POVERTY RANKINGS OF OPPORTUNITY PROFILES

VITO PERAGINE°, ERNESTO SAVAGLIO*, AND STEFANO VANNUCCI+

Abstract. We address the problem of ranking distributions of opportunity sets in terms of poverty. In order to accomplish this task, we identify a suitable notion of 'multidimensional poverty line' and characterize axiomatically several poverty rankings of opportunity-sets profiles. Among them, the Head-Count and the Opportunity-Gap poverty rankings, which are the natural counterparts of the most widely used income poverty indices.

JEL classification: D31; D63; I31.

Keywords and phrases: Multidimensional Poverty, Opportunity sets, Head-Count, Opportunity-Gap, Basic Needs.

1. Introduction

The present work proposes and characterizes poverty criteria in a setting where individuals are endowed with a finite set of opportunities. In so doing, it develops a new approach to the problem of measuring poverty in a society. Indeed, the current economic literature on poverty measurement is mainly concerned with the comparison of *univariate* indices of poverty. However, the 'univariate approach' is considered an inadequate basis for comparing individual disparities because people differ in many aspects and the analysis of a *set* of different individual opportunities is crucial to understand and evaluate poverty.

Motivation. Poverty reduction plays a prominent role in political debates in many countries, and methods and techniques to make poverty comparisons are necessary tools in order to design and to evaluate policies aimed at decreasing poverty. Indeed, since the publication of Sen's (1976) pioneering paper on poverty measurement, in the last quarter century a great deal has been written on this subject and several measures of poverty are now available in the literature. However, most of the existing literature on poverty measurement regards income or consumption expenditures as the only relevant explanatory dimension of poverty. This approach now appears as insufficient and incomplete because various issues interact to impact on poverty such as education, health, housing, income, food security and access to the decision making process that goes on in politics. The problem of poverty therefore permeates many dimensions of human life, namely it is essentially a multidimensional phenomenon and the exclusive reliance on just one indicator can hide crucial aspects of economic deprivation.

In that respect, many scholars like Rawls (1971), Sen (1985, 1991), Roemer (1996) have defended in their influential works the necessity to move from an income-based evaluation of social inequities

---

Date: July 19, 2014.

We thank Marc Fleurbaey and Nicolas Gravel for their most useful criticisms and comments. The usual disclaimer applies.

towards the more comprehensive domain of opportunities, agreeing on the fact that two societies with the same distribution of monetary earnings can hardly be considered as equivalent in terms of poverty if in one of them a fraction of the population is denied a number of basic rights and liberties such as the right to vote, freedom of speech, freedom of movement and so on.

We agree on this point and we focus on the specific problem of poverty measurement when the explanatory variable is a *set* of individual opportunities (rather than a scalar, as it is the case with income or consumption). As a consequence, our problem becomes that of ranking different distributions of opportunity sets on the basis of poverty, i.e. a multidimensional evaluation exercise that is by no means straightforward, at least from a theoretical point of view.

Contents. Up to our knowledge, the issue of ranking different distributions of opportunities in terms of the poverty they exhibit has never been addressed before. The aim of the present paper is therefore to fill this gap. In order to address the problem, we keep the approach as general as possible and the notion of "opportunity" is treated in an abstract way: we define an opportunity set as any finite set in some arbitrary space. Opportunities may be thought of as non-welfare characteristics of agents such as basic liberties, political rights, and individual freedoms, or as access to certain welfare enhancing traits. A further interpretation is in terms of functionings à la Sen (such as being educated, being well-nourished, avoiding premature mortality, and so on): in this case the opportunity set corresponds to the capability set of an individual (see Sen (1985)).

A natural approach towards devising a poverty ranking for opportunity-sets distributions is try to extend the familiar notion of "poverty line" and the most well-known income poverty measures into our richer setting. In order to identify the different value systems involved in the use of different poverty criteria we use the axiomatic methodology: we propose a number of properties that a poverty-ranking relation on the possible distributions (profiles) of finite opportunity sets should satisfy, and study their logical implications. We then proceed by following Sen's approach which divides the evaluation of poverty into two steps: (i) the identification step, in which the poor are identified in a given society; (ii) the aggregation step, in which the data about who is poor according to one (or several) variable(s) are brought together into an overall measure in order to obtain an assessment of the poverty in a society.

In the unidimensional context, the identification step is solved by choosing a poverty line that divides the population into two sets: the poor and the non-poor. The identification of the poverty line can follow an absolute or a relative approach. While with an absolute approach the poverty line is defined in an exogenous way and it is the same across distributions, with a relative approach the poverty line in a distribution is a function of the distribution itself (e.g. the poverty line can be fixed at half the median income level in that society).

In the present multidimensional context there are two different choices to be made. The first is the choice of a threshold for each relevant dimension. The second is the aggregation along the different dimensions in order to evaluate the poverty of each individual. As for the first problem, our choice is implicit by the domain we are working with: each dimension is modelled as a binary variable. One individual does have access to a specific opportunity or he does not; there are not levels, either cardinal or ordinal, of access at a given dimension of well-being. Hence we treat all the opportunities as a dichotomous variable. While this domain restriction, if referred to variable such as education, health, income, may be seen as a loss of information, it allows to treat in a unified (and admittedly simplified) framework both variables such as basic liberties, rights, individual freedoms, which are binary by nature, and other variable which would admit richer representation.[1].

The second choice concerns the aggregation of the different dimensions. There are two main approaches in the existing literature: the *union approach*, which declares one person as poor if he is below the threshold at least in a single dimension, and the *intersection approach* which instead regards one person as poor if he is below the threshold in all the relevant dimensions.[2] We propose a solution based on the concept of *essential opportunities*. We identify a set of essential opportunities within the universal set, and then we declare one person as poor if she does not have access to all such essential opportunities. Hence we follow the *union approach to identification*, but we restrict it to the essential alternatives that in such a case must be considered as goods that are not substitute. One possible interpretation of such a multidimensional poverty line is linked to the *essential needs* approach: having access to all essential alternatives in this interpretation means being able to satisfy all basic needs. An alternative interpretation suggests that the essential alternatives could represent certain *basic functionings* (see Sen (1985)) such as, for example, life expectancy, literacy, and so on, or a set of *primary goods* (see Rawls (1971)).[3] Once we have identified the poor, we need to amalgamate information on the deprivation suffered by the poor in order to answer to the question: 'when is one person poorer (richer) than another person in terms of opportunities?' In other words, we need to