Keynesian stabilization policy, instead of a "narrow" stationary VAR(1) of minimal number of dimensions equal only to the number of predetermined variables.

We break a *thirty years old convention* stating that one can use any of the infinite number of representations of optimal policy rule, *except one*, where the rule depends on private sector's variables (Miller and Salmon (1985), Levine and Currie (1987)). This convention has always contradicted linear algebra: a mathematically and observationally equivalent system of equations *including initial boundary conditions* describe optimal policy under commitment with this representation of the rule! Optimal policy under commitment has opposite policy recommandation with negative feedback rule parameters which are always different from positive feedback "simple" and "optimal simple" rule parameters. Thirty years is incredibly long!

Can policy makers lean against bubbles and crashes of prices, output, asset prices and debt diverging from a "good" local economic equilibrium? At the same time, can policy makers determine unique anchors for the current values of forward looking variables, such as the price level? The new Keynesian and dynamic stochastic general equilibrium (DSGE) models, "simple" Taylor rule, provides the current standard answer which is negative. They use a "*non-optimal policy maker's determinacy hypothesis*" (our terminology): policy makers should not use negative feedback rule parameters stabilizing the private sector's saddlepoint equilibrium into a converging sink. Policy maker's should use positive feedback rule parameters so that private sector's equilibrium remains a saddlepoint equilibrium. Hence, there is a unique anchor of forward variables, such as inflation, output and asset prices on predetermined variables, such as the stocks of public debt, of private debt, of capital and autoregressive shocks.

However, policy-makers and economists do not have a perfect knowledge of structural parameters such that the intertemporal elasticity of substitution (Havranek et al. (2015), Havranek (2013)) and the inflation output gap sensitivity in the new-Keynesian Phillips curve (Mavroeidis et al. (2014)). The "*non-optimal policy maker's determinacy hypothesis*" is not robust to misspecification errors of the value of structural parameters. The initial anchor of forward on the unique stable path of a equilibrium with slightly misspecified parameters corresponds to a jump on an "*out-of-equilibrium*" unstable path of the correctly specified good local equilibrium, with a large loss of welfare. Potential diverging paths (bubbles) of DSGE models maintained with "non-optimal policy maker's determinacy hypothesis" have *a probability equal one to be real world effective bubbles* with large welfare costs of leaving a good local equilibrium neighborhood.

Section two compares optimal and non-optimal determinacy hypothesis for the new-Keynesian four-equations model and Matthes (2015) versus Clarida, Gali, Gertler (2000) estimations since Volcker's Fed. Section three does the same for Taylor's (1999) principle with *opposite* sign restrictions than in the new-Keynesian model. Section four obtains general determinacy and policy rule identification results for DSGE models comparing policy maker's models. The conclusion uses alternative criteria than determinacy to compare the Ramsey (1927) determinacy hypothesis with respect to the non-optimal policy maker's determinacy hypothesis.

2 New-Keynesian Four-Equations Model

(1) The monetary policy transmission mechanism

The new-Keynesian private sector's four-equations model is written with all variables as deviations of an equilibrium. In equations 1 and 2, two non-controllable exogenous stationary and predetermined variables $z_{x,t}$ and cost-push $z_{\pi,t}$ are auto-regressive of order one ($0 < |\rho_{z,x}| < 1$ and $0 < |\rho_{z,\pi}| < 1$) where $\varepsilon_{g,t}$ and $\varepsilon_{z,t}$ are zero-mean, normally, independently and identically distributed additive disturbances. Initial values of predetermined forcing variables are given. The equilibrium is for output gap: $y_t = 0$, for inflation: $\pi_t = 0$ and $i_t = i^*$: the nominal rate is equal to an optimal real rate of interest i^* with zero inflation at the equilibrium. In the representative household's consumption Euler equation (equation 1), *expected* output gap is equal to current output gap plus an increasing linear function of the real rate of interest, nominal rate i_t (written as deviation from i^*) minus *expected* inflation $E_t\pi_{t+1}$, with a intertemporal elasticity of substitution $\gamma > 0$. This is an intertemporal substitution (IS) equation. It is a mistake to call it an investment-savings equation as the representative consumer consumes all its current income at all dates: investment and savings are equal to zero (Gali (2015)). In the new-Keynesian Phillips curve (equation 2), β discounted expected inflation is equal to current inflation plus a negative linear function of current output gap with a sensitivity $-\kappa$. *Sign restrictions* are such that parameters γ, β, κ are all strictly positive. The new-Keynesian private sector's four-equations system is:

$$z_{x,t} = \rho_{z,x} z_{x,t-1} + \varepsilon_{x,t} \text{ where } \varepsilon_{x,t} \text{ is iid } N\left(0, s_x^2\right), \; z_{x,0} \text{ given,} \tag{1}$$

$$z_{\pi,t} = \rho_{z,\pi} z_{\pi,t-1} + \varepsilon_{\pi,t} \text{ where } \varepsilon_{x,t} \text{ is iid } N\left(0, s_\pi^2\right), \; z_{\pi,0} \text{ given.} \tag{2}$$

$$x_t = E_t x_{t+1} - \gamma\left(i_t - E_t \pi_{t+1}\right) + z_{x,t} \text{ where } \gamma > 0 \tag{3}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + z_{\pi,t} \text{ where } \beta > 0 \text{ and } \kappa > 0 \tag{4}$$

The system is written in a vector auto-regressive of order one (VAR(1)) state space form, with first predetermined variables $\mathbf{z}_t$ and second forward variables $\mathbf{q}_t$ (Giordani and Söderlind (2004)):

$$\begin{pmatrix} z_{x,t+1} \\ z_{\pi,t+1} \\ E_t x_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \begin{pmatrix} \rho_{z,x} & 0 & 0 & 0 \\ 0 & \rho_{z,\pi} & 0 & 0 \\ -1 & \frac{\gamma}{\beta} & 1+\frac{\gamma\kappa}{\beta} & -\frac{\gamma}{\beta} \\ 0 & -\frac{1}{\beta} & -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} z_{x,t} \\ z_{\pi,t} \\ x_t \\ \pi_t \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \gamma \\ 0 \end{pmatrix} i_t + \begin{pmatrix} \sigma_{z,x} & 0 \\ 0 & \sigma_{z,\pi} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{z,x,t+1} \\ \varepsilon_{z,\pi,t+1} \end{pmatrix}, \tag{5}$$

with block notations, in Kalman controllable staircase form:

$$\mathbf{z}_t = \begin{pmatrix} z_{x,t} \\ z_{\pi,t} \end{pmatrix}, \mathbf{q}_t = \begin{pmatrix} x_t \\ \pi_t \end{pmatrix}, \mathbf{z}_0 \text{ given, } \mathbf{q}_0 \text{ free.} \tag{6}$$

$$\begin{pmatrix} \mathbf{z}_{t+1} \\ E_t \mathbf{q}_{t+1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{zz} & \mathbf{0}_{zq} \\ \mathbf{A}_{qz} & \mathbf{A}_{qq} \end{pmatrix} \begin{pmatrix} \mathbf{z}_t \\ \mathbf{q}_t \end{pmatrix} + \begin{pmatrix} \mathbf{0}_z \\ \mathbf{B}_q \end{pmatrix} \mathbf{i}_t + \begin{pmatrix} \mathbf{\Sigma}_t \\ \mathbf{0}_{qq} \end{pmatrix} \varepsilon_t. \tag{7}$$

Reduced form elements of the transmission mechanism matrices $\mathbf{A}$ and $\mathbf{B}_q$ are functions of the vector of structural parameters denoted $\theta_1 = (\beta, \sigma, \kappa, \rho_{z,x}, \rho_{z,\pi})$. *Two* policy targets (output gap x_t and inflation π_t) are *two*-time-steps Kalman-controllable by a *single* monetary policy instrument i_t, because rank$(\mathbf{B}_q, \mathbf{A}_{qq}\mathbf{B}_q) = 2$. The full system $(\mathbf{A}, \mathbf{B})$ is stabilizable because the non-controllable variables are assumed to be asymptotically stable: $0 < |\rho_{z,x}| < 1$ and $0 < |\rho_{z,\pi}| < 1$.

With respect to the general case of section 4, this model does not include endogenous predetermined variables, because *the assumptions of a representative household and a zero net supply of one-period debt guarantee that public debt is zero for all dates* (Gali (2015), p.20, footnote 3). This eliminates the stock-flow dynamic equation of public debt. Then, forward variables cannot be initially anchored on predetermined public debt, but on *ad hoc* exogenous state variables $\mathbf{z}_t$ with auto-regressive parameters. In this *ad hoc* corner equilibrium, the interpretation of the intertemporal substitution equation is slightly absurd. Although interest payments are zero because the representative household does not hold public debt, the interest rate on zero public debt determines the household growth rate of its consumption, .

The laissez-faire ("open loop") private sector's model is described by the Fed following a fixed interest rate target or peg: $i_t - i^* = 0$. The transition matrix $\mathbf{A}_{qq}$ has one eigenvalue less than one and the other eigenvalue is larger than one. The matrix $\mathbf{A}_{zz}$ eigenvalues are stable: $0 < |\rho_{z,x}| < 1$ and $0 < |\rho_{z,\pi}| < 1$. The laissez-faire equilibrium has three stable eigenvalues and one unstable eigenvalue for two predetermined variables and two forward variables. It faces indeterminacy according to Blanchard and Kahn's (1980) condition.

(2) Comparing optimal versus non-optimal policy maker's determinacy hypothesis

Ramsey (1927) policy maker minimizes a quadratic loss function with respect to the policy rate, inflation and the output gap in order to find optimal rule parameters, using this following representation $i_t = \mathbf{F}_q^* (\theta) \mathbf{q}_t + \mathbf{F}_z^* (\theta) \mathbf{z}_t$, with a positive weights matrix $Q \geq 0$, with a *strictly* positive adjustment cost parameter $R > 0$ on the volatility of her policy instrument and a discount factor β equal to one:

$$L = -E_t \sum_{t=0}^{+\infty} \beta^t \left[\pi_t^2 + Q_{xx} x_t^2 + 2Q_{\pi z} \pi_t z_{\pi,t} + 2Q_{xz} x_t z_{x,t} + Q_{zz} z_t^2 + R i_t^2 \right] \quad (8)$$

subject to the private sector's new-Keynesian four equations model (equations (1) to (4)), with initial conditions for predetermined state variables and natural boundary conditions for forward variables (see section 4). There are possible policy maker's restrictions $Q_{xz} = Q_{\pi z} = Q_{zz} = 0$. For a Ramsey planner, additional restrictions constrain preference parameters $\mathbf{Q}$ to be functions of structural parameters of the private sector (Levine, Pearlman, Pierse (2008)). The policy maker's chooses optimal policy while taking private sector's behavior and initial conditions as constraints (equations 1,2,3,4).

Table 1 compares optimal versus non-optimal policy maker's determinacy hypothesis with an *ad hoc* interest rule $i_t = \mathbf{F}_q \mathbf{q}_t + \mathbf{F}_z \mathbf{z}_t$. .

Table 1: Optimal versus non-optimal policy maker's determinacy hypothesis.

PM:	Optimal	Non-optimal PM
game	Stackelberg	against nature
P $\mathbf{z}_{t+1} =$	$\mathbf{A}_{zz}\mathbf{z}_t + \mathbf{\Sigma}_t\varepsilon_{t+1}$	$\mathbf{A}_{zz}\mathbf{z}_t + \mathbf{\Sigma}_t\varepsilon_{t+1}$
F $\mathbf{q}_{t+1} =$	$\left(\mathbf{A}_{qz} + \mathbf{B}_q\mathbf{F}_z^*\right)\mathbf{z}_t$ $+\left(\mathbf{A}_{qq} + \mathbf{B}_q\mathbf{F}_q^*\right)\mathbf{q}_t$	$\mathbf{N}_z\left(\mathbf{F}\right)\mathbf{z}_{t+1}$
L $\mu_{q,t} =$	$\mathbf{P}_{qz}\mathbf{z}_t + \mathbf{P}_{qq}\mathbf{q}_t$	$\mathbf{0}$, Lucas critique
R $\mathbf{i}_t =$	$\mathbf{F}_z^*\mathbf{z}_t + \mathbf{F}_q^*\mathbf{q}_t$	$\mathbf{F}_z\mathbf{z}_t + \mathbf{F}_q\mathbf{q}_t$
$\mathbf{F}$	$\mathbf{F}^*\left(\theta\right)$, structural parameters	ad hoc, Lucas critique
I $\mathbf{z}_0 =$	given	given
I $\mathbf{q}_0 =$	$-\mathbf{P}_{qq}^{-1}\mathbf{P}_{qz}\mathbf{z}_0$	$\mathbf{N}_z\left(\mathbf{F}\right)\mathbf{z}_0$
P	$2+2$: $\mathbf{z}_t, \mu_{q,t}$	2: $\mathbf{z}_t$
F	2: $\mathbf{q}_t$	2: $\mathbf{q}_t$
B. K.	4 stable: $\rho_{z,x}, \rho_{z,\pi}$, $\left\|0.39 \pm 0.13i\right\| = 0.41$.	2 stable: $\rho_{z,x}, \rho_{z,\pi}$.
B. K.	2 unstable: $\left\|a \pm bi\right\| = 1/0.41 = 2.44$	2 unstable: $\left\|a \pm bi\right\| = 1.27$
total	6 dimensions	4 dimensions
PS	sink 4/0	saddlepoint 2/2
PM	saddlepoint 4/2	Lucas critique 0/0
identif. restrict $\mathbf{F}$	zero	2: e.g. $\mathbf{F}_z = 0$ as $\mathbf{i}_t =$ $\left(\mathbf{F}_q + \mathbf{F}_k\mathbf{N}_z^{-1}\right).\mathbf{q}_t$
Matthes/CGG $\mathbf{F}_z$	$F_{zx} = 1.15, F_{z\pi} = 5.1$	$F_{zx} = 0, F_{z\pi} = 0$
set $\mathbf{F}_q$	$D_O \subset D_S$	$D_{NO} \subset D_U$
Feedback $\mathbf{F}_q$	Negative, Stabilizing	Positive, Destabilizing
Sign restriction F_x	$F_x < 0$	$F_x > 0$
Taylor principle	$F_x > \frac{\kappa}{1-\beta}\left(1 - F_\pi\right)$	$F_x > \frac{\kappa}{1-\beta}\left(1 - F_\pi\right)$
Matthes/CGG $\mathbf{F}_q$	$F_x = -1.61, F_\pi = 2.79$	$F_x = 0.93, F_\pi = 2.15$
$\pi_t^* \rightarrow 0$ (for $Q_\pi \rightarrow +\infty$)	$\mathbf{F}_y^* \in D_O$ bounded with $\left\|\lambda\left(\mathbf{F}_y^*\right)\right\| \rightarrow 0$	$F_\pi \rightarrow \pm\infty, F_x$ bounded $\left\|\lambda\left(\mathbf{F}_y\right)\right\| \rightarrow +\infty$

B.K. Blanchard Kahn determinacy condition, PS: private sector, PM policy maker, P predetermined, F forward, L Lagrange multiplier, R rule, I initial conditions and initial anchor.

Both rational expectations solutions are exactly described by the system of equations (PFLRI) in table 1. For optimal policy under commitment, forward variables (equation F) are recursive and belongs to the VAR(1) part of the rational expections system, because of the credible commitment at the initial date of the policy maker. This is no longer the case with non-optimal policy maker, where private sector's forward variables are permanently anchored as linear functions of current predetermined variables. The elements of $\mathbf{N}_{NO}(\mathbf{F}_q, \theta)$ are computed as rational fractions of reduced form policy rule parameters $\mathbf{F}_q$ and of structural parameters $(\beta, \gamma, \kappa, \mathbf{\Sigma}_t, \mathbf{A}_{zz})$ in appendix 1.1.

Policy maker's Lagrange multipliers (equation L) corresponds to the optimal jump equations of rational expectations system, as a result of infinite horizon transversality conditions (cf. section 4). They are set to zero for all dates in the case of non-optimal policy, which faces the Lucas critique.

The optimal rule parameters (equation R) depends on policy maker's preferences and on the private sector's monetary transmission mechanism structural parameters. In the non-optimal policy maker's determinacy hypothesis, changes of structural parameters $\theta = (\beta, \gamma, \kappa)$ of monetary policy transmission mechanism do not imply a change of reduced form policy rule parameters (F_π, F_x): the non-optimal policy maker's rule faces the Lucas critique (Hurtado (2014)).

A unique optimal initial anchor (equation I) of forward variables is chosen by the policy maker. It is usually distinct from the initial and permanent anchor of non-optimal policy maker.

The unique optimal anchor is found minimizing the loss function at the initial date with respect to initial value of forward variables. This *predetermines to zero* at the initial date the two policy maker's Lagrange multipliers of the two forward variables, which are equal to the marginal loss function. Hence, the policy maker's Hamiltonian system includes then 4 predetermined variables and 2 forward, whereas the non-optimal policy maker's determinacy hypothesis includes only 2 predetermined variables and 2 forward. Hence, the Blanchard Kahn determinacy condition are different. The number of stable eigenvalues required for determinacy is 4 for the Ramsey (1927) optimal policy maker. It is equal to 2 in the case of non-optimal determinacy.

The representation of the Ramsey (1927) dynamics in the four dimensions stable subspace using private sector's variables is now a stable sink with 4 stable eigenvalues, whereas the open loop model included 3 stable and 1 unstable eigenvalues. There remains 2 unstable eigenvalues for the policy maker's saddlepoint equilibrium, which includes 6 dimensions. Optimal policy rule used stabilizing negative feedback in order to turn the open loop unstable private sector dynamics into a closed loop stable sink. The closed loop matrix $\mathbf{A}_{yy} + \mathbf{B}_q\mathbf{F}_q$ has two stable eigenvalues.Numerical eigenvalues are given for Matthes (2015) estimation of structural parameters.

By contrast, the number of stable dimensions required for determinacy is 2 with the non-optimal policy maker's determinacy hypothesis. With this policy, the private sector shifts from an open loop dynamics 3 stable and 1 unstable eigenvalue to a closed loop dynamics including 2 stable and 2 unstable eigenvalues. Non-optimal policy rule parameters use destabilizing positive feedback in order to increase the number of unstable dimensions with respect to the open loop system. The closed loop matrix $\mathbf{A}_{qq} + \mathbf{B}_q \mathbf{F}_q$ has two unstable eigenvalues. The *out-of-equilibrium* monetary transmission mechanism related to the vector auto-regressive of order one (VAR(1)) $\mathbf{q}_t = (\mathbf{A}_{qz} + \mathbf{B}_q \mathbf{F}_z)\mathbf{z}_{t-1} + (\mathbf{A}_{qq} + \mathbf{B}_q \mathbf{F}_q)\mathbf{q}_{t-1}$ is exploding. It is substituted by the instantaneous jump of forward variables on predetermined variables for all periods: $\mathbf{q}_t = \mathbf{N}_{NO}(\mathbf{F}_q, \theta) .\mathbf{z}_t$.

Two identification restrictions are required for the non-optimal policy rule, as the VAR(1) consists now only of two equations (equation P), and two variables are linear function of two others (equation F). The non-optimal policy rule can include at most two parameters (cf. section 4). We set to zero the two rule parameters on autoregressive shocks. with identification restrictions such that the policy rule does not react to auto-regressive shocks $\mathbf{z}_t$ ($\mathbf{F}_z = \mathbf{0}$).

Optimal negative feedback rule parameters related to controllable variables versus non-optimal positive feedback rule parameters belongs to two distinct sets denoted D_O and D_{NO}.

(3) Volcker's optimal policy under commitment versus non-optimal policy maker's determinacy hypothesis

Volcker's US disinflationary monetary policy is often related to the private sector's belief of the Fed's credible commitment to lean against inflation spiral despite a recession (Goodfriend and King (2005)). Matthes (2015) uses Söderlind (1999) algorithms to estimate the private sector's probability of discretionary policy versus optimal policy under commitment from 1960 to 2005. He assumes restrictions on policy maker's restrictions $Q_{xz} = Q_{\pi z} = Q_{zz} = 0$. Additional restrictions assuming Volcker-Greenspan to be Ramsey's (1927) planners would have constrained their preference parameter Q_{xx} to be a function of structural parameters of the private sector.

Matthes (2015) finds that the pre-Volcker period fits a stationary discretionary policy estimating Oudiz and Sachs (1985) model. He finds that optimal policy under commitment fits Volcker's period. We use Matthes (2015, table 1) calibrated discount factor $\beta = 0.99$ and posterior estimates for optimal policy under commitment, starting from priors satisfies the sign restrictions of the new-Keynesian four-equations model: $\kappa = 0.7/(1+\beta) = 0.7/(1.99) = 0.35$ (prior $0.3/(1.99) = 0.15$), $\gamma = 1/\sigma = 1/1.61 = 0.62$ (prior $1/2$), $\rho_{z,x} = 0.4$,$\rho_{z,\pi} = 0.57$,$\rho_{z,x\pi} = 0.61$,$R = 0.11$, $Q_x = 0.07$, $Q_\pi = 1$, $Q_{jz} = 0$. Both priors for κ and γ corresponds to large positive values with respect to Havranek et al. (2015), Havranek (2013) and Mavroeidis (2014) surveys. The large posterior size of inflation/output sensitivity $\kappa = 0.35$ (more than the double of the prior 0.15) is an extreme positive value very likely to be revised in future replications.

Optimal policy implies four stable eigenvalues: two complex conjugate $0.39 - 0.13i, 0.39 + 0.13i$ with absolute value of 0.41, two auto-regressive parameters $(0.3, 0.6)$ of forcing variables $\mathbf{z}_t$. It includes two unstable complex conjugate eigenvalues which mirror the two complex conjugate, with absolute value $1/0.41 = 2.4$ which are ruled out by policy maker's transversality conditions leading to matrix $\mathbf{P}_q$.

We compute the linear relation between policy maker's Lagrange multipliers and private sectors variables. The optimal initial anchor of forward variables corresponds to the Fed credibility structural break in 1981. In Matthes 2015 (figure 2), the probability of commitment regime rises over 0.6 in 1981:

$$\mu_t = \mathbf{P}_q\left(\beta, \kappa, \gamma, Q_x, R\right)\mathbf{q}_t + \mathbf{P}_z\left(\beta, \kappa, \gamma, Q_x, R, Q_{xz}, Q_{\pi z}, \mathbf{A}_{zz}\right)\mathbf{z}_t \quad (9)$$

$$\begin{pmatrix} \mu_{x,t} \\ \mu_{\pi,t} \end{pmatrix} = \begin{pmatrix} 0.66 & -1.22 \\ -1.22 & 3.67 \end{pmatrix} \begin{pmatrix} x_t \\ \pi_t \end{pmatrix} + \begin{pmatrix} 0.40 & 1.92 \\ -0.83 & -4.01 \end{pmatrix} \begin{pmatrix} z_{x,t} \\ z_{\pi,t} \end{pmatrix} \quad (10)$$

$$\begin{pmatrix} \mu_{x,1981} \\ \mu_{\pi,1981} \end{pmatrix} = 0 \Longleftrightarrow \begin{pmatrix} x_{1981} \\ \pi_{1981} \end{pmatrix} = \begin{pmatrix} -0.52 & -2.31 \\ 0.05 & 0.33 \end{pmatrix} \begin{pmatrix} z_{x,1981} \\ z_{\pi,1981} \end{pmatrix} \quad (11)$$

The matrix $\mathbf{P}_q$ is the unique solution of a discrete time Ricatti equation (Sargent and Ljungqvist (2012), chapter 19) and $\mathbf{P}_z$ is the unique solution of a Sylvester equation. They allow to compute unique optimal rule parameters $\mathbf{F}_q$ and $\mathbf{F}_z$ (code in appendix 1) presented in table 2.

In the general case (section 4), during a period where structural parameters do not change, there is an infinite number of representations of the optimal policy rule which are observationally equivalent to the representation as a *non-inertial simple rule function of private sectors current variables with* with fixed coefficients belonging to an optimal set D_O. This representation is *exactly* Kalman's (1960) representation of the optimal rule for the linear quadratic regulator which is programmed in MATLAB and SCILAB. Among the infinity of representation $\mathbf{F}'$ and of their determinacy sets $D_S^{'}$, this representation is the simplest to be presented by policy advisers to policy makers and by policy makers to the real world economic agents.

Considered in isolation, all these policy rules are completely different. Considered within the optimal policy maker's Hamiltonian system of equations including boundary conditions $\{P, F, L, R, I\}$ for all dates, they are all equivalent representations found by linear substitution using other equations of the system, for that the new representations of the rule R' belonging to a mathematically and observationally equivalent system of equations $\{P', F', L', R', I'\}$ for all dates:

$$\{P, F, L, R, I\} \text{ for all dates } t \Leftrightarrow \{P', F', L', R', I'\} \text{ for all dates } t \tag{12}$$

The optimal set $D_O^{'}$ of rule parameters $\mathbf{F}'$ of the rule R' may appear very different from the optimal set D_O of rule parameters $\mathbf{F}$, *but this does not change the solution.* Table 2 compares different representations of policy rules:

Table 2: Representations of rule parameters during Volcker's-Greenspan period.

Rule:	F_x	F_π	F_{z_x}	F_{z_π}	F_{μ_x}	F_{μ_π}	$F_{z_x(-1)}$	$F_{z_\pi(-1)}$	$F_{\pi(t-1)}$	$F_{i(t-1)}$
CGG	0.93	2.15	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	0.79
M	-1.61	2.79	1.15	5.10	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
M.1981	-3.92	2.03	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
M.SL	n.a.	n.a.	2.14	9.73	-2.69	-0.13	n.a.	n.a.	n.a	n.a.
M.1981	n.a.	n.a.	2.14	9.73	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
M3.HD	n.a.	n.a.			n.a.	n.a.			n.a.	
M.apdx	n.a.	n.a.	0.41	-0.56	?	?	n.a.	n.a.	-1.27	n.a.

n.a.: not available, ?: values not reported by Matthes (2015), SL: Sargent Ljunqgvist textbook representation, HD: Sargent Ljunqgvist history dependent representation, apdx: Matthes (2015) online appendix.

The first rule is Clarida, Gali and Gertler (2000): $F_\pi = 2.15$, $F_x = 0.93$ using *limited information* generalized method of moments (GMM) to estimate an inertial Taylor rule but not the four other equations of the new-Keynesian model. Using the same data for the same period, Mavroeidis (2010, figure 2) found very large robust confidence intervals for Clarida, Gali, Gertler (2000) estimates which *overlaps to negative values* for both rule parameters F_π and F_x.

By contrast, Matthes (2015) is a *full information* Bayesian estimation of the new-Keynesian four equations model with an estimation of Fed preference parameters. Matthes (2015) estimated inflation target π^* is 1.76 which is lower than 3.58 found by Clarida, Gali and Gertler (2000) and his estimated long run interest rate is $i^* = 1.03\%$.

For estimated structural parameters, which did not change during Volcker-Greenspan period, this *reduced form* representation of the policy rule of optimal policy under commitment is a *non-inertial simple rule with a small number (four) of fixed rule parameters.* Volcker's optimal policy under commitment is estimated such that the inflation rule parameter is large (2.79) and the output gap parameter is large in absolute value and negative (-1.61) and the interest rate also respond to auto-regressive shocks. This *seemingly pro-cyclical* monetary policy rule is counter-cyclical in this model (see next section).

For the year 1981, we substitute the optimal initial anchor of inflation and output gap in 1981. The credibility structural break of the initial jump of forward variables is *observationally equivalent* to an initial value of the optimal policy rate in 1981 with a response to inflation in 1981 (2.03) and a sharp negative response to output gap in 1981 (-3.92).

Another observationally equivalent representation of Matthes (2015) optimal rule is Sargent and Ljunqvist's (2012) textbook representation as a function of the policy maker's predetermined variables. It is given on the fourth row of table 2. Its observationally equivalent representation for the year of the structural break of Fed's credibility is immediately found setting $\mu_{x,1981} = \mu_{\pi,1981} = 0$. However, its rule parameters cannot be compared with Clarida, Gali and Gertler (2000). It can hardly be understood and communicated to policy maker's and by policy maker's to the general public, because the interest rate responds only to variables which are not reported national accounts statistics: policy maker's Lagrange multipliers $(\mu_{x,t}, \mu_{\pi,t})$ and auto-regressive forcing variables $(z_{x,t}, z_{\pi,t})$. We suggest to label this representation "implicit" instead of "explicit" as proposed by Svensson (2003). Conversely, we suggest to label the observationally equivalent representation of the rule as a function of private sector's variables as "explicit" instead of "implicit" as proposed by Svensson (2003).

Sargent and Ljunqvist (2012, chapter 19) and Woodford (2003) propose an observationally equivalent history dependent representation of optimal policy rule, after linear substitutions using period $t-1$ equations of the system (FPLRI) valid at all dates. With this "history dependent" representation, appears the lagged interest rate in the rule. "*This insight partly motivated Woodford (2003) to use his model to interpret empirical evidence about interest rate in the United States*" (footnote 10, p.774). Unfortunately, this representation includes 5 reduced form parameters instead of 4 reduced form parameters for the two other observationally equivalent rules which are "non-inertial". This implies that one identification restriction on reduced form parameters is *required* for this representation of the rule. Without this identification restriction, *the auto-regressive parameter in this "history-dependent" rule is not identified*. Contrary to Woodford's (2003) interpretation, including a cost of the volatility of the policy rate ($\mathbf{R} > \mathbf{0}$) in the policy maker's loss function *is not* a micro-foundation of an "inertial" policy rule including a lagged value of policy rate. This representation is observationally equivalent to non-inertial representations of optimal policy with fewer reduced form parameters. Larger smoothing parameters ($\mathbf{R} > \mathbf{0}$) implies *lower* optimal policy rule parameters $\mathbf{F}_q$ and $\mathbf{F}_z$ and implies a *more inertial rule*, with a parameter equal to zero for lagged instrument i_{t-1}, but with a policy rate which responds less to deviations of policy targets from their equilibrium!

The last row is a representation of Matthes' (2015) rule in Matthes online appendix (equation 4), which excludes the output gap. He does not report the numerical values of two rule parameters. This representation includes 5 reduced form parameters instead of 4 reduced form parameters for the two previous observationally equivalent rules. This implies that one identification restriction on reduced form parameters is required for this representation of the rule.

Finally, an infinity of *observationally equivalent* forward and backward representations of optimal rule where the policy rate responds to lagged or expected values of inflation are computed by linear substitutions using previous or future periods equations P and F of the PFLAI system of equations:

$$i_t = \mathbf{F}(\mathbf{A} + \mathbf{BF})^k.(\mathbf{q}_{t+k}, \mathbf{z}_{t+k}) \text{ for } k \in \mathbb{Z} \tag{13}$$

(4) Optimal versus non-optimal determinacy sets

Proposition 2.1. Optimal versus non-optimal determinacy sets

(i) The determinacy set D_O of optimal rule parameters (F_π, F_x) obtained when varying $(Q_{xx}, R) \in [0, +\infty[\times]0, +\infty[$ is included in the stable set D_S which imply two stable eigenvalues for the closed loop system excluding non-controllable variables $q_t = (A_{yy} + B_q F_q) q_{t-1}$.

(ii) This set has no intersection with the non-optimal policy maker determinacy set D_{NO} which imply two unstable eigenvalues. It is usually restricted to an implementable intersection with $F_x \in [0, 3]$ and $F_\pi \in [1, 3]$ (Schmitt-Grohé and Uribe (2007). This rectangle is included in the unstable set of rule parameters D_U:

$$D_O \subset D_S \text{ and } D_{NO} \subset D_U \text{ with } D_S \cap D_U = \varnothing$$

(iii) Both sets D_O and D_{NO} satisfies the new-Keynesian Taylor principle for F_π: $F_x > \frac{\kappa}{1-\beta}(1 - F_\pi)$. However, the optimal policy rule parameter for the output gap is always strictly negative $F_x < 0$ for the optimal determinacy set D_O whereas it is usually assumed to be at least equal to zero $F_x \geq 0$ for the non-optimal policy maker determinacy set D_{NO} (table 1).

Proof.

(i) is derived from infinite horizon transversality conditions, general case in section 3.

(ii) is a particular case of Wonham (1967) pole (eigenvalues) placement theorem: *for a controllable pair $(\mathbf{A}_{yy}, \mathbf{B}_q)$, the eigenvalues of $\mathbf{A}_{yy} + \mathbf{B}_q \mathbf{F}_q$ can be arbitrarily located in the complex plane (complex eigenvalues, however, occur in complex conjugate pairs) by choosing a policy rule $\mathbf{F}_q$ accordingly.*

(iii) see appendix 1 and general case in section 3.

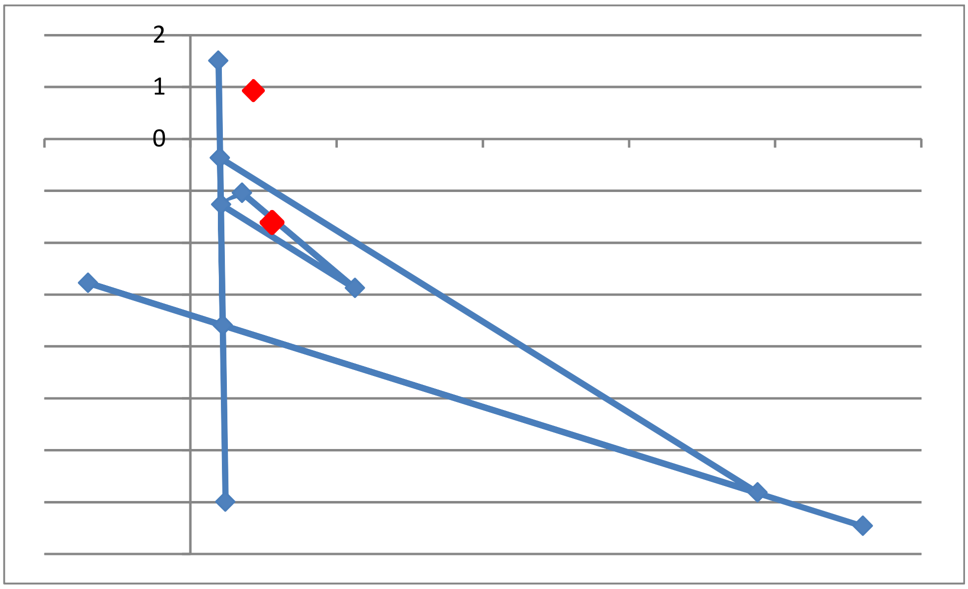

Figure 1: Optimal determinacy set (small triangle) D_0 of rule parameters included in upper corner of the large stable triangle D_s in the plane of rule parameters (F_π, F_x) for the new-Keynesian two-equations model, using Matthes (2015) estimates.

Table 3: Apexes of the optimal set and of the stability set of rule parameters on output gap and on inflation

Minimize:	R	Q_{xx}	$Q_{\pi\pi}$	$\lambda_1, \|\lambda_1\|$	$\lambda_2, \|\lambda_2\|$	F_π	F_x
Interest rate,	$1, +\infty$	0	0	0.41	0.73	1.77	-1.05
Output gap,	10^{-7}	$1, (+\infty)$	0	10^{-7}	0.995	1.06	-1.27
Inflation.	10^{-7}	0	$1, (+\infty)$	10^{-3}	10^{-3}	5.63	-2.88
Matthes (2015)	0.11	0.07	1	0.41	0.41	2.79	-1.61
CGG (2000)	-	-	-	1.27	1.27	2.15	0.93
Saddlenode/Hopf:	-	-	-	1	1	1.01	-0.37
Saddlenode/Flip:	-	-	-	1	-1	1.10	-3.60
Hopf/Flip:	-	-	-	-1	-1	19.4	-6.82

In figure 1, the optimal determinacy set D_O for $(Q_{xx}, R) \in [0, +\infty[\times]10^{-7}, +\infty[$ corresponds to the small triangle within the stability triangle D_S. In table 3, vertexes of D_O corresponds to the Fed seeking only maximal inertia of the policy rate or miniminizing only the variance of output gap without notice of the zero lower bound ($R = 10^{-7}$) or miniminizing only the variance of inflation without notice of the zero lower bound ($R = 10^{-7}$).

Matthes (2015) rule parameters are represented by a red dot in the small triangle D_O and Clarida, Gali and Gertler (2000) rule parameters are represented by in red dot in the Schmitt-Grohé and Uribe (2007) implementable simple rule rectangle. Assuming non-optimal determinacy versus optimal determinacy amounts to cross the Hopf bifurcation top side of the stability triangle, which corresponds to two conjugate eigenvalues having absolute value equal to one (Barnett and Dhuzak's (2008)). The private sector's economy shifts from 4 stable eigenvalue, including two complex conjugate eigenvalues (plus two unstable for the policy maker's Lagrangian system) to 2 stable eigenvalues and 2 unstable complex conjugate eigenvalues (four unstable eigenvalues for the policy maker's Lagrangian system, which violates the policy maker's infinite horizon transversality conditions).

The near-vertical left side of the stability triangle D_S sets the border of "*Taylor principle saddle-node bifurcations*". It is such that 1 is an eigenvalue ($P(1) = 0$). The eigenvector related to this unit root eigenvalue has a large weight on inflation with respect to the output gap. In 1973, the Fed shifted from active monetary policy (Taylor principle is satisfied F_x larger than one, on the right of this near-vertical side) to passive monetary policy on the left of this near-vertical side. In 1981, Volcker's Fed shifted back to stability from passive monetary policy to active monetary policy with a structural break of the credibility of the Fed commitment to lean against inflation. The private sector's economy shifts from 4 stable eigenvalue (plus two unstable for the policy maker's Lagrangian system) to 3 stable eigenvalue and one unstable eigenvalue (three unstable eigenvalue for the policy maker's Lagrangian system).

In 1973, the Fed may have had a relative large weight Q_{xx} on output, due an excessively large error on the measure of the output gap following an unexpected downward break of the trend of output growth. In the limit case where the Fed minimizes only the output gap volatility without notice of zero lower bound constraint on policy rate ($R \approx 0$), the left apex D_O corresponds to rule parameters ($F_\pi = 1.06, F_x = -1.27$) with a Taylor rule parameter very close to the Taylor principle lower bound (the saddle-node bifurcation line) and inflation close to unit root (eigenvalue), while output gap may be close to zero root (eigenvalue). The ability to stabilize output without stabilizing inflation is due to this assumption of the monetary transmission mechanism: interest has an effect on one-time-step output gap and no direct effect on one-time-step inflation. Being so close to the Taylor principle saddle-node bifurcation border, this optimal policy was not robust to small misspecification of private sector's structural parameters (γ, κ, β). Then, the Fed crossed the Taylor principle saddlenode bifurcation side of the stability triangle in 1973.

In 1981, Matthes (2015) found Volcker's rule parameters far from the Taylor principle saddle-node bifurcation border: ($F_\pi = 2.76, F_x = -1.61$). It is half way to the extreme case where the Fed preferences minimizes only inflation volatility without notice of zero lower bound constraint ($R \approx 0$), with rule parameters ($F_\pi = 5.63, F_x = -2.88$) with related two complex conjugate eigenvalues with absolute value close to zero ($|\lambda_1| = |\lambda_2| = 0.001$). This case corresponds to the right vertex of the optimal determinacy triangle D_O. Policies with a relative large weight on inflation are necessarily also stabilizing output, because the policy rate does not have a direct one-time-step effect on inflation and because there is an indirect (pass-through) transmission channel effect which occurs from one-time-step output gap to two-time-step inflation. By comparison, Schmitt-Grohé and Uribe (2007) "optimal simple" rule with a loss function minimizing only the square of inflation would find zero inflation for positive or negative infinite value of the inflation rule parameter F_π and an infinite eigenvalues.

Finally, the bottom side of the stability triangle corresponds a flip bifurcation (-1 is an eigenvalue) when the output gap parameter F_x is too large in absolute value and negative. Because the optimal optimal determinacy triangle D_O is far from this border, large errors on structural parameters would be required to observe such a bifurcation. The private sector's economy shifts from 4 stable eigenvalue (plus two unstable for the policy maker's Lagrangian system) to 3 stable eigenvalue and one unstable eigenvalue (three unstable eigenvalue for the policy maker's Lagrangian system).

With opposite sign restrictions $\gamma < 0$ and $\kappa < 0$, triangles of figure 1 are found by symmetry with respect to the horizontal axis in the positive quadrant of (F_x, F_π). A similar story holds this time with a positive sign of the output gap parameter $F_x > 0$ for optimal policy under commitment and a negative sign of the output gap parameter $F_x < 0$ for Clarida, Gali, Gertler (2000).

Proposition 2.2. The signs of negative feedback rule parameters (F_x, F_π) depends on the signs restrictions on monetary policy transmission mechanism (γ, κ) according to table 4.

Table 4: Sign restrictions on policy transmission implies signs restrictions on *negative* feedback optimal rule parameters

$\frac{\partial E_t(\pi_{t+2})}{\partial i_t}$ $= -\kappa(\gamma)$	NKIS: $\gamma > 0$ $\underset{>0}{\frac{\partial E_t(x_{t+1})}{\partial i_t}} F_x < 0$ $\Rightarrow F_x = \frac{\partial i_t}{\partial x_t} < 0$	Taylor (1999): $\underset{=0}{\frac{\partial E_t(x_{t+1})}{\partial i_t}} F_x < 0$ $\Rightarrow F_x = 0$ $\frac{\partial x_t}{\partial i_t} = \sigma < 0$	Bilbiie (2007): $\underset{<0}{\frac{\partial E_t(x_{t+1})}{\partial i_t}} F_x < 0$ $\Rightarrow F_x = \frac{\partial i_t}{\partial x_t} > 0$
NKPC $\frac{\partial E_t(\pi_{t+2})}{\partial E_t(x_{t+1})}$ $= -\kappa < 0$	New-Keynesian $\underset{<0}{\frac{\partial E_t(\pi_{t+2})}{\partial E_t(x_{t+1})}} \underset{>0}{\frac{\partial E_t(x_{t+1})}{\partial i_t}} F_\pi < 0$ $\Rightarrow F_\pi = \frac{\partial i_t}{\partial \pi_t} > 0$	$\underset{<0}{\frac{\partial E_t(\pi_{t+1})}{\partial x_t}} \underset{<0}{\frac{\partial x_t}{\partial i_t}} F_\pi < 0$ $\Rightarrow F_\pi = \frac{\partial i_t}{\partial \pi_t} < 0$	$\underset{<0}{\frac{\partial E_t(\pi_{t+2})}{\partial E_t(x_{t+1})}} \underset{<0}{\frac{\partial E_t(x_{t+1})}{\partial i_t}} F_\pi < 0$ $\Rightarrow F_\pi = \frac{\partial i_t}{\partial \pi_t} < 0$
Fed $\frac{\partial E_t(\pi_{t+2})}{\partial E_t(x_{t+1})}$ $= -\kappa > 0$	$\underset{>0}{\frac{\partial E_t(\pi_{t+2})}{\partial E_t(x_{t+1})}} \underset{>0}{\frac{\partial E_t(x_{t+1})}{\partial i_t}} F_\pi < 0$ $\Rightarrow F_\pi = \frac{\partial i_t}{\partial \pi_t} < 0$	Taylor (1999): $\underset{>0}{\frac{\partial E_t(\pi_{t+1})}{\partial x_t}} \underset{<0}{\frac{\partial x_t}{\partial i_t}} F_\pi < 0$ $\Rightarrow F_\pi = \frac{\partial i_t}{\partial \pi_t} > 0$	Fed statements: $\underset{>0}{\frac{\partial E_t(\pi_{t+2})}{\partial E_t(x_{t+1})}} \underset{<0}{\frac{\partial E_t(x_{t+1})}{\partial i_t}} F_\pi < 0$ $\Rightarrow F_\pi = \frac{\partial i_t}{\partial \pi_t} > 0$

NKIS: new-Keynesian intertemporal substitution equation, NKPC: new-Keynesian Phillips curve, Fed: Fed statements, Bilbiie (2007) and Bilbiie and Straub (2013): limited asset market participation theory.

Proof.

A negative feedback rule imply absolute values of closed-loop eigenvalues of $\mathbf{A}_{yy} + \mathbf{B}_q \mathbf{F}_q$ smaller than the ones of open loop matrix $\mathbf{A}_{yy}$. For positive open-loop eigenvalues for $\mathbf{A}_{yy} > 0$, then one has $\mathbf{B}_q \mathbf{F}_q < 0$. Hence, the rule parameter $\mathbf{F}_q$ should have the opposite sign of the marginal effect $\mathbf{B}_q$ of the policy instrument on the one-time-step value of the target. This is *only a necessary condition*. A necessary and sufficient condition is that both rule parameters belong to the stability set for negative feedback rule.

In the new-Keynesian model transmission mechanism proceeds in two steps. On a first period, a rise of the policy rate leads to a rise of one-time-step output gap (with a strictly positive intertemporal substitution parameter γ). Hence, a stabilizing negative feedback rule parameter should be negative for a negative feedback of a positive current output gap shock $x_t > 0$.

There is no direct effect of the policy rate on one-time-step inflation, but only a two-time-step effect passing through output gap. If the marginal effect of the policy rate on two-time-step inflation is negative, hence, a stabilizing negative feedback rule parameter on inflation should be positive.

Conversely, the limited asset market participation allows the possibility of a negative intertemporal elasticity of substitution. The monetary policy transmission mechanism is in line with Fed statements: a rise of the policy rate leads to a fall of one-time-step output gap. When the current output gap is above equilibrium $x_t > 0$, a negative feedback response of the policy rate stabilizing the output gap is necessarily positive $F_x > 0$. In the new-Keynesian Phillips curve, the sign restriction is such that a rise of one-time-step output gap *decreases* two-time-step inflation, which is the opposite of

Fed statements. The two signs of the transmission channel through the effect on the output gap are at each period the opposite of Fed statements. Because minus times minus is equal to plus, a rise of the interest rate leads to decrease two-time-step inflation by a factor $-\kappa\gamma$ in the new-Keynesian model and for the Fed statements. When current inflation is above equilibrium $\pi_t > 0$, a stabilizing negative feedback response of the policy rate is necessarily positive $F_\pi > 0$.

The empirical evidence of sign restrictions has been recently reviewed using thousands of single equation (limited-information) estimates in Havranek et al. (2015) and in Mavroeidis et al. (2014)). Both surveys highlight a massive publication bias in favor of positive coefficients ($\gamma > 0, \kappa > 0$) (Havranek (2013) and Mavroeidis et al. (2014), table 5). Both surveys found a large uncertainty of estimates of (γ, κ) so that strictly positive, equal to zero or strictly negative signs are not necessarily rejected. Using positive prior in Bayesian estimation, Matthes (2015) found positive posterior estimate: $\gamma = 0.62$, (with a prior of $\gamma = 0.5$) using limited asset market participation (LAMP) for a theoretical micro-foundation of an aggregate negative elasticity of substitution γ, Bilbiie and Straub (2013) found $\gamma = -0.6$ during the Volcker-Greenspan period in the USA.

3 Taylor's (1999) principle

Cochrane (2011) highlighted that Taylor (1999) used negative feedback rule only because output gap and inflation are assumed to be predetermined. But it is possible to assume that inflation is a forward variable in Taylor's (1999) model. Optimal policy under commitment uses negative feedback rule and finds again the Taylor principle, with Taylor (1999) opposite sign restrictions $\kappa < 0$ and an intertemporal elasticity of substitution $\gamma = 0$ than with the new-Keynesian model ($\kappa > 0$,$\gamma > 0$). Taylor's model includes the new-Keynesian Philips curve with negative sign: $\kappa < 0$ and auto-regressive cost-push shock $z_{\pi,t}$, assuming $\beta = 1$. With an intertemporal elasticity of substitution equal to zero, output gap is a predetermined variable with depends negatively ($-\delta$) on the current policy rate minus current inflation, instead of expected inflation, which is debatable. It also depends on an auto-regressive shock $z_{x,t}$ with the same auto-regressive parameter $\rho_{z,x} = \rho_{z,\pi}$ than the cost-push shock $z_{\pi,t}$.

$$x_t = -\delta\left(i_t - \pi_t\right) + z_{x,t} \text{ where } \delta > 0$$

The current period output gap can be eliminated, so that the private sector's monetary policy transmission mechanism is described by an inflation equation diverging at growth rate σ which is Kalman controllable by the policy rate ($\partial E_t \pi_{t+1} / \partial i_t = \sigma = -\kappa\delta \neq 0$) and a single aggregated AR(1) predetermined stationary shock z_t. The policy maker's quadratic loss function includes structural parameters $Q \geq 0$ and $R > 0$:

$$\min -E_t \sum_{t=0}^{+\infty} \beta^t \left[\pi_t^2 + 2Q_{\pi z} \pi_t z_{\pi,t} + Q_{zz} z_t^2 + R i_t^2 \right] \tag{14}$$

$$E_t \pi_{t+1} = (1+\sigma) \pi_t - \sigma (i_t - r) + z_t \tag{15}$$

$$z_t = \rho z_{t-1} + \varepsilon_{z,t} \tag{16}$$

$$z_t = -\kappa z_{x,t} + z_{\pi,t}, \varepsilon_{z,t} = -\kappa \varepsilon_{y,t} + \varepsilon_{\pi,t} \tag{17}$$

In the laissez-faire ("open loop") private sector's model, the Fed follows a fixed nominal interest rate target: $i_t = i^*$. The system includes one forward variable (inflation π_t diverging at growth rate $\sigma > 0$) and one exogenous predetermined variable and is a saddlepoint equilibrium. Blanchard and Kahn (1980) condition is satisfied for determinacy with $\pi_t = \frac{-1}{1+\sigma-\rho} z_t$. Table 5 compares solutions for both determinacy hypothesis, .

Figure 2. Taylor's (1999) model phasis diagramma: inflation response after a -10% cost-push shock, a a function of the shock with 0.9 auto-correlation coefficient.

Table 5. Taylor (1999) model.

Optimal PM determinacy	Non optimal PM determinacy
(P) $z_t = \rho z_{t-1} + \Sigma_t \varepsilon_t$	$z_t = \rho z_{t-1} + \Sigma_t \varepsilon_t$
(F) $\pi_t = (1 + \sigma - \sigma F^*_\pi)\,\pi_{t-1} + (1 - \sigma F^*_z)\, z_{t-1}$	$\pi_t = \frac{-1}{1+\sigma-\sigma F_\pi - \rho} z_t$
(L) $\mu_{\pi,t} = P_\pi \pi_t + P_{\pi z} z_t$	$\mu_{\pi,t} = 0$: Lucas critique
(R) $i_t = F^*_\pi\left(\underset{-}{\sigma},\underset{-}{R}\right)\pi_t + F^*_z\left(\underset{-}{\sigma},\underset{-}{R},\underset{-}{\rho},\underset{+}{Q_{\pi z}}\right) z_t$	$i_t = F_\pi \pi_t$ and $F_z = 0$
$F^*_\pi\left(\underset{-}{\sigma},\underset{-}{R}\right)$, $F^*_z\left(\underset{-}{\sigma},\underset{-}{R},\underset{-}{\rho},\underset{+}{Q_{\pi z}}\right)$	F_π: ad hoc, Lucas critique
(A) $\pi^*_0\left(\underset{+}{\sigma},\underset{+}{R},\underset{-}{\rho},\underset{-}{Q_{\pi z}}\right) = -P_\pi^{-1} P_{\pi z} z_0 \Leftrightarrow \mu_{\pi,0} = 0$	$\pi_0 = \frac{-1}{1+\sigma-\sigma F_\pi - \rho} z_0$, z_0 given
Predetermined: 2: $(z_t, \mu_{\pi,t})$	1: z_t
Forward: 1: π_t	1: π_t
Blanchard Kahn: 2 stable $\rho, \lambda(F^*_\pi)$,	1 stable eigenvalue ρ,
1 unstable eigenvalue: $1/\lambda(F^*_\pi)$.	1 unstable eigenvalue $\lambda(F_\pi)$.
$0 < R < +\infty \Rightarrow 0 < \lambda(F^*_\pi) = 1 + \sigma - \sigma F^*_\pi < \frac{1}{1+\sigma}$	$\lvert\lambda(F_\pi)\rvert = \lvert 1 + \sigma - \sigma F_\pi\rvert > 1$
Taylor: $F^*_\pi \in D_O =]1 + \frac{1}{1+\sigma}, 1 + \frac{1}{\sigma}[\subset]1, 1 + \frac{2}{\sigma}[$	$F_\pi \in D_{NO} =]-\infty, 1[\cup]1 + \frac{2}{\sigma}, +\infty[$
Negative feedback, Stabilizing	Positive feedback, Destabilizing
$\partial \pi_t / \partial F^*_\pi < 0$, if $\pi_{t-1} > 0$	$\partial \pi_t / \partial F_\pi < 0$
If $Q_\pi \to +\infty \Rightarrow F^*_\pi\left(\underset{-}{\sigma},\underset{-}{R}\right) \to 1 + \frac{1}{\sigma}$ $\Rightarrow \lvert\lambda(F^*_\pi)\rvert \to 0 \Rightarrow \pi^*_t \to 0$	For $Q_\pi \to +\infty : \lvert F_\pi\rvert \to +\infty$ $\Rightarrow \lvert\lambda(F_\pi)\rvert \to +\infty \Rightarrow \pi \to 0$

The optimal inflation rule parameter $F^*_\pi\left(\underset{-}{\sigma},\underset{-}{R}\right)$ decreases with the monetary transmission parameter σ and with the Fed's relative weight R on the volatility of the policy rate. When varying Fed's preference $0 < R < +\infty$ for given transmission σ, $F^*_\pi\left(\underset{-}{\sigma},\underset{-}{R}\right)$ varies within the optimal set $D_O =]1 + \frac{1}{1+\sigma}, 1 + \frac{1}{\sigma}[$ included in the stability interval $D_S =]1, 1 + \frac{2}{\sigma}[$. The rule parameter F^*_π is a linear decreasing function of the inflation growth factor (eigenvalue λ_1) which varies between zero and the inverse of the open-loop growth factor for maximal inertia of the Fed ($R \to +\infty$): $]0, \frac{1}{1+\sigma}[$.

If the Fed minimizes only the variance of inflation down to zero (growth factor $\lambda_1 = 0$) regardless of zero lower bound constraint (the cost of changing policy rate tends to zero: $R \to 0$), the *maximal negative feedback* value of the optimal rule parameter is reached: $F^*_\pi = 1 + \frac{1}{\sigma}$.

For non optimal policy maker's determinacy hypothesis, this is obtained in minimizing the jump of inflation on the predetermined variable. This minimizes the slope of the eigenvector of the eigenvalue ρ where the rule parameter F_π is at the denominator of a rational fraction. This implies an *infinite out-of-equilibrium positive feedback* rule coefficient F_π, policy rate

i_t and inflation growth factor λ_1. The Fed should have a perfect knowledge of structural parameters σ and ρ. *For this optimal policy assuming the non-optimal policy maker's determinacy condition (F_π), an infinitesimal measurement error by the Fed on σ or ρ would instantanously blow up inflation π_t with an infinite discontinuity, from zero to infinity. This implies an infinite lack of robustness of the stability of this equilibrium path to the misspecification of structural parameters.*

This result is frequently found in DSGE models seeking with simulation grid optimal rule parameters $\mathbf{F}$ in the non-optimal policy maker's determinacy hypothesis set D_{NO}. For example, Schmitt-Grohé and Uribe (2007) find numerically $F_\pi = 332$ (footnote 8, p.1712). Indeed, Ramsey (1927) planner seeks *optimal* rule parameters $\mathbf{F}$ in the *optimal* Ramsey (1927) planner determinacy set D_O which has no intersection with the non-optimal policy maker's determinacy set D_{NO}.

Figure 2 represents inflation response after a -10% cost-push shock as a function of a cost-push shock with an auto-correlation parameter $\rho = 0.9$ for several cases: a laissez-faire equilibrium path, a laissez-faire out-of-equilibrium path, an optimal path under commitment on date $t = 0$, an optimal path under commitment optimizing on date $t = 3$, and the line $\pi_t = -P_\pi^{-1} P_{\pi z} z_t$ in the out-of-equilibrium case of the Fed reneging commitment at all dates and changing the optimal initial anchor with a new optimisation at all dates (cf. section 5). The "optimal policy minimizing only inflation within the non-optimal policy maker's determinacy set" is such that $\pi_t = 0$ and $z_t = \rho^t z_0$, with dots on the horizontal axis. Its related out-of-equilibrium paths corresponds to $\pi_t = \pm\infty$: they cannot be seen on the diagram.

4 Linear quadratic policies: general case

The policy transmission mechanism is described by private sector's linearized dynamics. In the linear quadratic framework including additive gaussian random variables, the certainty equivalence property (Simon (1956), Kalman (1960), Hansen and Sargent (2013)) is validated for linear first order conditions, so that the rational expectations solutions of optimal rule parameters is equivalent to a *perfect foresight solution with unknown initial conditions for forward variables:* they finally do not depend on an appropriate vector of random gaussian variables which can be added or omitted. The system can be written in a Kalman controllable staircase form:

$$\begin{pmatrix} \mathbf{z}_{t+1} \\ E_t\mathbf{y}_{t+1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{zz}(\theta_1) & \mathbf{0}_{zy} \\ \mathbf{A}_{yz}(\theta_1) & \mathbf{A}_{yy}(\theta_1) \end{pmatrix} \begin{pmatrix} \mathbf{z}_t \\ \mathbf{y}_t \end{pmatrix} + \begin{pmatrix} \mathbf{0}_z \\ \mathbf{B}_y(\theta_1) \end{pmatrix} \mathbf{i}_t \tag{18}$$

where $\mathbf{y}_t = (\mathbf{k}_t^T, \mathbf{q}_t^T)^T$ is an $(n_c + m) \times 1$ vector, $\mathbf{k}_t$ is an $n_c \times 1$ vector of controllable predetermined variables at time $t = 0$ with initial conditions $\mathbf{k}_0$ given, $\mathbf{q}_t$ is an $m \times 1$ vector of controllable non-predetermined "forward" variables (including at least one forward variable $m \geq 1$), $\mathbf{z}_t$ is an $(n - n_c) \times 1$ vector of exogenous non-controllable predetermined stationary variables (such as auto-regressive forcing variables), $\mathbf{i}_t$ is a $p \times 1$ vector of the policy maker's instruments. We assume that *all* m forward variables are controllable and that the number of controllable predetermined variables is n_c with $0 \leq n_c \leq n$. All variables are expressed as absolute or proportional deviations about a steady state. $\mathbf{A}(\theta_1)$ is $(n + m) \times (n + m)$ matrix which depends on a vector of structural parameters θ_1 belonging to a set $\Theta_1 \subseteq \mathbb{R}^{n_{\theta_1}}$. $\mathbf{B}(\theta_1)$ is the $(n + m) \times p$ matrix of the marginal effects of policy instruments $\mathbf{i}_t$ on next period policy targets $\mathbf{y}_{t+1}$.

In the *laissez-faire* equilibrium, the private sector's dynamics is an *open loop* system governed by the transition matrix $\mathbf{A}(\theta_1)$: there is no policy intervention: $\mathbf{i}_t = \mathbf{0}$. Else the private sector's dynamics is a *closed loop* system with a *transmission mechanism of economic policy* given by the series $\mathbf{A}^t\mathbf{B},$ $t \in \mathbb{N}$. Kalman's (1960) defines that a closed loop system is *t-time-steps controllable* if, from any start state $\mathbf{y}_0$ we can reach any desired state $\mathbf{y}^*$ at time t. Kalman's (1960) controllability is a generalization of Tinbergen's (1952) rule for static models: achieving the desired values of a number of policy targets in t-time-steps requires the policy maker to control an equal number of policy instrument per time-steps (see appendix).

Ramsey's (1927) policy maker is a Stackelberg leader of a decentralized economy uses only policy instrument under his control (taxes, subsidies, policy interest rate) which are not under the control of the decentralized private sector. The Ramsey policy maker's preferences are represented by a quadratic loss function subject to private sectors linear conditions. The policy maker's preferences can be derived from private sector's utility: in this case, the policy maker is a Ramsey planner. An approximation of non-linear non-quadratic programs into a linear-quadratic program is obtained using the method proposed by Levine, Pearlman and Pierse (2008). The policy maker minimizes her quadratic loss function subject to the private sector dynamics by finding a sequence of decision rules for policy instruments $\mathbf{i}_t$ (Hansen and Sargent (2013)):

$$\max_{\{\mathbf{i}_t\}} L = -\frac{1}{2}\sum_{t=0}^{+\infty}\beta^t \left(\mathbf{y}_t^T \mathbf{Q}_{yy}\left(\theta_1,\theta_2\right)\mathbf{y}_t + 2\mathbf{y}_t^T \mathbf{Q}_{yz}\left(\theta_1,\theta_2\right)\mathbf{z}_t + \mathbf{z}_t^T \mathbf{Q}_{zz}\left(\theta_1,\theta_2\right)\mathbf{z}_t + \mathbf{i}_t^T \mathbf{R}\left(\theta_2\right)\mathbf{i}_t\right) \tag{19}$$

where β is the policy maker's discount factor and her policy preference are the relative weights included matrices $\mathbf{Q},\mathbf{R}$. $\mathbf{Q} \geq \mathbf{0}$ is a $(n+m)\times(n+m)$ positive symmetric semi-definite matrix, $\mathbf{R} > \mathbf{0}$ is a $p \times p$ *strictly* positive symmetric definite matrix so that policy maker's has at least a small concern for the volatility of policy instruments.

Matrices $\mathbf{Q}$ and $\mathbf{R}$ define the policy maker's preference which depend on an vector of structural parameters θ_2 belonging to a set $\Theta_2 \subseteq \mathbb{R}^{n_{\theta_2}}$. If the Ramsey policy maker is a Ramsey planner, her preference matrices $\mathbf{Q}$ also depends on the private sector's structural parameters θ_1 governing the private sectors equation (1). The weights $\mathbf{R}$ on the volatility of the macroeconomic policy instruments may represent the private sector preference, they always belong to the set Θ_2 because they do not show up in equation (1) describing the private sector's micro-level decisions, which excludes the optimal decisions of macroeconomic policy instruments.

With Ramsey (1927) optimal policy under commitment, Lyapunov asymptotic stability of the closed loop private sector's equilibrium (whereas the open loop "laissez-faire" private sector's equilibrium may be a saddlepoint equilibrium) jointly obtained with the determinacy of the initial value of private sector's forward variables (proposition 4) is *an immediate consequence* of $2(n+m)$ boundary conditions determining the policy maker's Lagrangian system with $2(n+m)$ variables (y_t,μ_t), with μ_t the policy maker's Lagrange multipliers.

Table 6: Boundary conditions: stability and determinacy of optimal policy under commitment

Number: $2(n+m) =$	Boundary conditions
$n - n_c$	$\lim\limits_{t\rightarrow+\infty} \beta^t \mathbf{z}_t = 0$, $\mathbf{z}_t$ bounded
$+n_c + m$	$\lim\limits_{t\rightarrow+\infty} \frac{\partial L}{\partial \mathbf{y}_t} = 0 = \lim\limits_{t\rightarrow+\infty} \beta^t \mu_t$, μ_t bounded
$+n$	$\mathbf{k}_0$ and $\mathbf{z}_0$ predetermined
$+m$	$\mathbf{q}_0 = \mathbf{q}_0^* \Leftrightarrow \frac{\partial L}{\partial \mathbf{q}_0} = 0 = \mu_{\mathbf{q},t=0}^*$ predetermined

First, besides the assumption that exogenous variables $\mathbf{z}_t$ are asymptotically stable, transversality conditions at the final period, taken as limits in the infinite horizon, are assumed to seek asymptotic stability of the policy maker's Lagrange multipliers. This provides (n_c+m) constraints which forces stabilization of controllable variables of the decentralized private sector's saddle point equilibrium into a sink. Boundary conditions at the initial date are required for determinacy. First, n boundary conditions are the initial conditions of predetermined variables $\mathbf{k}_0$ and $\mathbf{z}_0$ given (called "essential boundary conditions" (Bryson and Ho (1975)).

Second, m "natural boundary conditions" are such that the policy maker's, as a consistent Stackelberg leader, uses the criteria of optimality to select unique optimal initial values (or "jumps", or "anchors") of private sectors forward variables. Bryson and Ho ((1975), p.55) explains "natural boundary conditions" as follows: "*If* q_t *is not prescribed at* $t = t_0$, *it does not follow that* $\delta q(t_0) = 0$. *In fact, there will be an optimum value for* $q(t_0)$ *and it will be such that* $\delta L = 0$ *for arbitrary small variations of* $q(t_0)$ *around this value. For this to be the case, we choose* $\frac{\partial L}{\partial q(t_0)} = \mu_{q,t_0} = 0$ *(1) which simply says that small changes of the optimal initial value of the forward variables* $q(t_0)$ *on the loss function is zero. We have simply traded one boundary condition:* $q(t_0)$ *given, for another, (1). Boundary conditions such as (1) are sometimes called "natural boundary conditions" or transversality conditions associated with the extremum problem.*" The policy maker's Lagrange multipliers of private sector's forward (Lagrange multipliers) variables are *predetermined at the value zero:* $\mu_{\mathbf{q},t=0} = 0$ in order to determine the unique optimal initial value $\mathbf{q}_0 = \mathbf{q}_0^*$ of private sector's forward variables. Table 7 compares three policy maker's model.

Table 7: Comparing three policy maker's models

PM:	Optimal	Discretion	Non-optimal PM
game	Stackelberg	Nash	against nature
P $\mathbf{z}_{t+1} =$	$\mathbf{A}_{zz}\mathbf{z}_t + \mathbf{\Sigma}_t\varepsilon_{t+1}$	$\mathbf{A}_{zz}\mathbf{z}_t + \mathbf{\Sigma}_t\varepsilon_{t+1}$	$\mathbf{A}_{zz}\mathbf{z}_t + \mathbf{\Sigma}_t\varepsilon_{t+1}$
P $\mathbf{k}_{t+1} =$	$\left(\mathbf{A}_{kk} + \mathbf{B}_y\mathbf{F}_k^*\right)\mathbf{k}_t$ $+\left(\mathbf{A}_{kq} + \mathbf{B}_y\mathbf{F}_q^*\right)\mathbf{q}_t$ $+\left(\mathbf{A}_{kz} + \mathbf{B}_y\mathbf{F}_z^*\right)\mathbf{z}_t$	$\left(\mathbf{A}_{k,D} + \mathbf{B}_{k,D}\mathbf{F}_{k,D}^*\right)\mathbf{k}_t$ $+\left(\mathbf{A}_{kz,D} + \mathbf{B}_{k,D}\mathbf{F}_{z,D}^*\right)\mathbf{z}_t$	$\left(\mathbf{A}_{kk} + \mathbf{B}_y\mathbf{F}_y\right)\mathbf{k}_t$ $+\left(\mathbf{A}_{kq} + \mathbf{B}_y\mathbf{F}_z\right)\mathbf{q}_t$ $+\left(\mathbf{A}_{kz} + \mathbf{B}_y\mathbf{F}_z\right)\mathbf{z}_t$
F $\mathbf{q}_{t+1} =$	$\left(\mathbf{A}_{qk} + \mathbf{B}_y\mathbf{F}_k^*\right)\mathbf{k}_t$ $+\left(\mathbf{A}_{qq} + \mathbf{B}_y\mathbf{F}_q^*\right)\mathbf{q}_t$ $+\left(\mathbf{A}_{qz} + \mathbf{B}_y\mathbf{F}_z^*\right)\mathbf{z}_t$	$\mathbf{N}_{k,D}^*.\mathbf{k}_{t+1}$ $+\mathbf{N}_{z,D}^*\mathbf{z}_{t+1}$	$\mathbf{N}_k\left(\mathbf{F}\right)\mathbf{k}_{t+1}$ $+\mathbf{N}_z\left(\mathbf{F}\right)\mathbf{z}_{t+1}$
L $\mu_{k,t} =$	$\mathbf{P}_{kk}\mathbf{k}_t + \mathbf{P}_{kz}\mathbf{z}_t$ $+\mathbf{P}_{kq}\mathbf{q}_t$	$\mathbf{P}_{kk,D}\mathbf{k}_t + \mathbf{P}_{kz,D}\mathbf{z}_t$.	$\mathbf{0}$
L $\mu_{q,t} =$	$\mathbf{P}_{qk}\mathbf{k}_t + \mathbf{P}_{qz}\mathbf{z}_t$ $+\mathbf{P}_{qq}\mathbf{q}_t$	$\mathbf{0}$	$\mathbf{0}$
R $\mathbf{i}_t =$	$\mathbf{F}_k^*\mathbf{k}_t + \mathbf{F}_z^*\mathbf{z}_t$ $+\mathbf{F}_q^*\mathbf{q}_t$	$\mathbf{F}_{k,D}^*\mathbf{k}_t + \mathbf{F}_{z,D}^*\mathbf{z}_t$.	$\mathbf{F}_k\mathbf{k}_t + \mathbf{F}_k\mathbf{z}_t$ $+\mathbf{F}_q\mathbf{q}_t$
$\mathbf{F}$	$\mathbf{F}^*\left(\theta\right)$	$\mathbf{F}_D^*\left(\theta\right)$	Lucas critique
I $\mathbf{z}_0 =$	given	given	given
I $\mathbf{k}_0 =$	given	given	given
I $\mathbf{q}_0 =$	$-\mathbf{P}_{qq}^{-1}\mathbf{P}_{qk}\mathbf{k}_0$ $-\mathbf{P}_{qq}^{-1}\mathbf{P}_{qz}\mathbf{z}_0$	indeterminacy	$\mathbf{N}_k\left(\mathbf{F}\right)\mathbf{k}_0$ $+\mathbf{N}_z\left(\mathbf{F}\right)\mathbf{z}_0$
P	$n+m$: $\mathbf{z}_t, \mathbf{k}_t, \mu_{q,t}$	n: $\mathbf{z}_t, \mathbf{k}_t$	n: $\mathbf{z}_t, \mathbf{k}_t$
F	$m+n_c$: $\mathbf{q}_t, \mu_{k,t}$	$m+n_c$: $\mathbf{q}_t, \mu_{k,t}$	m: $\mathbf{q}_t$
B. K.	$n+m$ stable	at least n	n stable,
B. K.	$m+n_c$ unstable	at most $m+n_c$	m unstable
total	$n+n_c+2m$	$n+m+n_c$	$n+m$
PS	converging sink	saddlepoint; sink	saddlepoint
PM	saddlepoint	saddlepoint	Lucas critique
restrict	zero	$\mathbf{F}_{q,D} = \mathbf{0}$, $\mathbf{N}_{q,D} = \mathbf{0}$	$\mathbf{N}_{q,N} = \mathbf{0}$
identif. of $\mathbf{F}$	at most: $n+m$	at most: n	at most n: $\mathbf{i}_t =$ $\left(\mathbf{F}_k + \mathbf{F}_q\mathbf{N}_k\right).\mathbf{k}_t$ $+\left(\mathbf{F}_k + \mathbf{F}_q\mathbf{N}_z\right).\mathbf{z}_t$
set $\mathbf{F}_y$	$D_O \subset D_S$	D_D	$D_{NO} \subset D_U$
feed-back	negative stabilizing	indeterminacy	positive destabilizing

B.K. Blanchard Kahn determinacy condition, PS: private sector, PM policy maker, P predetermined, F forward, L Lagrange multiplier, R rule, I initial conditions and initial anchor.

Proposition 4.1: Determinacy and stability.

If Ramsey policy maker's preference satisfy: $\mathbf{Q} \geq \mathbf{0}$ *and* $\mathbf{R} > \mathbf{0}$, *if there is controllability of all forward variables and of* n_c *predetermined variables and* *if there is asymptotic stability of the remaining $n - n_c$ exogenous variables:*

(i) There is determinacy (uniqueness) of policy rule parameters

(ii) There is determinacy of optimal initial values of private sectors forward variables.

(iii) The policy maker's Blanchard and Kahn's (1980) determinacy condition is the number of stable eigenvalues of the policy maker's Hamiltonian system is equal the number of the policy maker's predetermined variables, which is equal to the number of the private sector's predetermined and forward variables.

(iv) The representation of policy maker's optimal rule functions of private sector's variables is a negative feedback which transforms the private sector's saddle point equilibrium into a stable sink, with a unique optimal jump of forward variables.

(v) Rule parameters in optimal policy under commitment always belong a distinct set D_O without intersection with the set D_{NO} of the non-optimal policy maker's determinacy hypothesis.

Proof.

(i) The policy maker's Hamiltonian system is exactly the linear quadratic regulator Hamiltonian system. There is uniqueness of the optimal policy rule parameters if private sector's variables are controllable by the policy maker and if policy maker's preference satisfy: $\mathbf{Q} \geq \mathbf{0}$ and $\mathbf{R} > \mathbf{0}$ (Kalman (1960)).

(ii) If the forward variables are controllable, it is possible to determine their optimal initial value with a first order condition such that the predetermined Lagrange multipliers of the private sector's forward variables are set to zero (Bryson and Ho (1975), p.55-59). Xie (1997) provides a counter-example when a forward variable is not controllable and the initial date transversality condition does not hold.

(iii) The number of private sector's controllable and predetermined variable is n_c and their policy maker's Lagrange multipliers are forward variables n_c. If If all the m private sectors' forward variables are controllable, the related m policy maker's Lagrange multipliers are all *predetermined.* The total number of policy maker's predetermined and controllable variables is equal to n_c $+m$ which is also the number of policy maker's forward variables. The policy maker's Blanchard and Kahn's (1980) determinacy condition is that there are exactly $n_c + m$ stable eigenvalues for the policy maker's Lagrangian system. The policy maker's saddle point equilibrium has a stable manifold of dimension $n_c + m$ in a space of dimension $2(n_c + m)$. The private sector's saddle point equilibrium has a stable manifold of dimension n_c in a space of dimension $n_c + m$. There are *two distinct* Blanchard and Kahn's (1980) determinacy conditions for the closed loop policy maker's Lagrangian system and for the open loop *laissez-faire* private sector's Lagrangian system.

(iv) The representation of optimal policy rule as a non-inertial linear function of the current values of private sectors variables is the standard solution proposed by the LQR problem (Kalman (1960), Sargent and Ljungqvist (2012), chapter 19). Optimal rule parameters do not depend on initial conditions. Optimal rule parameter do not depend on shock nor on initial conditions (Kalman (1960)). Optimal rule parameters are *exactly the same functions of structural parameters* if all variables are predetermined (old-Keynesian stabilization) or if at least one or more controllable variables is forward with unknown initial values (rational expectations)! *This rather obvious interpretation has never been clearly stated for more than thirty years of optimal policy under commitment. The compulsory convention to use alternative representations of optimal rules appears to be designed "as if" this property was unconceivable.*

(v) For given structural parameters θ, the representation of the optimal policy rule as a function of private sector's variables has constant reduced form coefficients $\mathbf{F}$. It is a non-inertial linear simple rule. These rule parameters cannot be in the non-optimal policy maker's determinacy hypothesis set (table 4). It is an immediate consequence of Wonham (1967) pole placement theorem for controllable pair $(\mathbf{A}, \mathbf{B})$. The set of rule parameters corresponding to $n_c + m$ stable closed loop eigenvalues has no intersection with the set of rule parameters corresponding to n_c stable closed loop eigenvalues and m unstable closed loop eigenvalues for a given private sector's system of equations.

Q.E.D.

Proposition 4.2: An infinite number of observationally equivalent representations of reduced form optimal rule

The representation of optimal policy rule as a non-inertial linear function of current values of private sector's variables included in the policy maker's optimal system of equations (which also includes boundary conditions, such as the optimal initial anchor) faces an infinite number of observationally representations of policy rules obtained using linear substitutions from other equations of the optimal system of equations.

Proof:

Considered in isolation, all these policy rules are completely different. Considered within the optimal policy maker's Hamiltonian system of equations *including boundary conditions* $\{P, F, L, R, I\}$ for all dates, they are all equivalent representations found by linear substitution using other equations of the system, for that the new representations of the rule R' belonging to a mathematically and observationally equivalent system of equations $\{P', F', L', R', I'\}$ for all dates:

$$\{P, F, L, R, I\} \text{ for all dates } t \Leftrightarrow \{P', F', L', R', I'\} \text{ for all dates } t \quad (20)$$

The optimal set D_O' of rule parameters $\mathbf{F}'$ of the rule R' may appear very different from the optimal set D_O of rule parameters $\mathbf{F}$, *but this does not change the solution.*

In Levine and Currie (1987), "over-stable" rules R use the representation of optimal policy rule as a function of the current values of the private sector's variables, but *they are assumed* not to satisfy the optimal initial anchor condition $I : \mathbf{q}_0 = -\mathbf{P}_{qq}^{-1}\mathbf{P}_{qk}\mathbf{k}_0 - \mathbf{P}_{qq}^{-1}\mathbf{P}_{qz}\mathbf{z}_0$. Hence, *they can never represent optimal policy under commitment*, because, by assumption (by *petitio principii*), they are never optimally anchored and always lead to "sunspots" equilibria. Levine and Currie (1987) equations surrounding their theorem 5 *are proving* that a representation of optimal rules as a function of current private sector's variables (R) within an optimal system *including the necessary initial optimal boundary conditions* for forward variables (I: $\mathbf{q}_0 = \mathbf{q}_0^*$) represents optimal policy under commitment. From the point of view of the Kalman's (1960) policy maker, as a Stackelberg leader in the policy game, this is the private sector's model which is *under-stable* in the policy maker's Lagrangian system, taking into account the number of its infinite horizon transversality conditions (see additional information in the appendix), not the policy maker's optimal rule which is "stable" and not "over-stable".

There is nothing "unconceivable" with this result which just falls from the boundary conditions of table 6. If one wishes to maintain the private saddlepoint equilibrium, *one should not assume* the policy maker's as a Ramsey (1927) Stackelberg leader in Kalman linear-quadratic framework, because, by Kalman's design of infinite horizon transversality conditions, the open loop saddlepoint equilibrium will always be transformed into a converging sink by an optimal negative feedback rule, within a policy maker's Hamiltonian system saddlepoint equilibrium of larger dimension. Omitting some boundary conditions for ad hoc "over-stable" rules is not the correct answer for "protecting" private sector's saddlepoint equilibrium.

A comparison with time-consistent discretionary policy and "simple" rule non-optimal policy maker's determinacy hypothesis.

Under the non-optimal policy maker's determinacy hypothesis, policy maker's Lagrange multipliers μ_t of controllable variables are *set to zero for all periods.* This reduces the number of predetermined variables to n instead of $n + m$. This excludes m predetermined optimal policy maker's Lagrange multipliers $\mu_{q,t}$ of m private sector's forward variables $\mathbf{q}_t$. This reduces the number of forward variables to m instead of $m + n_c$. This excludes n_c forward optimal policy maker's Lagrange multipliers $\mu_{k,t}$ of n_c private sector's predetermined and controllable variables $\mathbf{k}_t$. Rule parameters always belong to distinct set with optimal determinacy set D_O (for varying policy maker's preferences) of negative feedback rules versus non-optimal determinacy set D_{NO} of positive feedback rules.

Both "simple" rules and optimal policy without commitment faces time inconsistency (Calvo (1978), Miller and Salmon (1985), Currie and Levine (1987)). If the policy maker optimizes again at a future date T without a change of structural parameters, the optimal policy rule parameters $\mathbf{F}$ and the optimal $\mathbf{P}$ providing weight matrix for the initial anchor do not change, because they do not depend on initial conditions. The *time-inconsistent* change of current date anchor is due to the fact that the current values of predetermined variables $\mathbf{k}_T$ and $\mathbf{z}_T$ changed with respect to their values at the initial date $\mathbf{k}_0$ and $\mathbf{z}_0$. The optimal value at date T when optimizing again at date T ($\mathbf{q}_T^{*,t=T}$) is different from the optimal value at date T chosen at date zero ($\mathbf{q}_T^{*,t=0}$):

$$\mathbf{q}_0^{*,t=0} = -\mathbf{P}_{qq}^{-1}\mathbf{P}_{qk}\mathbf{k}_0 - \mathbf{P}_{qq}^{-1}\mathbf{P}_{qz}\mathbf{z}_0\text{: optimal } t=0 \text{ initial anchor} \quad (21)$$

$$\mathbf{q}_T^{*,t=T} = -\mathbf{P}_{qq}^{-1}\mathbf{P}_{qk}\mathbf{k}_T - \mathbf{P}_{qq}^{-1}\mathbf{P}_{qz}\mathbf{z}_T \neq \mathbf{q}_T^{*,t=0}\text{: optimal } t=T \text{ initial anchor} \quad (22)$$

$$\mathbf{q}_T^{*,t=0} = (\mathbf{A}_{qk} - \mathbf{B}_q\mathbf{F}_k)\,\mathbf{k}_{T-1} + (\mathbf{A}_{qq} - \mathbf{B}_q\mathbf{F}_k)\,\mathbf{q}_{T-1} + (\mathbf{A}_{qz} - \mathbf{B}_q\mathbf{F}_z)\,\mathbf{z}_{T-1} \quad (23)$$

In time-consistent discretionary policies (Oudiz and Sachs (1985)), *both* the private sector and the policy maker know that the policy maker is not credible to stick to the promised paths of private sector's forward variables. Exclusion restrictions on rule parameters of the private sectors ($\mathbf{N}_{q,D} = \mathbf{0}$) and of the policy makers ($\mathbf{F}_{q,D} = \mathbf{0}$) related to forward variables define time-consistent rule. Private sector's and policy maker's time-consistent policy rules depend only on predetermined variables $\mathbf{k}_t$ and $\mathbf{z}_t$.

Are excluded from time-consistent policy rule all forward variables (inflation, output gap, asset prices, systemic risk indicators), which of usually Fed's targets. Although time-consistent rules are appealing in theory, this are not for policy maker's because they acknowledge the failure of Fed's credibility to anchor and to drive private sector's expectations. A second major problem highlighted recently is that time-consistent discretionary policy faces indeterminacy (Blake and Kirsanova (2012)). Depending on initial condition, Oudiz and Sachs (1985) algorithm may converge to distinct equilibria.

With non-optimal policy maker's determinacy hypothesis, the private sector's knows that the policy maker's lacks credibility to stick to her management of private sector's forward variables. Hence, the private sector's optimal policy only depends on predetermined variables ($\mathbf{N}_{q,N} = \mathbf{0}$). But the policy maker is myopic. She does not notice that she is not credible for the private sector. She designs a policy rule function of forward variables ($\mathbf{F}_{q,N} \neq \mathbf{0}$) as if she is credible to determine the future path of private sector's expectations. The non-optimal rule without commitment faces time-inconsistency because it depends on forward variables.

Proposition 4.3. Identification with the non-optimal policy maker's determinacy hypothesis (Henry, Levine, Pearlman (2012)).

The number of rule parameters that can be identified is at most equal to the number n of private sector's predetermined endogenous (k_t) and exogenous (z_t) variables

Proof: It follows from the policy maker's myopic belief that she is credible while she is not perceived to be credible by the private sector! Substituting private sector's forward variables by their anchor into the policy maker's rule leads to an observationally equivalent policy which includes a number of non-zero rule parameters at most equal to the number of private sector's predetermined variables (table 5, identification row for non-optimal column). If the number of policy maker's rule parameters exceeds the number of private sector's predetermined variables, identification restrictions are missing. If there is no predetermined variable, zero rule parameter can be identified. This is the case of the new-Keynesian two-equation model without auto-correlated shocks (Cochrane (2011, section V)).

New-Keynesian example: Mavroeidis (2010, online appendix, model 2) checked with rank condition that if the auto-correlation coefficient is equal to zero $\rho_{z,\pi} = 0$, an identification restriction is needed to identify the two policy rule parameters in $\mathbf{F}_q$. Cochrane (2011, section V) found that identification of rule parameters is not possible if $\rho_{z,\pi} = \rho_{z,x} = 0$ ($\mathbf{A}_{zz} = 0$). The new-Keynesian model boils down to a certainty equivalent two-equations model. In this case, there is no longer a predetermined variable causing transitory dynamics that could be used as an anchor of forward variables. For any shock ε_t, forward variables inflation π_t and output gap x_t instantaneously jump on the long run equilibrium $\pi^* = 0 = x^*$, which is surrounded by out-of-equilibrium unstable paths. There is no predicted variation in right hand side variables of the policy rule $i_t = \mathbf{F}_q \mathbf{q}_t$. The policy rule parameters cannot be estimated and are not identified (Cochrane (2011), section V). This case is described by Burmeister (1980): "*Sometimes, the dynamic equilibrium point of models having no state variables is completely unstable; it is then convenient to interpret such dynamic equilibrium points as "degenerate saddle points" having a convergent manifold of dimension zero, i.e. the convergent manifold simply coincides with the dynamic equilibrium point*".

Proposition 4.4. Identification of optimal policy under commitment policy preferences.

If all (m) forward variables are controllable, if the number of controllable predetermined variables is such that $1 \leq n_c \leq n$, if all the remaining $(n - n_c)$ non-controllable predetermined variables have stable eigenvalues and if $n_{\theta_2} - 1$ defines the number of autonomous policy maker preference parameters:

(i) If $(n+m)p \leq n_{\theta_2} - 1$, identifying the optimal policy rule parameters amounts to identify a subset of the autonomous parameters of the policy maker's preference in the loss function.

(ii) If $(n+m)p - n_{\theta_2} - 1 = k > 0$, identifying the optimal policy rule parameters requires to set k identification restrictions on private sector deep parameters (for example, to calibrate their values instead of estimating them) and on covariances between disturbances.

Proof.

The rule parameters are reduced form parameters which are non-linear functions of the structural parameters $\mathbf{F}(\theta_1, \theta_2)$. It is possible to identify and estimate $p(n+m)$ reduced form rule parameters and to restrict (calibrate) the values of the same number of structural parameters. The conditions compare the number of structural parameters of the policy maker's preference with the number of reduced form rule parameters, which may be known after estimation of the policy rules. The number of reduced form parameters in the optimal policy rule $\mathbf{F}$ may be smaller than the number of structural form policy maker's preference parameters in $\mathbf{Q}(\theta_1, \theta_2)$ and $\mathbf{R}(\theta_2)$. The same, observationally equivalent, rule $\mathbf{F}$ can correspond to different matrices $\mathbf{Q}(\theta_1, \theta_2)$ and $\mathbf{R}(\theta_2)$. Identification restrictions on $\mathbf{Q}(\theta_1, \theta_2)$ and $\mathbf{R}(\theta_2)$ may be required. For example, $\mathbf{Q}$ and $\mathbf{R}$ may be constrained to be diagonal. The minimal number of policy rule parameters limits the number of policy maker preference parameters that can be estimated.

Proposition 4.5. Identification of discretionary policy preferences

If $n \geq 1$ is the number of predetermined variables and if $n_{\theta_2} - 1$ defines the number of autonomous policy maker preference parameters:

(i) If $np \leq n_{\theta_2} - 1$, at most n parameters for each of the time-consistent rules related to each of the p policy instruments can identify a subset of the autonomous parameters of the policy maker preference in the loss function.

(ii) If $np - n_{\theta_2} - 1 = k > 0$, identifying the time-consistent optimal policy rule parameters requires to set k restrictions on private sector structural parameters and on the covariances of disturbances.

Proof.

The reduced form n parameters of the p rules are non linear functions of the structural parameters (θ_1, θ_2). The identification conditions compare the number $n_{\theta_2} - 1$ of unknown structural parameters of policy maker's preference θ_2 to the total number np of reduced form parameters of the p rules.

5 Conclusion

A given DSGE model of the private sector faces at least three ways to be solved depending on assumptions regarding the policy maker: optimal policy under commitment, discretionary policy and non-optimal policy maker's determinacy hypothesis. As optimal policy maker's determinacy is the opposite of non-optimal policy maker's determinacy, *we need other criteria than only determinacy* in order to select which hypothesis of policy maker's determinacy is the most useful for normative and positive macroeconomic research. Table 6 lists other criteria besides determinacy, ordered by our personal preferences with respect to their importance. We rank very high the size of the stationary endogenous VAR(1) after optimal policy under commitment, because a major econometric problem of the two other solutions is that forward variables (flows such as consumption, investment, prices, asset prices) are forced to be linear function of very inertial endogenous predetermined stocks (debt, capital) which is obviously not a good fit. This usually justify adding several ad hoc exogenous auto-regressive forcing variables to fit the data. Their auto-regressive component may be removed where all forward and predetermined endogenous variables belong to the economy stationary VAR(1).

Table 8: Criteria for deciding on policy maker's models

Criteria	Commitment, 1978	Discretion 1985	Non-optimal 1998
Determinacy	If controllable	No	If controllable
Identification of rule	At most $n+m$	At most n	At most n
Endogenous VAR(1)	n_c+m	n_c	n_c
Stabilization robustness	Yes	No	No
Optimal normative	Yes	No	No
Lucas critique OK	Yes	Yes	No
Certainty equivalence	Yes	Yes	No
Time consistency	No	Yes, no forward	No
Fed statements OK	Yes	No	No
Computational time	Fast, Ricatti	Indeterminacy	Grid, it depends
Total score	9/10	3/10	1/10
% DSGE 2014	2%, 45 papers	2%, 45 papers	96%, 3410 papers

Despite their superiority concerning these nine criteria, DSGE solved with optimal rules under commitment (e.g. Woodford (2003), Levine, McAdam and Pearlman (2008), Matthes (2015)) represent around 2% of recent DSGE papers with respect to DSGE solved with non-optimal policy maker's determinacy hypothesis. This percentage, perhaps under-estimated, is a rounded ratio of number of answers to the keywords "optimal policy under commitment" and to the keyword "DSGE" in Google scholar data base for 2014: $56/3580 = 1.6\%$.

A drawback of optimal policy under commitment was the lack of an interpretation of policy rules functions of policy maker's Lagrange multipliers, when giving advice to policy makers. With the representation of the policy rule proposed in this paper, this is no longer the case. An additional step is to adjust optimal policy under commitment with optimal control robust to misspecification of the private sector's model (Hansen and Sargent (2007), Walsh (2003), Giordani and Söderlind (2004), Giannoni (2007), Levine and Pearlman (2010)) with this new representation of optimal policy.

We highlight a conflict between *opposite optimal versus non-optimal determinacy* hypothesis. This is not a conflict between simple rule versus optimal rule. A simple rule is defined as a linear function of deviations from equilibrium of a small number of private sector's variables with fixed parameters. During a period with unchanged structural parameters, a reduced form rule of optimal policy under commitment *is* a particular simple rule. Linear quadratic regulator including zero restrictions on rule parameters also sets boundary conditions seeking private sector's stable sink equilibrium with determinacy (Holly and Hughes-Hallett (1989)). Those zero restrictions do *not* necessarily imply the non-optimal policy maker's determinacy hypothesis. An more demanding definition states that a "simple" rule *implies necessarily* the non-optimal policy maker's determinacy hypothesis. Such a demanding statement is *not* Taylor's (1999) definition of a simple rule.

Thousands of new papers can be published solving and estimating already published DSGE models using (robust) optimal policy under commitment instead of the non-optimal policy maker's determinacy hypothesis. *These new publications will suggest opposite stabilization results and opposite policy recommendations with distinct rule parameters than the original papers.* Svensson (2003) argues: "*Monetary policy by the world's most advanced central banks these days is at least as optimizing and forward-looking as the behavior of most rational agents. I find it strange that a large part of the literature on monetary policy still prefers to represent central bank behavior with the help of mechanical instrument rules*" and, we add, with the help of non-optimal policy maker's determinacy hypothesis.

References

[1] Anderson E.W., Hansen L.P., McGrattan E.R. and Sargent T.J. [1996]. Mechanics of Forming and Estimating Dynamic Linear Economies. in

Amman H.M., Kendrick D.A. and Rust J. (editors) *Handbook of Computational Economics*, Elsevier, Amsterdam, 171-252.

[2] Barnett W.A. and Duzhak E.A. [2008]. Non-Robust Dynamic Inferences from Macroeconometric Models: Bifurcation stratification of Confidence Region. *Physica A,* 387, pp. 3817-3825.

[3] Blake A.P. and Kirsanova T. [2012]. Discretionary Policy and Multiple Equilibria in LQ RE models. *Review of Economic Studies,* 79(4), pp. 1309-1339.

[4] Blanchard O.J. and Kahn C. [1980]. The solution of linear difference models under rational expectations. *Econometrica,* 48, pp. 1305-1311.

[5] Burmeister E. [1980]. On some conceptual issues in rational expectations modeling. *Journal of Money, Credit and Banking,* 12(4), part 2: rational expectations, pp. 800-816.

[6] Bryson A.E. and Ho Y.C. [1975]. *Applied Optimal Control*, John Wiley and Sons, New York.

[7] Bilbiie F.O. [2008]. Limited asset market participation, monetary policy and (inverted) aggregate demand logic. *Journal of Economic Theory,* 140(1), 162-196.

[8] Bilbiie F.O. and Straub R. [2013]. Asset Market Participation, Monetary Policy Rules, and the Great Inflation. *Review of Economics and Statistics,* 95(2), 377-392.

[9] Calvo G.A. [1978]. On the Time Consistency of Optimal Policy in a Monetary Economy, *Econometrica*, 46(6), pp. 1411-1428.

[10] Clarida R., Gali J. and Gertler M. [1999]. The science of monetary policy: a new-Keynesian perspective. *Journal of Economic Literature,* 37, pp. 1661-1707.

[11] Clarida R., Gali J., Gertler M. [2000]. Monetary Policy Rules and Macroeconomic Stability. *Quarterly Journal of Economics*, 115, pp. 147-180.

[12] Cochrane J.H. [2011]. Determinacy and identification with Taylor Rules. *Journal of Political Economy,* 119(3), pp. 565-615.

[13] Currie D., Levine P. [1987]. The Design of Feedback Rules in Linear Stochastic Rational Expectations Models. *Journal of Economic Dynamics and Control*, 11(1), pp. 1–28.

[14] Feynman R.P, Leighton R.B., Sands M. [1964]. *The Feynman lectures on Physics: Volume II.* Addison-Wesley, Reading, Mass.

[15] Gali J. [2015]. *Monetary Policy, Inflation and the Business Cycle.* 2nd edition, Princeton University Press, Princeton.

[16] Giannoni M.P. [2007]. Robust Optimal Monetary Policy in a Forward-Looking Model with Parameter and Shock Uncertainty. *Journal of Applied Econometrics*, 22, pp. 179-213.

[17] Giannoni M.P. [2014]. Optimal Interest-Rate Rules and Inflation Stabilization versus Price-Level Stabilization. *Journal of Applied Dynamics and Control*, 41, pp. 110-129.

[18] Giordani P. and Söderlind P. [2004]. Solutions of macromodels with Hansen-Sargent robust policies: some extensions. *Journal of Economics Dynamics and Control*, 28(12), pp. 2367-2397.

[19] Goodfriend M. and King R.G. [2005]. The incredible Volcker Disinflation. *Journal of Monetary Economics*, 52, pp. 981-1015.

[20] Hansen L.P. and Sargent T. [2008]. *Robustness*, Princeton University Press, Princeton.

[21] Hansen L.P. and Sargent T. [2013]. *Recursive Models of Dynamic Linear Economies*, Princeton University Press, Princeton.

[22] Havranek T., Horvath R., Irsova Z. and Ruznak M. [2015]. Cross-country heterogeneity in intertemporal substitution. *Journal of International Economics*, 96(1), pp. 100-118.

[23] Havranek T. [2013]. Publication bias in measuring intertemporal substitution. IES working paper 15.

[24] Henry B., Levine P. and Pearlman J. [2012]. Determinacy and identification with optimal rules. Paper presented at Birkbeck, University of London economics and finance seminar. 17th may 2012.

[25] Holly S. and Hughes-Hallett A. [1989]. *Optimal Control, Expectations and Uncertainty*, Cambridge University Press, Cambridge.

[26] Hurtado S. [2014]. DSGE models and the Lucas critique. *Economic Modelling,* 44(S1), S12-S19.

[27] Kalman R.E. [1960]. Contributions to the Theory of Optimal Control. *Boletin de la Sociedad Matematica Mexicana*, 5, pp.102-109.

[28] Levine P., McAdam P. and Pearlman J. [2008]. Quantifying and sustaining welfare gains from monetary commitment. *Journal of Monetary Economics*, 55(7), 1253-1276.

[29] Levine P., Pearlman J. and Pierse R. [2008]. Linear-quadratic Approximation, external habit and targeting rules. *Journal of Economic Dynamics and Control*, 32(10), pp. 3315–3349.

[30] Levine P., Pearlman J. [2010]. Robust Monetary Rules under Unstructured Model Uncertainty. *Journal of Economic Dynamics and Control*, 34, pp. 456–471.

[31] Ljungqvist L. and Sargent T.J. [2012]. *Recursive Macroeconomic Theory.* 3rd edition. The MIT Press. Cambridge, Massaschussets.

[32] Mavroeidis S. [2010]. Monetary Policy Rules and Macroeconomic Stability: Some New Evidence. *American Economic Review,* 100(1), 491-503.

[33] Mavroeidis S., Plagbord-Moller M. and Stock J. [2014]. Empirical Evidence of Inflation Expectations in the new-Keynesian Phillips Curve. *Journal of Economic Literature*, 52(1), 124–188.

[34] Matthes C. [2015]. Figuring out the FED - Beliefs about policymakers and gains from transparency. *Journal of Money, Credit and Banking*, 47(1), 1-29.

[35] Miller M. and Salmon M. [1985]. Dynamic Games and the Time Inconsistency of Optimal Policy in Open Economies, *Economic Journal*, 95 supplement: conference papers, 124-137.

[36] Oudiz G. and Sachs J.D. [1985], International policy coordination in dynamic macroeconomic models, in Buiter W.H. and Marston R.C. (editors), *International economic policy coordination*, Cambridge U.K., Cambridge University, pp. 275-319.

[37] Ramsey F.P. [1927]. A Contribution to the Theory of Taxation. *Economic Journal*, 37(145), pp. 47-61.

[38] Ramsey F.P. [1928]. A Mathematical Theory of Saving. *Economic Journal*, 38(152), pp. 543-559.

[39] Schmitt-Grohé S. and Uribe M. [2007]. Optimal simple and implementable monetary and fiscal rules. *Journal of Monetary Economics*, pp. 1702-1723.

[40] Söderlind P. [1999]. Solution and Estimation of RE Macromodels with Optimal Policy. *European Economic Review*, 43, 813-23.

[41] Simon H.A. [1956]. Dynamic Programming under Uncertainty with a Quadratic Criterion Function. *Econometrica*, 24(1), 74-81.

[42] Svensson L.E. [2003]. What is wrong with Taylor rules? Using Judgment in Monetary Policy through Targeting Rules. *Journal of Economic Literature.* 41(2), pp.426-477.

[43] Taylor J.B. [1999]. The Robustness and Efficiency of Monetary Policy Rules as Guidelines for Interest Rate Setting by the European Central Bank. *Journal of Monetary Economics.* 43, pp.655-679.

[44] Tinbergen J. (1952). *On the theory of economic policy.* North Holland.

[45] Walsh (2003) Implications of a Changing Economic Structure for the Strategy of Monetary Policy, in *Monetary Policy and Uncertainty: Adapting to a Changing Economy*, Jackson Hole Symposium, Federal Reserve Bank of Kansas City, 2003, 297-348.

[46] Wonham W.N. [1967]. On pole assignment in multi-input controllable linear system. *IEEE transactions on automatic control.* 12(6), pp 660-665.

[47] Woodford M. [2003]. Optimal Interest-Rate Smoothing. *Review of Economic Studies.* 70, pp. 861-886.

[48] Xie D. [1997]. On time consistency: a technical issue in Stackelberg differential games. *Journal of Economic Theory.* 76, pp. 412-430.

6 Appendix 1: New Keynesian model

1.1. Matrix $\mathbf{N}_{NO}$

The matrix $\mathbf{N}_{NO}$ is found seeking two stable eigenvectors using the non-optimal determinacy hypothesis which are necessarily the auto-regressive term of stationary exogenous forcing variables. Then, the lower diagonal block should include two unstable eigenvalues:

$$\begin{pmatrix} z_{x,t+1} \\ z_{\pi,t+1} \\ E_t x_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \begin{pmatrix} \rho_{z,x} & 0 & 0 & 0 \\ 0 & \rho_{z,\pi} & 0 & 0 \\ -1 & \frac{\gamma}{\beta} & 1+\frac{\gamma\kappa}{\beta} & -\frac{\gamma}{\beta} \\ 0 & -\frac{1}{\beta} & -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} z_{x,t} \\ z_{\pi,t} \\ x_t \\ \pi_t \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \gamma \\ 0 \end{pmatrix} i_t + \begin{pmatrix} \sigma_{z,x} & 0 \\ 0 & \sigma_{z,\pi} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{z,x,t+1} \\ \varepsilon_{z,\pi,t+1} \end{pmatrix}, \tag{24}$$

with block notations, in Kalman controllable staircase form:

$$\mathbf{z}_t = \begin{pmatrix} z_{x,t} \\ z_{\pi,t} \end{pmatrix}, \ \mathbf{q}_t = \begin{pmatrix} x_t \\ \pi_t \end{pmatrix}, \ \mathbf{z}_0 \text{ given, } \mathbf{q}_0 \text{ free.} \tag{25}$$

$$\begin{pmatrix} \mathbf{z}_{t+1} \\ E_t \mathbf{q}_{t+1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{zz} & \mathbf{0}_{zq} \\ \mathbf{A}_{qz} & \mathbf{A}_{qq} \end{pmatrix} \begin{pmatrix} \mathbf{z}_t \\ \mathbf{q}_t \end{pmatrix} + \begin{pmatrix} \mathbf{0}_z \\ \mathbf{B}_q \end{pmatrix} \mathbf{i}_t + \begin{pmatrix} \mathbf{\Sigma}_t \\ \mathbf{0}_{qq} \end{pmatrix} \varepsilon_t. \tag{26}$$

Let us define:

$$\mathbf{A}+\mathbf{BF} = \begin{pmatrix} \mathbf{A}_{zz} & \mathbf{0}_{zq} \\ \mathbf{A}_{qz}+\mathbf{B}_q\mathbf{F} & \mathbf{A}_{qq}+\mathbf{B}_q\mathbf{F} \end{pmatrix} = \begin{pmatrix} \rho_{z,x} & 0 & 0 & 0 \\ 0 & \rho_{z,\pi} & 0 & 0 \\ -1 & \frac{1}{\beta}\gamma & \gamma F_x + \frac{\kappa}{\beta}\gamma + 1 & \gamma F_\pi - \frac{1}{\beta}\gamma \\ 0 & -\frac{1}{\beta} & -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{pmatrix}$$

,

The matrix $\mathbf{C}$ ($\mathbf{C}^{-1}=\mathbf{D}$) includes first two columns eigenvectors $\mathbf{v}_{s,t}$ related to stable eigenvalues $\rho_{z,x}$ and $\rho_{z,\pi}$ in diagonal matrix $\mathbf{\Lambda}_s$ and second two column eigenvectors $\mathbf{v}_{u,t}$ related to the two unstable eigenvalues in diagonal matrix $\mathbf{\Lambda}_u$

$$\mathbf{M} = \begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{\Lambda}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_u \end{pmatrix} \begin{pmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{21} & \mathbf{D}_{22} \end{pmatrix} \tag{27}$$

$$\begin{pmatrix} \mathbf{z}_t \\ \mathbf{q}_t \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{v}_{s,t} \\ \mathbf{v}_{u,t} \end{pmatrix}, \ \begin{pmatrix} \mathbf{v}_{s,t} \\ \mathbf{v}_{u,t} \end{pmatrix} = \begin{pmatrix} \mathbf{\Lambda}_s^t & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_u^t \end{pmatrix} \begin{pmatrix} \mathbf{v}_{s,0} \\ \mathbf{v}_{u,0} \end{pmatrix} \tag{28}$$

The Blanchard and Kahn's (1980) stable subspace unique solution is given by:

$$\begin{pmatrix} \mathbf{z}_t \\ \mathbf{q}_t \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{11} \\ \mathbf{C}_{21} \end{pmatrix} \mathbf{\Lambda}_s^t \left(\mathbf{D}_{11}\mathbf{z}_0 + \mathbf{D}_{12}\mathbf{q}_0\right) + \begin{pmatrix} \mathbf{C}_{12} \\ \mathbf{C}_{22} \end{pmatrix} \mathbf{\Lambda}_u^t \left(\mathbf{D}_{21}\mathbf{z}_0 + \mathbf{D}_{22}\mathbf{q}_0\right) \tag{29}$$

$$\begin{pmatrix} \mathbf{z}_t \\ \mathbf{q}_t \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{11} \\ \mathbf{C}_{21} \end{pmatrix} \mathbf{\Lambda}_s^t \left(\mathbf{D}_{11}\mathbf{z}_0 + \mathbf{D}_{12}\mathbf{q}_0\right) \text{ when } \mathbf{q}_0 = \mathbf{P}\mathbf{z}_0 \text{ with } \mathbf{P} = -\mathbf{D}_{22}^{-1}\mathbf{D}_{21} \tag{30}$$

$$\mathbf{z}_t = \mathbf{C}_{11}\mathbf{v}_{s,t},\ \mathbf{q}_t = \mathbf{C}_{21}\mathbf{v}_{s,t} = \mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{z}_t = \mathbf{P}\mathbf{z}_t \text{ with } \mathbf{P} = \mathbf{C}_{21}\mathbf{C}_{11}^{-1} \tag{31}$$

In this case:

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_{11} & \mathbf{O}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix} \text{ with } \mathbf{C}_{11} = \begin{pmatrix} \frac{1}{D(\rho_{z,x})}\left(1-\beta\rho_{z,x}\right) & 0 \\ 0 & \frac{1}{D(\rho_{z,\pi})}\left(\gamma F_x + 1 - \rho_{z,\pi}\right) \end{pmatrix} \text{ and} \tag{32}$$

$$D\left(\rho_{z,i}\right) = \left(1-\kappa\rho_{z,i}\right)\gamma F_x + \kappa\gamma\left(F_\pi - \rho_{z,i}\right) + \left(1-\rho_{z,i}\right)\left(1-\beta\rho_{z,i}\right) \text{ for } i = x,\pi, \text{ so that:} \tag{33}$$

$$D\left(\rho_{z,i}\right) = \beta\rho_{z,x}^2 + \gamma F_x - \beta\rho_{z,x} - \rho_{z,x} + \kappa\gamma F_\pi - \kappa\gamma\rho_{z,x} - \beta\gamma F_x\rho_{z,x} + 1 \tag{34}$$

$$\mathbf{C}_{11} = \begin{pmatrix} \frac{1}{\kappa}D_x & 0 \\ 0 & \frac{1}{\gamma F_x - \rho_{z,\pi}+1}D_\pi \end{pmatrix}$$

$$\mathbf{C}_{12}\mathbf{C}_{11}^{-1} = \begin{pmatrix} -\frac{1}{\kappa}\left(\beta\rho_{z,x}-1\right) & -\frac{\gamma F_\pi - \gamma\rho_{z,\pi}}{\gamma F_x - \rho_{z,\pi}+1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\kappa}D_x & 0 \\ 0 & \frac{1}{\gamma F_x - \rho_{z,\pi}+1}D_\pi \end{pmatrix}^{-1}$$

$$\mathbf{C}_{12}\mathbf{C}_{11}^{-1} = \begin{pmatrix} -\frac{1}{D_x}\left(\beta\rho_{z,x}-1\right) & -\frac{1}{D_\pi}\left(\gamma F_\pi - \gamma\rho_{z,\pi}\right) \\ \frac{\kappa}{D_x} & \frac{1}{D_\pi}\left(\gamma F_x - \rho_{z,\pi}+1\right) \end{pmatrix}$$

An infinitely large value of the output gap rule parameter F_x leads to an immediate jump on the long run equilibrium for the output gap $x_t = 0$ (the parameters anchoring output gap x_t on predetermined variables $\mathbf{z}_t$ tends to zero). An infinitely large value of the output gap rule parameter F_π leads to an immediate jump of inflation on its long run equilibrium $\pi_t = 0$ (the parameters anchoring inflation π_t on predetermined variables $\mathbf{z}_t$ tends to zero).

$$\lim_{F_x\to+\infty}\mathbf{N}_{NO} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{1-\kappa\rho_{z,i}} \end{pmatrix} \text{ and } \lim_{F_\pi\to+\infty}\mathbf{N}_{NO} = \begin{pmatrix} 0 & \frac{1}{\kappa} \\ 0 & 0 \end{pmatrix} \tag{35}$$

Mavroeidis (2010) online appendix checks the rank condition for the identification of the policy rule parameters taking into account the other linear equations of the system including $\mathbf{y}_t = \mathbf{N}_{NO}\mathbf{z}_t$.

1.2. Stability set (triangle) D_S of rule parameters and bifurcations (figure 1)

The two rule parameters are an affine function of the trace T and the determinant D of the closed loop matrix $\mathbf{A}_{yy} + \mathbf{B}_y\mathbf{F}_y$ for control variables.

$$\mathbf{A}_{yy} + \mathbf{B}_y\mathbf{F}_y = \begin{pmatrix} 1 + \frac{\gamma\kappa}{\beta} + \gamma F_x & -\frac{\gamma}{\beta} + \gamma F_\pi \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{pmatrix} \tag{36}$$

$$0 = P(X) = X^2 - T.X + D \text{ with } T = 1 + \frac{1+\gamma\kappa}{\beta} + \gamma F_x \tag{37}$$

$$D = \frac{1}{\beta}\left(\gamma F_x + 1 + \gamma\kappa F_\pi\right) < 1 \tag{38}$$

$$\begin{pmatrix} T \\ D \end{pmatrix} = \begin{pmatrix} 1 + \frac{1+\gamma\kappa}{\beta} \\ \frac{1}{\beta} \end{pmatrix} + \begin{pmatrix} \gamma & 0 \\ \frac{\gamma}{\beta} & \frac{\gamma\kappa}{\beta} \end{pmatrix} \begin{pmatrix} F_x \\ F_\pi \end{pmatrix} \tag{39}$$

$$\Delta \leq 0 \Leftrightarrow T^2 - 4D \leq 0 \Leftrightarrow D \geq \frac{T^2}{4} \tag{40}$$

A stability triangle with bifurcation borders and a quadratic function delimiting complex conjugate versus non-complex solutions (discriminant $\Delta \leq 0$) is described in the plane (T, D). Figure 1 obtains a stability triangle and quadratic function in the plane (F_π, F_x) using the affine transformation between both pair of variables (F_π, F_x). Figure 1 expands Mavroeidis (2010) figure 2 for negative values of the output gap rule parameter F_x. Three inequality conditions delimits the stability triangle, the equality corresponds to a bifurcation to instability. The saddle-node bifurcation inequality condition is $P(1) > 0$ with one limit eigenvalue equal to 1. It corresponds to the Taylor principle border in Mavroeidis (2010). The Hopf bifurcation inequality condition is $D < 1$ computed by Barnett and Duzhak (2008) with a limit pair of complex conjugate solution of absolute value equal to one. We complement the figure by the flip bifurcation inequality $P(-1) > 0$ with one limit eigenvalue equal to -1. The stable set D_S for rule parameters (F_x, F_π) is delimited in a triangle such that the inflation rule parameters is larger than one (it satisfies the Taylor principle) but such that the output gap parameter is strictly negative for the sign restrictions $\gamma > 0$ and $\kappa > 0$:

$$F_x > \frac{\kappa}{1-_\beta} - \frac{\kappa}{1-\beta} F_\pi \text{ for } P(1) > 0\text{: If } F_x < 0 \Rightarrow F_\pi > 1 \tag{41}$$

$$F_x < \frac{\beta - 1}{\gamma} - \kappa F_\pi \text{ for } D < 1\text{: If } F_\pi > 0 \Rightarrow F_x < \frac{\beta - 1}{\gamma} < 0. \tag{42}$$

$$F_x > -\frac{2}{\gamma} - \frac{\kappa}{1+\beta} - \frac{\kappa}{1+\beta} F_\pi \text{ for } P(-1) > 0 \tag{43}$$

As $0 < \beta \approx 0.99 < 1$, (e.g. $\beta \approx 0.99$), the line ($D = 1$) is over the line ($P(-1) = 0$) for $F_\pi > 1$ and $F_x < 0$:

$$\frac{\beta - 1}{\gamma} > -\frac{2}{\gamma} - \frac{\kappa}{1+\beta_\pi} \text{ and } -\kappa > -\frac{\kappa}{1+\beta} \tag{44}$$

Intersection between conditions provides the three vertexes, with Matthes (2015) estimates values: $A\,(T = 2, D = 1) = (F_x = -0.37, F_\pi = 1.01)$, $B\,(T = 0, D = -1) = (-3.6, 1.10)$, $C\,(T = -2, D = 1) = (-6.82, 19.4)$ of the stability triangle, ordered by rising values of F_π and falling values of F_x. The case of one-time-step full stabilization with two identical zero eigenvalues corresponds to "central" point $E\,(T = 0, \Delta = 0 = D) = (-3.59, 5.66)$, on the horizontal line for $F_x = -\frac{1}{\gamma} - \frac{1+\gamma\kappa}{\gamma\beta}$ with the intersection with the parabola $\Delta = 0$ with horizontal axis on figure 1. Complex solutions are given by F_π being a quadratic function of F_x and crossing the triangle at the two intersection points A and C, with a minimal value on point A, so that the full segment $D = 1$ joining vertexes A to C of the triangle corresponds to an Hopf bifurcation.

Over the triangle and on the left of the quasi-vertical line $P(1) = 0$ is an area with two unstable eigenvalue (source), as well as on the right of the quasi-vertical line $P(1) = 0$ after point B and below the rather flat line $P(-1) = 0$ with two negative rule parameters. These two areas corresponds to the non-optimal policy maker's determinacy area. By convention, negative signs for both policy rules are are never considered. The remaining areas outside the triangle and the non-optimal policy maker's determinacy corresponds to the saddle point case, with one stable and one unstable eigenvalue. The origin $F_\pi = F_x = 0$ is the open-loop "laissez-faire" case: it belongs to one of these two areas.

1.3. Optimal policy under commitment set D_O of rule parameters (figure 1)

The upper vertex ($F_x = -1.05, F_\pi = 1.77$) corresponds to maximal inertia with an infinite cost of changing the policy rate: ($Q_{xx} = Q_\pi = 0, R = 1$), with eigenvalues related to open-loop eigenvalues as follows: $\lambda_1 = \lambda_{1,OL} = 0.41$, $\lambda_2 = 0.73 = 1/\lambda_{2,OL} = 1/1.37$. The right vertex ($F_x = -2.88, F_\pi = 5.63$)

corresponds to an infinite weight on inflation, that is a minimal inertia with a negligible cost of changing the policy rate and zero weight on the output gap: ($Q_{xx} = 0$, $Q_{\pi} = 1$, $R = 10^{-7}$), with two conjugate closed loop eigenvalues close to zero $|\lambda_1| = |\lambda_2| = 0.001$. The left vertex ($F_x = -1.27, F_{\pi} = 1.06$) close to the limit line of the Taylor principle for the inflation parameter, corresponds to an infinite weight on the output gap, that is a minimal inertia with a negligible cost of changing the policy rate and zero weight on inflation: ($Q_{xx} = 1$, $Q_{\pi} = 0$, $R = 10^{-7}$), with one closed loop eigenvalue nearly zero $\lambda_1 = 10^{-7}$ and the other one close to one: $\lambda_2 = 0.995$. This is due to the particular structure of the model. The output gap can be stabilized in one-time-step effect of the interest rate independently of inflation. Inflation depends only on two-times-step effects of the interest rate, as explained in the next section. Curves joining the vertexes are drawn using a numerical grid for varying (Q_{xx}, R) $\in [0, +\infty[\times]0, +\infty[$. These curves are quasi-linear. The optimal set for the linear quadratic regulator keeps some distance from the bifurcation limit lines (except in the case where ($Q_{xx} = 1$, $Q_{\pi} = 0$, $R = 10^{-7}$). This is a well known feature of the robustness of the linear quadratic regulator optimal policy. However, when the LQR is coupled with learning using the Kalman filter taking into account additive quadratic Gaussian errors, the linear quadratic Gaussian is not robust to misspecification, with a non-negligible probability of LQR rules leading to destabilization with eigenvalues outside the stability triangle. This is the reason why robust optimal control is a useful next step for these optimal policy (Hansen and Sargent (2007)).

The linear quadratic regulator determinacy set is much smaller than the stability triangle set and keeps some distance from bifurcations borders of the stability triangle set (another feature of robustness). Decreasing values of R always imply a larger (in absolute value) output gap rule parameter F_x. When Q_x is relatively small, decreasing values of R always imply a larger inflation rule parameter F_{π}. For $R = 10^{-7}$, $Q_x = 0$, $Q_{\pi} = 1$ imply two complex conjugate eigenvalues with absolute value 10^{-3} very close to zero (one-time-step stabilization). When Q_x is relatively very large, decreasing values of R imply a smaller inflation rule parameter F_{π}. For $R = 10^{-7}$, $Q_x = 1$, $Q_{\pi} = 0$, this leads to one eigenvalue close to zero and another one close to one. This is related to the structure of the transmission mechanism: the interest rate can stabilize the output gap in one time-step with only an effect of the output gap in two time step on inflation.

1.4. Scilab code

Download the free software Scilab, copy and paste the following code in the central window, and get the results (pay attention the character for transpose should be a straight line: '). The code enters Matthes (2015) estimates.

A discount factor β in the loss function implies to solve equations with matrices $\sqrt{\beta}\mathbf{A}$ and $\sqrt{\beta}\mathbf{B}$ instead of $\mathbf{A}$ and $\mathbf{B}$. Then, the code solves for $\mathbf{P}_y$ and rule parameters $\mathbf{F}_y$ using lqr (linear quadratic regulator) command. It then solves for $\mathbf{P}_z$ using sylv (Sylvester equation) command. Finally it computes $\mathbf{F}_z$ using Hansen and Sargent (2008) formulas (4.5.7). Their formulas (4.5.7) for $\mathbf{F}_y$ and $\mathbf{F}_y$, formula (4.5.10) for $\mathbf{P}_z$ and (4.3.13) for $\mathbf{P}_y$ are:

$$\mathbf{P}_y = \mathbf{Q}_{yy} + \beta\mathbf{A}_{yy}^{'}\mathbf{P}_y\mathbf{A}_{yy} - \beta\mathbf{A}_{yy}^{'}\mathbf{P}_y\mathbf{B}_y\left(\mathbf{R} + \beta\mathbf{B}_y^{'}\mathbf{P}_y\mathbf{B}_y\right)^{-1}\beta\mathbf{B}_y^{'}\mathbf{P}_y\mathbf{A}_{yy} \tag{45}$$

$$\mathbf{F}_y = \left(\mathbf{R} + \beta\mathbf{B}_y^{'}\mathbf{P}_y\mathbf{B}_y\right)^{-1}\beta\mathbf{B}_y^{'}\mathbf{P}_y\mathbf{A}_{yy} \tag{46}$$

$$\mathbf{P}_z = \mathbf{Q}_{yz} + \beta\left(\mathbf{A}_{yy} + \mathbf{B}_y\mathbf{F}_y\right)'\mathbf{P}_y\mathbf{A}_{yz} + \beta\left(\mathbf{A}_{yy} + \mathbf{B}_y\mathbf{F}_y\right)'\mathbf{P}_z\mathbf{A}_{zz} \tag{47}$$

$$\mathbf{F}_z = \left(\mathbf{R} + \beta\mathbf{B}_y^{'}\mathbf{P}_y\mathbf{B}_y\right)^{-1}\beta\mathbf{B}_y^{'}\left(\mathbf{P}_y\mathbf{A}_{yy} + \mathbf{P}_z\mathbf{A}_{zz}\right) \tag{48}$$

Several linear substitutions are included at the end of the code.

```
beta1=0.99; gamma1=1/1.61; kappa=0.7/(1+beta1);
rho1=0.40; rho2=0.57; rho12=0.61;
R=0.11;
Qx=0.07; Qpi=1; Qxpi=0;
Qxrho1=0; Qpirho1=0;
Qxrho2=0; Qpirho2 =0;
A1=[1-(kappa*gamma1/beta1) -gamma1/beta1 ; -kappa/beta1 1/beta1]
;
A=sqrt(beta1)*A1;
B1=[gamma1 ; 0];
B=sqrt(beta1)*B1;
Q=[Qx Qxpi ;Qxpi Qpi ];
Big=sysdiag(Q,R);
[w,wp]=fullrf(Big);C1=wp(:,1:2);D12=wp(:,3:$);
M=syslin('d',A,B,C1,D12);
[Fy,Py]=lqr(M);
A+B*Fy;
AS=(A+B*Fy)';
Fy
spec(A+B*Fy)
Ayz=[-1 gamma1/beta1 ; 0 -1/beta1 ];
Azz=[rho1 rho12; rho12 rho2 ];
Qyz=[Qxrho1 Qpirho1 ; Qxrho2 Qpirho2 ];
BS=-Azz;
CS=Qyz+AS*Py*Ayz;
```

```
Pz=sylv(AS, BS, CS, 'd');
AS*Pz*BS+Pz-CS;
norm (AS*Pz*BS+Pz-CS);
N=-inv(Py)*Pz;
Fz=inv(R+B'*Py*B)*B'*(Py*Ayz + Pz*Azz);
rho1=0.3; rho2=0.6;
sp1=spec(A+B*Fy)
sp1t=sp1'
Spectrum=[sp1t rho1 rho2 ]
F=[Fy Fz ]
N
Fy*N+Fz
Fy+Fz*inv(N)
Py
Pz
Fy*inv(Py)
Fz-Fy*inv(Py)*Pz
```

1.5. Taylor's (1999) model

For $\sigma > 0$, $0 < \rho < 1$, $R > 0$,, $Q_{\pi z} > 0$, using Hansen and Sargent (2008) formulas, one has closed form solutions:

$$P_\pi\left(\underset{-}{\sigma}, \underset{+}{R}\right) = \frac{R}{\sigma} + \frac{\sigma R + \sigma + \sqrt{(2R + \sigma R + \sigma)^2 + 4R}}{2\sigma} \tag{49}$$

$$F_\pi\left(\underset{-}{\sigma}, \underset{-}{R}\right) = \frac{\sigma P_\pi + \sigma^2 P_\pi}{R + \sigma^2 P_\pi} = \frac{1 + \frac{1}{\sigma}}{1 + \frac{1}{\sigma}\left(\frac{R}{\sigma P_\pi}\right)} \tag{50}$$

$$P_{\pi z}\left(\underset{-}{\sigma}, \underset{+}{R}, \underset{+}{\rho}, \underset{+}{Q_{\pi z}}\right) = \frac{Q_{\pi z} + (1 + \sigma - \sigma F_\pi) P_\pi}{1 - (1 + \sigma - \sigma F_\pi)\rho} \tag{51}$$

$$F_z\left(\underset{-}{\sigma}, \underset{-}{R}, \underset{-}{\rho}, \underset{+}{Q_{\pi z}}\right) = \frac{\sigma P_\pi + \rho\sigma P_{\pi z}}{R + \sigma^2 P_\pi} \tag{52}$$

$$\pi_0^*\left(\underset{+}{\sigma}, \underset{+}{R}, \underset{-}{\rho}, \underset{-}{Q_{\pi z}}\right) = -P_\pi^{-1} P_{\pi z} z_0 \tag{53}$$

7 Appendix 2: Optimal policy under commitment (not for publication).

Summary: The controllability of all forward variables and of at least one predetermined variables is required for the determinacy of optimal rules under commitment. The policy maker's Lagrangian includes $n+m$ stable eigenvalues and $n+m$ unstable eigenvalues. The minimal loss value matrix $\mathbf{P}$ is also a Blanchard and Kahn (1980) projection matrix ruling out unstable paths of the Hamiltonian system. The Hamiltonian system has equivalent representations with the policy maker's predetermined variables $(k_t^T,\mu_{q,t}^T)^T$ or with the private sector predetermined and forward variables $(k_t^T,q_t^T)^T$. Optimal policy under commitment *always stabilizes* the private sector saddle point equilibrium $(k_t^T,q_t^T)^T$ into a stable sink. Optimal initial conditions of forward variables $(\mu_{q,t}^T,q_t^T)^T$ are computed.

7.1 Controllability

Predetermined and forward variables: If ${}_t\mathbf{q}_t$ is the agents expectations

at date t of $\mathbf{q}_{t+1}$ defined as follows :

$$ {}_t\mathbf{q}_{t+1} = E_t\left(\mathbf{q}_{t+1} \mid \Omega_t\right). \tag{54} $$

Ω_t is the information set at date t (it includes past and current values of all endogenous variables and may include future values of exogenous variables). According to Blanchard and Kahn (1980), a *predetermined* variable is a function only of variables known at date t so that $\mathbf{k}_{t+1} = {}_t\mathbf{k}_{t+1}$ whatever the realization of the variables in Ω_{t+1}. Predetermined variables are state variables such as the stocks of capital, wealth and debt of entrepreneurs, wage-earners, banks and government and the shadow prices of forward variables in the policy maker optimal program. A *non-predetermined* ("forward" in what follows) variable can be a function of any variable in Ω_{t+1}, so that we can conclude that $\mathbf{q}_{t+1} = {}_t\mathbf{q}_{t+1}$ only if the realization of all variables in Ω_{t+1} are equal to their expectations conditional on Ω_t. Forward variables are flow variables (such as consumption, output gap, investment), price variables (consumer price, asset prices, credit interest rate), and the shadow prices of predetermined variables.

Stability condition modified by the policy maker's discount factor. Because of the discounted quadratic loss function, the stability criterion for eigenvalues of the dynamic system is such that $\left|(\beta\lambda_i^2)^t\right| < |\beta\lambda_i^2| < 1$, so that stable eigenvalues are such that $|\lambda_i| < 1/\sqrt{\beta}$.

Kalman's (1960) controllability. The matrix pair $(\sqrt{\beta}\mathbf{A}_{n_c+m,n_c+m}$ $\sqrt{\beta}\mathbf{B}_{n_c+m,1})$ is controllable if the Kalman (1960) controllability matrix has full rank:

$$ \text{rank}\left(\sqrt{\beta}\mathbf{B} \;\; \beta\mathbf{AB} \;\; \beta^{\frac{3}{2}}\mathbf{A}^2\mathbf{B} \;\; ... \;\; \beta^{\frac{n_c+m}{2}}\mathbf{A}^{n_c+m-1}\mathbf{B}\right) = n_c + m \tag{55} $$

Hint. A system is t-time-steps controllable if from any start state y_0 we can reach any target state y^* at time t. For a linear time-invariant system, we have

$$\mathbf{y}_t = \mathbf{A}^t\mathbf{y}_0 + \mathbf{A}^{t-1}\mathbf{B}\mathbf{i}_0 + \mathbf{A}^{t-2}\mathbf{B}^2\mathbf{i}_1 + ... + \mathbf{A}\mathbf{B}\mathbf{i}_{t-2} + \mathbf{B}\mathbf{i}_{t-1} \tag{56}$$

Hence, the system is t-time-steps controllable if and only if the above linear system of equations in the sequence of policy instruments $\mathbf{i}_0$,$\mathbf{i}_1$,...,$\mathbf{i}_{t-1}$ has a solution for all choices of $\mathbf{y}_0$ and $\mathbf{y}^*$. With n the dimension of the state space, this is the case if and only if:

$$\text{rank } \begin{pmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & ... & \mathbf{A}^{t-1}\mathbf{B} \end{pmatrix} = n. \tag{57}$$

The Cayley-Hamilton theorem states that for all $\mathbf{A}$**,** for all dates $t \geq n$:

$$\exists \mathbf{w} \in \mathbb{R}^n, \, \mathbf{A}^t = \sum_{i=0}^{i=n-1} w_i \mathbf{A}^i$$

Hence, we obtain that the system $(\mathbf{A}\ \mathbf{B})$ is controllable for all times $t \geq n$ if and only if

$$\text{rank } \begin{pmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & ... & \mathbf{A}^{n-1}\mathbf{B} \end{pmatrix} = n \tag{58}$$

Kalman's controllability matrix $(\mathbf{B} \ \ \mathbf{AB} \ \ \mathbf{A}^2\mathbf{B} \ \ ... \ \ \mathbf{A}^{n-1}\mathbf{B})$ is the interaction over n periods of matrix $\mathbf{B}$ with matrix $\mathbf{A}$ describes the transmission mechanism of control policies.

Kalman's (1960) controllability is a precise statement of Tinbergen's (1952) rule. In the scalar case, controllability means $\partial y_{t+1}/\partial i_t = b \neq 0$: a single policy instrument has a non-zero marginal effect on the future value of a single policy target. For n policy targets, Kalman's (1960) controllability defines exact conditions for Tinbergen's rule: n policy instruments *with non-collinear marginal effects* on the n policy targets, so that rank($\mathbf{B}$)$= n$, are required to bring back a system to equilibrium policy targets $\mathbf{y}^*$after a shock $\mathbf{y}_0$ in only *one* period. Kalman's controllability states a generalized Tinbergen's rule. *Only one* policy instrument can bring back a system of n policy targets to equilibrium policy targets $\mathbf{y}^*$after a shock $\mathbf{y}_0$ in n periods. There are n instrument *per period* with non-collinear effects over time on the n policy targets. *Stabilization of multiple policy targets can be achieved by a single policy instrument at a lower pace (it takes more time).* Including more than a single instruments with non-collinear effects over time is then only a matter of reducing the *minimal* time necessary to bring back the system to equilibrium and fastening the convergence to equilibrium. Tinbergen's (1952) intuition to allocate each instrument to a distinct target is not exactly necessary. Kalman's (1960) point is that policy instruments should not have collinear effects over time on the policy targets.

Pole placement theorem (Wonham (1967)). *If the matrix pair* $(\sqrt{\beta}\mathbf{A}_{n_c+m,n_c+m}\ \sqrt{\beta}\mathbf{B}_{n_c+m,1})$ *is controllable, i.e. if the Kalman (1960) controllability matrix has full rank:*

$$\operatorname{rank}\left(\sqrt{\beta}\mathbf{B}\ \ \beta\mathbf{AB}\ \ \beta^{\frac{3}{2}}\mathbf{A}^2\mathbf{B}\ \ ...\ \ \beta^{\frac{n_c+m}{2}}\mathbf{A}^{n_c+m-1}\mathbf{B}\right)=n_c+m \tag{59}$$

the eigenvalues of $\sqrt{\beta}(\mathbf{A}+\mathbf{BF})$ *can be arbitrarily located in the complex plane (complex eigenvalues, however, occur in complex conjugate pairs) by choosing a policy rule matrix* $\mathbf{F}$ *accordingly.*

7.2 Optimal policy under commitment

7.2.1 Optimal program

The loss function is:

$$\max_{\{\mathbf{i}_t,\mathbf{y}_{t+1}\}_{t=0}^{t=+\infty}} -\frac{1}{2}\sum_{t=0}^{+\infty}\beta^t\left(\mathbf{y}_t^T\mathbf{Q}_{yy}(\theta_1,\theta_2)\mathbf{y}_t+2\mathbf{y}_t^T\mathbf{Q}_{yz}(\theta_1,\theta_2)\mathbf{z}_t+\mathbf{z}_t^T\mathbf{Q}_{zz}(\theta_1,\theta_2)\mathbf{z}_t+\mathbf{i}_t^T\mathbf{R}(\theta_2)\mathbf{i}_t\right) \tag{60}$$

subject to:

$$\begin{pmatrix}\mathbf{z}_{t+1}\\ E_t\mathbf{y}_{t+1}\end{pmatrix}=\begin{pmatrix}\mathbf{A}_{zz}(\theta_1) & \mathbf{0}_{zy}\\ \mathbf{A}_{yz}(\theta_1) & \mathbf{A}_{yy}(\theta_1)\end{pmatrix}\begin{pmatrix}\mathbf{z}_t\\ \mathbf{y}_t\end{pmatrix}+\begin{pmatrix}\mathbf{0}_z\\ \mathbf{B}_y(\theta_1)\end{pmatrix}\mathbf{i}_t \tag{61}$$

where $\mathbf{z}_t$ are non-controllable predetermined variables and where $\mathbf{y}_{t+1}$ are controllable forward and predetermined variables. It is also subject to boundary conditions:

Number: $2(n+m)=$	Boundary conditions
$n-n_c$	$\lim_{t\to+\infty}\beta^t\mathbf{z}_t=0$, $\mathbf{z}_t$ bounded
$+n_c+m$	$\lim_{t\to+\infty}\frac{\partial L}{\partial\mathbf{y}_t}=0=\lim_{t\to+\infty}\beta^t\mu_t$, μ_t bounded
$+n$	$\mathbf{k}_0$ and $\mathbf{z}_0$ predetermined
$+m$	$\mathbf{q}_0=\mathbf{q}_0^*\Leftrightarrow\frac{\partial L^*(\mathbf{y}_0)}{\partial\mathbf{q}_0}=0=\mu^*_{\mathbf{q},t=0}$ predetermined

The solution proceeds in three steps.

Step 1 finds the optimal value function matrix $\mathbf{P}_y$ and rule parameters $\mathbf{F}_y$ obtained when setting the exogenous non-controllable variables to zero at all dates ($\mathbf{z}_t=0$). This is the linear quadratic regulator solution.

Step 2 finds the optimal value function matrix $\mathbf{P}_z$ and the optimal rule parameters $\mathbf{F}_z$ of the exogenous non-controllable variables. This is the augmented linear quadratic regulator solution. Those two steps provide rule parameters which do not depend on random additive normal shocks nor on initial conditions (Simon (1956) Kalman (1960)). The rule parameters are thus identical for old-Keynesian model (the number of forward variables is zero) or for new-Keynesian rational expectations model (the number of forward variable is at least one).

Step 3 finds the optimal initial anchor of private sector's forward variables $\mathbf{q}_0^*$ on the current value of predetermined variables $\mathbf{k}_0$ and $\mathbf{z}_0$. Step 3 is specific to rational expectations forward variables with an optimal initial anchor decided by the policy maker. Without commitment, another period optimal anchor $\mathbf{q}_t^*$ changes because on the current value of predetermined variables $\mathbf{k}_t$ and $\mathbf{z}_t$ is different from their initial values $\mathbf{k}_0$ and $\mathbf{z}_0$.

We present a first way to find the unique solution of optimal policy under commitment with Lagrange multipliers and a second way without Lagrange multipliers substituting directly the private sector's recursive dynamics directly in the Bellman's equation loss function. We then demonstrate observationally equivalent optimal solution including optimal initial values of forward variables.

7.2.2 First way to find the unique solution of optimal policy under commitment with Lagrange multipliers

Besides providing options with numerical computations, the main interest of this solution that it allows an interpretation in line with Radon (1928) and Blanchard and Kahn (1980) condition for unique solution of a saddlepoint Lagrangian or Hamiltonian system. The Lagrangian solution involves costate variables (Lagrange multipliers) for the policy maker. The policy maker's Lagrangian includes $n+m$ stable eigenvalues and n_c+m unstable eigenvalues. The matrix $\mathbf{P}$ is interpreted as a projection matrix on the stable subspace of dimension $n+m$ satisfying the infinite horizon transversality conditions. Consistent to the size of the stable subspace and Blanchard and Kahn's (1980) determinacy of the initial conditions of forward variables, The third step implies that the m policy maker's costate variables, Lagrange multiplier of forward variables, are predetermined to zero at the initial date. The n_c policy maker's costate variables of controllable predetermined variables are forward (jump) variables.

Step 1. Lagrangian system

Because of the certainty equivalence principle for determining optimal policy in the linear quadratic regulator, the expectations of random variables

ε are equal to zero and do not show up in the Lagrangian. Hence, the program is identical to solving for optimal policy rule parameters with perfect foresight dynamics such as $E_t\mathbf{y}_{t+1} = \mathbf{y}_{t+1}$. Her Lagrangian includes $n_c + m$ Lagrange multipliers on controllable predetermined and forward variables: $2\beta^{t+1}\mu_{y,t+1}$, $\mu'_{y,t+1} = (\mu_{k,t+1}, \mu_{q,t+1})$ and denote $y'_t = \left(\mathbf{k}_t^T, \mathbf{q}_t^T\right)^T$:

$$\max_{\{\mathbf{i}_t, \mathbf{y}_{t+1}\}_{t=0}^{t=+\infty}} -\mathcal{L} = \sum_{t=0}^{+\infty} \beta^t \left[\begin{array}{c} \frac{1}{2}\left(\mathbf{y}_t'\mathbf{Q}_y\mathbf{y}_t + 2\mathbf{y}_t'\mathbf{Q}_{yz}\mathbf{z}_t + \mathbf{z}_t'\mathbf{Q}_{zz}\mathbf{z}_t + \mathbf{i}_t'\mathbf{R}\mathbf{i}_t\right) \\ +2\beta^{t+1}\mu'_{t+1}\left[\mathbf{A}_{yy}\mathbf{y}_t + \mathbf{A}_{yz}\mathbf{z}_t + \mathbf{B}_y i_t - \mathbf{y}_{t+1}\right] \end{array} \right]$$

Following Hansen and Sargent (2007), the evolution of the forcing sequence $\mathbf{z}_{t+1} = \mathbf{A}_{zz}\mathbf{z}_t$ can be temporarilly set aside of the Lagrangian, as it cannot be controlled by the policy maker's instrument. There is no first order condition with respect to the non-controllable variables $\mathbf{z}_t$ nor Lagrange multipliers.

First order necessary conditions for the maximization of the Lagrangian with respect to the policy instrument and to the state variables are:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{y}_{t+1}} = \mathbf{0} \Rightarrow \beta\mathbf{A}'_{yy}\mu_{t+1} = \mu_t - \mathbf{Q}_{yy}\mathbf{y}_t - \mathbf{Q}_{yz}\mathbf{z}_t \tag{62}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{i}_t} = \mathbf{0} \Rightarrow \beta\mathbf{B}'_y\mu_{t+1} = -\mathbf{R}\mathbf{i}_t \tag{63}$$

We eliminate temporarily the policy instrument substituting it by the Lagrange multipliers (using its marginal condition) in the private sector law of motion:

$$E_t\mathbf{y}_{t+1} = \mathbf{A}_{yz}\mathbf{z}_t + \mathbf{A}_{zz}\mathbf{y}_t + \beta\mathbf{B}_y\mathbf{R}^{-1}\mathbf{B}'_y\mu_{t+1} \tag{64}$$

Adding the marginal conditions on the policy targets ($n_c + m$ Lagrange multipliers dynamics) to the private sector's model leads to this Lagrangian system:

$$\mathbf{L}^a \begin{pmatrix} \mathbf{y}_{t+1} \\ \mu_{t+1} \\ \mathbf{z}_{t+1} \end{pmatrix} = \mathbf{N}^a \begin{pmatrix} \mathbf{y}_t \\ \mu_t \\ \mathbf{z}_t \end{pmatrix} \tag{65}$$

It includes $(n_c + m)$ states and their $(n_c + m)$ costates, the $2(n_c + m)$ dimensions state-costate evolution equations is:

$$\underbrace{\begin{pmatrix} \mathbf{I} & \beta\mathbf{B}(\theta_1)\mathbf{R}(\theta_2)^{-1}\mathbf{B}'(\theta_1) \\ \mathbf{0} & \beta\mathbf{A}'(\theta_1) \end{pmatrix}}_{=\mathbf{L}} \begin{pmatrix} \mathbf{y}_{t+1} \\ \mu_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{A}(\theta_1) & \mathbf{0} \\ -\mathbf{Q}(\theta_1) & \mathbf{I} \end{pmatrix}}_{=\mathbf{N}} \begin{pmatrix} \mathbf{y}_t \\ \mu_t \end{pmatrix} \tag{66}$$

If $\mathbf{L}$ and in particular $\mathbf{A}$ are invertible (non singular), the state-costate evolution equation with discrete time Hamiltonian matrix $\mathbf{M}$ is:

$$\begin{pmatrix} \mathbf{y}_{t+1} \\ \mu_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{A}+\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{A}'^{-1}\mathbf{Q} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{A}'^{-1} \\ -\beta^{-1}\mathbf{A}^{-1}\mathbf{Q} & \beta^{-1}\mathbf{A}'^{-1} \end{pmatrix}}_{=\mathbf{M}=\mathbf{L}^{-1}\mathbf{N}} \begin{pmatrix} \mathbf{y}_t \\ \mu_t \end{pmatrix}$$

The infinite horizon transversality condition setting bounded discounted costates restrict a solution that stabilizes the state-costate vector for any initial values of $\mathbf{y}_0$ and $\mathbf{z}_0$. Hence, we seek a characterization of the multiplier μ_t of the form:

$$\mu_t = \mathbf{P}_y\mathbf{y}_t + \mathbf{P}_z\mathbf{z}_t$$

such that the resulting composite sequence $\left(\mathbf{y}_t^{'}, \mu_t^{'}, \mathbf{z}_t^{'}\right)$ is in the stable subspace of the Lagrangian system. This constraint on the Lagrange multipliers μ_t minimizes the optimal loss function:

$$L_t^* = \mathbf{y}_t\mathbf{P}_y\mathbf{y}_t + 2\mathbf{y}_t\mathbf{P}_z\mathbf{z}_t + \mathbf{z}_t\mathbf{P}_{zz}\mathbf{z}_t \text{ so that } \mu_t = \frac{\partial L_t^*}{\partial \mathbf{y}_t} = \mathbf{P}_y\mathbf{y}_t + \mathbf{P}_z\mathbf{z}_t \quad (67)$$

Step 2. Linear quadratic regulator

It includes (n_c+m) states and their (n_c+m) costates, the $2(n_c+m)$ dimensions state-costate evolution equations is:

$$\underbrace{\begin{pmatrix} \mathbf{I} & \beta\mathbf{B}(\theta_1)\mathbf{R}(\theta_2)^{-1}\mathbf{B}'(\theta_1) \\ \mathbf{0} & \beta\mathbf{A}'(\theta_1) \end{pmatrix}}_{=\mathbf{L}} \begin{pmatrix} \mathbf{y}_{t+1} \\ \mu_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{A}(\theta_1) & \mathbf{0} \\ -\mathbf{Q}(\theta_1) & \mathbf{I} \end{pmatrix}}_{=\mathbf{N}} \begin{pmatrix} \mathbf{y}_t \\ \mu_t \end{pmatrix} \quad (68)$$

If $\mathbf{L}$ and in particular $\mathbf{A}$ are invertible (non singular), the state-costate evolution equation with discrete time Hamiltonian matrix $\mathbf{M}$ is:

$$\begin{pmatrix} \mathbf{y}_{t+1} \\ \mu_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{A}+\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{A}'^{-1}\mathbf{Q} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{A}'^{-1} \\ -\beta^{-1}\mathbf{A}^{-1}\mathbf{Q} & \beta^{-1}\mathbf{A}'^{-1} \end{pmatrix}}_{=\mathbf{M}=\mathbf{L}^{-1}\mathbf{N}} \begin{pmatrix} \mathbf{y}_t \\ \mu_t \end{pmatrix}$$

The eigenvalues of $2(n+m)$ square matrix $\mathbf{M}$ come in reciprocal pairs. A stable eigenvalue is paired with a mirror unstable eigenvalue. For any real eigenvalue λ_i, eigenvalues of $\mathbf{M}$ includes both λ_i and $1/\lambda_i$. For a pair of complex conjugate stable eigenvalues with absolute value $|\lambda_i|$, eigenvalues of $\mathbf{M}$ also includes another pair of complex conjugate unstable values with absolute value $1/\left|\lambda_i\right|$.

A hint for this result is that the matrix $\mathbf{M}$ is a real symplectic matrix:

$$\mathbf{J}=\mathbf{MJM}' \text{ and } \mathbf{M}'=\mathbf{J}^{-1}\mathbf{M}^{-1}\mathbf{J} \text{ with } \mathbf{J}=\begin{pmatrix} \mathbf{0} & -\mathbf{I}_{n+m} \\ \mathbf{I}_{n+m} & \mathbf{0} \end{pmatrix}$$

Similar matrices define the same linear transformation but with different coordinate system. In this case, the change of coordinate is done using an anti-symmetric matrix $\mathbf{J}$. The transpose of $\mathbf{M}$ is similar to its inverse $\mathbf{M}^{-1}$, thus they share the same eigenvalues. For any matrix $\mathbf{M}$, the eigenvalues of $\mathbf{M}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{M}$, and the eigenvalues of $\mathbf{M}'$, the transpose of $\mathbf{M}$, are the same than the eigenvalues of $\mathbf{M}$. Hence, eigenvalues of M come in reciprocal pairs.

If we exclude the case with eigenvalues equal to one, a Jordan reduced form of the matrix $\mathbf{M}$ with the usual spectral saddle point factorization where $\mathbf{\Lambda}_s$ includes the first half $(n+m)$ of the stable eigenvalues, with eigenvectors basis of the stable invariant subspace of $\mathbf{M}$, and $\mathbf{\Lambda}_s^{-1}$ includes the second half $(n+m)$ of the unstable eigenvalues, with absolute values equal to the inverse of the eigenvalues of $\mathbf{\Lambda}_s$. The left eigenvector matrix is denoted C, its inverse (the right eigenvector matrix) is denoted $\mathbf{D}=\mathbf{C}^{-1}$ and canonical variables are denoted **v:**

$$\mathbf{M}=\begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix}\begin{pmatrix} \mathbf{\Lambda}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_s^{-1} \end{pmatrix}\begin{pmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{21} & \mathbf{D}_{22} \end{pmatrix} \tag{69}$$

$$\begin{pmatrix} \mathbf{y}_t \\ \mu_t \end{pmatrix}=\begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix}\begin{pmatrix} \mathbf{v}_{s,t} \\ \mathbf{v}_{u,t} \end{pmatrix},\ \begin{pmatrix} \mathbf{v}_{s,t} \\ \mathbf{v}_{u,t} \end{pmatrix}=\begin{pmatrix} \mathbf{\Lambda}_s^t & \mathbf{0} \\ \mathbf{0} & \left(\mathbf{\Lambda}_s^{-1}\right)^t \end{pmatrix}\begin{pmatrix} \mathbf{v}_{s,0} \\ \mathbf{v}_{u,0} \end{pmatrix} \tag{70}$$

The vector of costate variables is equal to the marginal value of the loss function:

$$L_t^*=\frac{1}{2}\mathbf{y}_t'\mathbf{P}\mathbf{y}_t=\frac{1}{2}\mu_t'\mathbf{y}_t \text{ with } \mu_t=\frac{\partial L^*}{\partial \mathbf{y}_t}=\mathbf{P}\mathbf{y}_t \tag{71}$$

$$\begin{pmatrix} \mu_{k,t} \\ \mu_{q,t} \end{pmatrix}=\begin{pmatrix} \mathbf{P}_{kk} & \mathbf{P}_{kq} \\ \mathbf{P}_{qk} & \mathbf{P}_{qq} \end{pmatrix}\begin{pmatrix} \widehat{k}_t \\ \widehat{q}_t \end{pmatrix}. \tag{72}$$

The minimal loss value matrix P is also a Blanchard and Kahn (1980) projection matrix used to set the initial conditions in order to rule out unstable paths of the discrete time Hamiltonian system. The optimal solution P of the matrix Riccati equation is also a function of stable left eigenvectors

($C_{.1}$) or stable right eigenvectors ($\mathbf{D}_{2.}$) of the Hamiltonian matrix $\mathbf{H}$ according to $\mathbf{P} = -\mathbf{D}_{22}^{-1}\mathbf{D}_{21} = C_{21}C_{11}^{-1}$. The matrix $\mathbf{P}$ can also be interpreted as a Blanchard and Kahn's (1980) determinacy projection matrix ruling out the explosive paths related to the $n+m$ unstable eigenvalue $\mathbf{\Lambda}_u$ of the Stackelberg leader's Lagrangian, because of the transversality conditions in the infinite horizon:

$$\begin{pmatrix} \mathbf{y}_t \\ \mu_t \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{11} \\ \mathbf{C}_{21} \end{pmatrix} \mathbf{\Lambda}_s^t \left(\mathbf{D}_{11}\mathbf{y}_0 + \mathbf{D}_{12}\mu_0\right) + \begin{pmatrix} \mathbf{C}_{12} \\ \mathbf{C}_{22} \end{pmatrix} \mathbf{\Lambda}_u^t \left(\mathbf{D}_{21}\mathbf{y}_0 + \mathbf{D}_{22}\mu_0\right) \tag{73}$$

$$\begin{pmatrix} \mathbf{y}_t \\ \mu_t \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{11} \\ \mathbf{C}_{21} \end{pmatrix} \mathbf{\Lambda}_s^t \left(\mathbf{D}_{11}\mathbf{y}_0 + \mathbf{D}_{12}\mu_0\right) \text{ when } \mu_0 = \mathbf{P}\mathbf{y}_0 \text{ with } \mathbf{P} = -\mathbf{D}_{22}^{-1}\mathbf{D}_{21} \tag{74}$$

$$\mathbf{y}_t = \ \mathbf{C}_{11}\mathbf{v}_{s,t},\ \mu_t = \mathbf{C}_{21}\mathbf{v}_{s,t} = \mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{y}_t = \mathbf{P}\mathbf{y}_t \text{ with } \mathbf{P} = \mathbf{C}_{21}\mathbf{C}_{11}^{-1} \tag{75}$$

One obtains the dynamics of the private sector state and costate variables ($\mathbf{k}_t$,$\mathbf{q}_t$) and the dynamics of the leader's costate multipliers in relation:

$$\begin{aligned} \mathbf{y}_t &= \begin{pmatrix} \mathbf{k}_t \\ \mathbf{q}_t \end{pmatrix} = \mathbf{C}_{11}e^{\mathbf{\Lambda}_s t}\mathbf{C}_{11}^{-1} \begin{pmatrix} \mathbf{k}_0 \\ \mathbf{q}_0 \end{pmatrix} = (\mathbf{A}+\mathbf{BF})^t \begin{pmatrix} \mathbf{k}_0 \\ \mathbf{q}_0 \end{pmatrix} \\ \mu_t &= \begin{pmatrix} \mu_{k,t} \\ \mu_{q,t} \end{pmatrix} = \mathbf{C}_{21}e^{\mathbf{\Lambda}_s t}\mathbf{C}_{21}^{-1} \begin{pmatrix} \mu_{k,0} \\ \mu_{q,0} \end{pmatrix} = \mathbf{P} \begin{pmatrix} \mathbf{k}_t \\ \mathbf{q}_t \end{pmatrix} \end{aligned}$$

Step 3: Augmented linear quadratic regulator.

We then solve for $\mathbf{P}_z$ as a Sylvester equation. The rule parameters are computed $\mathbf{F}_z$ using Hansen and Sargent (2008) formulas (4.5.7):

$$\mathbf{P}_z = \mathbf{Q}_{yz} + \beta\left(\mathbf{A}_{yy} + \mathbf{B}_y\mathbf{F}_y\right)'\mathbf{P}_y\mathbf{A}_{yz} + \beta\left(\mathbf{A}_{yy} + \mathbf{B}_y\mathbf{F}_y\right)'\mathbf{P}_z\mathbf{A}_{zz} \tag{76}$$

$$\mathbf{F}_z = \left(\mathbf{R} + \beta\mathbf{B}_y^{'}\mathbf{P}_y\mathbf{B}_y\right)^{-1}\beta\mathbf{B}_y^{'}\left(\mathbf{P}_y\mathbf{A}_{yy} + \mathbf{P}_z\mathbf{A}_{zz}\right) \tag{77}$$

Step 4: Optimal initial anchor of forward variables on predetermined variables

The marginal value of the optimal loss function is (it is also to Lagrange multipliers $\mu_{q,t}$ in the second solution):

$$L_t^* = \mathbf{y}_t\mathbf{P}_y\mathbf{y}_t + 2\mathbf{y}_t\mathbf{P}_z\mathbf{z}_t + \mathbf{z}_t\mathbf{P}_{zz}\mathbf{z}_t \ \Rightarrow\ \frac{\partial L_t^*}{\partial \mathbf{q}_t} = \mathbf{P}_{y,kq}\mathbf{k}_t + \mathbf{P}_{y,qq}\mathbf{q}_t + \mathbf{P}_z\mathbf{z}_t = \mu_{q,t} \tag{78}$$

Optimal initial anchor of forward variables on predetermined variables is:

$$\frac{\partial L_0^*}{\partial \mathbf{q}_0} = \mathbf{P}_{y,kq}\mathbf{k}_0 + \mathbf{P}_{y,qq}\mathbf{q}_0 + \mathbf{P}_z\mathbf{z}_0 = \mathbf{0} = \mu_{q,0} \Rightarrow \tag{79}$$

$$\mathbf{q}_0 = -\mathbf{P}_{y,qq}^{-1}\mathbf{P}_{y,kq}\mathbf{k}_0 - \mathbf{P}_{y,qq}^{-1}\mathbf{P}_z\mathbf{z}_0 \tag{80}$$

We may also compute the Lagrange multiplier on predetermined variables at the initial date:

$$\mu_{k,t=0} = \mathbf{P}_{kk}\mathbf{k}_0 + \mathbf{P}_{kq}\mathbf{q}_0 = \left(\mathbf{P}_{kk} - \mathbf{P}_{kq}\mathbf{P}_{qq}^{-1}\mathbf{P}_{qk}\right)\mathbf{k}_0$$

7.2.3 Second way to find the unique solution of optimal policy under commitment without Lagrange multipliers (Bellman's equation)

Step 1: Linear quadratic regulator

The optimal value function has the form $L_0^*(\mathbf{y}_0) = -\mathbf{y}_0'\mathbf{P}_y\mathbf{y}_0$ where it is to be demonstrated that $\mathbf{P}$ is solution of a discrete time matrix Ricatti equation. Associated with the optimal program problem assuming $(\mathbf{z}_t = 0)$ is the Bellmann equation:

$$-\mathbf{y}_t'\mathbf{P}\mathbf{y}_t = \max_{\{\mathbf{i}_t,\mathbf{y}_{t+1}\}_{t=0}^{t=+\infty}} -\left(\mathbf{y}_t'\mathbf{Q}\mathbf{y}_t + \mathbf{i}_t'\mathbf{R}\mathbf{i}_t + \beta\mathbf{y}_{t+1}'\mathbf{Q}\mathbf{y}_{t+1}\right)$$
$$\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t + \mathbf{B}\mathbf{i}_t$$

where $\mathbf{y}_{t+1}$ denotes next value of the state. The optimal rule is found using the transition law to eliminate next period state, the Bellman's equation becomes:

$$-\mathbf{y}_t'\mathbf{P}\mathbf{y}_t$$
$$= \max_{\{\mathbf{i}_t,\mathbf{y}_{t+1}\}_{t=0}^{t=+\infty}} -\left[\mathbf{y}_t'\mathbf{Q}\mathbf{y}_t + \mathbf{i}_t'\mathbf{R}\mathbf{i}_t + \beta\left(\mathbf{A}\mathbf{y}_t + \mathbf{B}\mathbf{i}_t\right)'\mathbf{P}\left(\mathbf{A}\mathbf{y}_t + \mathbf{B}\mathbf{i}_t\right)\right]$$

The formula for policy rule parameters of endogenous variables $\mathbf{F}_y$ is given by first order condition of the Bellman's equation:

$$\left(\mathbf{R} + \beta\mathbf{B}'\mathbf{P}\mathbf{B}\right)\mathbf{i}_t = -\beta\mathbf{B}'\mathbf{P}\mathbf{A}\mathbf{y}_t \Rightarrow$$
$$\mathbf{i}_t = -\left(\mathbf{R} + \beta\mathbf{B}'\mathbf{P}\mathbf{B}\right)^{-1}\beta\mathbf{B}'\mathbf{P}\mathbf{A}\mathbf{y} = \mathbf{F}_y\mathbf{y}_t \text{ with}$$
$$\mathbf{F}_y = -\beta\left(\mathbf{R} + \beta\mathbf{B}^T\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A}.$$

One also derives the matrix $\mathbf{P}(\theta_1, \theta_2)$ as a unique stabilizing solution of a discrete time matrix Ricatti equation with below its inverse matrix equation.

$$\mathbf{P}(\theta_1, \theta_2) = \mathbf{Q} + \beta \mathbf{A}^T \mathbf{P} \mathbf{A} - \beta \mathbf{A}^T \mathbf{P} \mathbf{B} \left(\mathbf{R} + \beta \mathbf{B}^T \mathbf{P} \mathbf{B} \right)^{-1} \beta \mathbf{B}^T \mathbf{P} \mathbf{A}. \tag{81}$$

$$\mathbf{P}(\theta_1, \theta_2) = \mathbf{Q} + \beta \mathbf{A}^T \left(\mathbf{P}^{-1} + \beta \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^T \right)^{-1} \beta \mathbf{A} \tag{82}$$

Step 2: Augmented linear quadratic regulator
See second way to solve for the unique solution.
Step 3: Optimal initial anchor of forward variables on predermined variables
The marginal value of the optimal loss function is (it turns to be equal to Lagrange multipliers $\mu_{q,t}$ in the first solution):

$$L_t^* = \mathbf{y}_t \mathbf{P}_y \mathbf{y}_t + 2\mathbf{y}_t \mathbf{P}_z \mathbf{z}_t + \mathbf{z}_t \mathbf{P}_{zz} \mathbf{z}_t \Rightarrow \frac{\partial L_t^*}{\partial \mathbf{q}_t} = \mathbf{P}_{y,kq} \mathbf{k}_t + \mathbf{P}_{y,qq} \mathbf{q}_t + \mathbf{P}_z \mathbf{z}_t = \mu_{q,t} \tag{83}$$

Optimal initial anchor of forward variables on predetermined variables is:

$$\frac{\partial L_0^*}{\partial \mathbf{q}_0} = \mathbf{P}_{y,kq} \mathbf{k}_0 + \mathbf{P}_{y,qq} \mathbf{q}_0 + \mathbf{P}_z \mathbf{z}_0 = \mathbf{0} \Rightarrow \tag{84}$$

$$\mathbf{q}_0 = -\mathbf{P}_{y,qq}^{-1} \mathbf{P}_{y,kq} \mathbf{k}_0 - \mathbf{P}_{y,qq}^{-1} \mathbf{P}_z \mathbf{z}_0 \tag{85}$$

Q.E.D.

7.2.4 Observationally equivalent representation of the unique optimal solution of optimal policy under commitment

The certainty equivalent Hamiltonian dynamics system including its boundary conditions in the stable manifold of dimension $n+m$ has equivalent representations when written using $n+m$ states or costates variables among the total set of $n+n_c+2m$ variables. They are forced to be function of $n+m$ stable eigenvalues $\mathbf{\Lambda}_s$. In particular, one may focus either on the private sector variables $(\mathbf{k}_t^T, \mathbf{q}_t^T)^T$ or on the policy maker's predetermined variables $(\mathbf{k}_t^T, \mu_{q,t}^T)^T$. The Hamiltonian system has equivalent representations with the policy maker's predetermined variables $(k_t^T, \mu_{q,t}^T)^T$ or with the private sector variables $(k_t^T, q_t^T)^T$. Sometimes, textbook solution uses a permutation matrix for solving the Hamiltonian directly in the space $(\mathbf{k}_t^T, \mu_{q,t}^T)^T$. A system of linear equation has identical solutions whatever the order of the equations, and rational expectations macroeconomic models cannot be an exception!

Eigenvalues and the stable invariant subspace do not change when reversing the order of the lines of a matrix. It is possible but *not* necessary to change the order of the Hamiltonian equations or the order of the lines of the Hamiltonian matrix $\mathbf{H}$ to solve for optimal rules under commitment in order to compute directly the dynamics for the pair of predetermined variables ($\mathbf{k}_t$,$\mu_{q,t}$). Let us compare the two representations of the unique solution:

PM:	Optimal R1	Optimal R2
P $\mathbf{z}_{t+1} =$	$\mathbf{A}_{zz}\mathbf{z}_t + \mathbf{\Sigma}_t\varepsilon_{t+1}$	$\mathbf{A}_{zz}\mathbf{z}_t + \mathbf{\Sigma}_t\varepsilon_{t+1}$
P $\mathbf{k}_{t+1} =$	$(\mathbf{A}_{kk} + \mathbf{B}_y\mathbf{F}_k^*)\,\mathbf{k}_t$ $+ \left(\mathbf{A}_{kq} + \mathbf{B}_y\mathbf{F}_q^*\right)\mathbf{q}_t$ $+ (\mathbf{A}_{kz} + \mathbf{B}_y\mathbf{F}_z^*)\,\mathbf{z}_t$	$\left(\mathbf{A}_{2,kk} + \mathbf{B}_{2,y}\mathbf{F}_{2,k}^*\right)\mathbf{k}_t$ $+ \left(\mathbf{A}_{2,kq} + \mathbf{B}_{2,y}\mathbf{F}_{2,\mu_{q,t}}^*\right)\mu_{q,t}$ $+ \left(\mathbf{A}_{2,kz} + \mathbf{B}_{2,y}\mathbf{F}_{2,z}^*\right)\mathbf{z}_t$
F $\mathbf{q}_{t+1} =$	$(\mathbf{A}_{qk} + \mathbf{B}_y\mathbf{F}_k^*)\,\mathbf{k}_t$ $+ \left(\mathbf{A}_{qq} + \mathbf{B}_y\mathbf{F}_q^*\right)\mathbf{q}_t$ $+ (\mathbf{A}_{qz} + \mathbf{B}_y\mathbf{F}_z^*)\,\mathbf{z}_t$	$\left(\mathbf{A}_{2,qk} + \mathbf{B}_{2,y}\mathbf{F}_{2,k}^*\right)\mathbf{k}_t$ $+ \left(\mathbf{A}_{2,qq} + \mathbf{B}_{2,y}\mathbf{F}_{2,q}^*\right)\mu_{q,t}$ $+ \left(\mathbf{A}_{2,qz} + \mathbf{B}_{2,y}\mathbf{F}_{2,z}^*\right)\mathbf{z}_t$
L $\mu_{k,t} =$	$\mathbf{P}_{kk}\mathbf{k}_t + \mathbf{P}_{kz}\mathbf{z}_t$ $+\mathbf{P}_{kq}\mathbf{q}_t$	$\mathbf{P}_{kk}\mathbf{k}_t + \mathbf{P}_{kz}\mathbf{z}_t$ $+\mathbf{P}_{kq}\mathbf{q}_t$
L $\mu_{q,t} =$	$\mathbf{P}_{qk}\mathbf{k}_t + \mathbf{P}_{qz}\mathbf{z}_t$ $+\mathbf{P}_{qq}\mathbf{q}_t$	$\mathbf{P}_{qk}\mathbf{k}_t + \mathbf{P}_{qz}\mathbf{z}_t$ $+\mathbf{P}_{qq}\mathbf{q}_t$
R $\mathbf{i}_t =$	$\mathbf{F}_k^*\mathbf{k}_t + \mathbf{F}_z^*\mathbf{z}_t$ $+\mathbf{F}_q^*\mathbf{q}_t$	$\mathbf{F}_{2,k}^*\mathbf{k}_t + \mathbf{F}_{2,z}^*\mathbf{z}_t$ $+\mathbf{F}_{2,q}^*\mu_{q,t}$
I $\mathbf{z}_0 =$	given	given
I $\mathbf{k}_0 =$	given	given
I $\mathbf{q}_0 =$	$-\mathbf{P}_{qq}^{-1}\mathbf{P}_{qk}\mathbf{k}_0$ $-\mathbf{P}_{qq}^{-1}\mathbf{P}_{qz}\mathbf{z}_0$	$\mu_{q,0} = \mathbf{0}$ $\mathbf{P}_{qq}^{-1}$ exists

The linear relation between private sector's forward variables and policy maker's Lagrange multipliers of private sector's forward variables can only be done within the stable subspace of the policy maker's Hamiltonian system $\mu = \mathbf{P}_y\mathbf{y} + \mathbf{P}_z\mathbf{z}$ because of the infinite horizong transversality condition of the policy maker's program. It is obtained as follows:

$$\begin{pmatrix} \mu_{k,t} \\ \mu_{q,t} \end{pmatrix} = \begin{pmatrix} \mathbf{P}_{y,kk} & \mathbf{P}_{y,kq} \\ \mathbf{P}_{y,qk} & \mathbf{P}_{y,qq} \end{pmatrix} \begin{pmatrix} \mathbf{k}_t \\ \mathbf{q}_t \end{pmatrix} + \mathbf{P}_z \mathbf{z}_t \text{ and } \mathbf{P}_{qq}^{-1} \text{ exists} \Rightarrow$$

$$\mathbf{q}_t = -\mathbf{P}_{y,qq}^{-1}\mathbf{P}_z\mathbf{z}_t - \mathbf{P}_{y,qq}^{-1}\mathbf{P}_{y,qk}\mathbf{k}_t + \mathbf{P}_{y,qq}^{-1}\mu_{q,t} \Leftrightarrow$$

$$\begin{pmatrix} \mathbf{z}_t \\ \mathbf{k}_t \\ \mathbf{q}_t \end{pmatrix} = \mathbf{D} \begin{pmatrix} \mathbf{z}_t \\ \mathbf{k}_t \\ \mu_{q,t} \end{pmatrix} \text{ with } \mathbf{D} = \begin{pmatrix} \mathbf{I}_{zz} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{kk} & \mathbf{0} \\ -\mathbf{P}_{y,qq}^{-1}\mathbf{P}_z & -\mathbf{P}_{y,qq}^{-1}\mathbf{P}_{y,qk} & \mathbf{P}_{y,qq}^{-1} \end{pmatrix} \text{ and}$$

$$\mathbf{D}^{-1} = \begin{pmatrix} \mathbf{I}_{zz} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{kk} & \mathbf{0} \\ \mathbf{P}_z & \mathbf{P}_{y,qk} & \mathbf{P}_{y,qq} \end{pmatrix}$$

The easiest equivalence is for the optimal initial condition of forward variables (I) of forward variables I $\mathbf{q}_0$:

$$\mathbf{q}_0 = -\mathbf{P}_{y,qq}^{-1}\mathbf{P}_z\mathbf{z}_0 - \mathbf{P}_{y,qq}^{-1}\mathbf{P}_{y,qk}\mathbf{k}_0 \Leftrightarrow \mu_{q,t=0} = \mathbf{0}$$

Surprisingly, it is actually the one which is "forgotten" to erroneously "prove" that the representation (1) leads necessarily to sunspots.

The equivalence for recursive dynamics (VAR(1) of minimal size) are:

$$\begin{pmatrix} \mathbf{z}_{t+1} \\ \mathbf{k}_{t+1} \\ \mathbf{q}_{t+1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{zz} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{kz} + \mathbf{B}_y\mathbf{F}_z^* & \mathbf{A}_{kk} + \mathbf{B}_y\mathbf{F}_k^* & \mathbf{A}_{kq} + \mathbf{B}_y\mathbf{F}_q^* \\ \mathbf{A}_{qz} + \mathbf{B}_y\mathbf{F}_z^* & \mathbf{A}_{qk} + \mathbf{B}_y\mathbf{F}_k^* & \mathbf{A}_{qq} + \mathbf{B}_y\mathbf{F}_q^* \end{pmatrix} \begin{pmatrix} \mathbf{z}_t \\ \mathbf{k}_t \\ \mathbf{q}_t \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} \mathbf{z}_{t+1} \\ \mathbf{k}_{t+1} \\ \mu_{q,t+1} \end{pmatrix} = \mathbf{D}^{-1} \begin{pmatrix} \mathbf{A}_{zz} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{kz} + \mathbf{B}_y\mathbf{F}_z^* & \mathbf{A}_{kk} + \mathbf{B}_y\mathbf{F}_k^* & \mathbf{A}_{kq} + \mathbf{B}_y\mathbf{F}_q^* \\ \mathbf{A}_{qz} + \mathbf{B}_y\mathbf{F}_z^* & \mathbf{A}_{qk} + \mathbf{B}_y\mathbf{F}_k^* & \mathbf{A}_{qq} + \mathbf{B}_y\mathbf{F}_q^* \end{pmatrix} \mathbf{D} \begin{pmatrix} \mathbf{z}_t \\ \mathbf{k}_t \\ \mu_{q,t} \end{pmatrix}$$

The equivalence for optimal rule parameters is:

$$\mathbf{i}_t = \begin{pmatrix} \mathbf{F}_z & \mathbf{F}_k & \mathbf{F}_q \end{pmatrix} \begin{pmatrix} \mathbf{z}_t \\ \mathbf{k}_t \\ \mathbf{q}_t \end{pmatrix} = \mathbf{FD} \begin{pmatrix} \mathbf{z}_t \\ \mathbf{k}_t \\ \mu_{q,t} \end{pmatrix} \Leftrightarrow$$

$\mathbf{F}_2$=$\mathbf{FD}$

Considered in isolation, these policy rules are completely different. Considered within the optimal policy maker's Hamiltonian system of equations *including boundary conditions* $\{P, F, L, R, I\}$ for all dates, they are equivalent representations found by linear substitution using other equations of the system, for that the new representations of the rule R' belonging to a mathematically and observationally equivalent system of equations $\{P', F', L', R', I'\}$ for all dates:

$$\{P, F, L, R, I\} \text{ for all dates } t \Leftrightarrow \{P', F', L', R', I'\} \text{ for all dates } t \quad (86)$$

Indeed, these linear substitutions do not change the closed, loop optimal path for the variables $(\mathbf{z}_t, \mathbf{k}_t, \mathbf{q}_t)$ obtained by the full system $\{P, F, L, R, I\}$ for all dates t. For the interpretation of optimal policy rule and for communication to and by policy maker's, representation 2 is completely useless. It is as if it was used to conceal the negative feedback stabilizing properties of optimal policy under commitment for thirty years. Q.E.D.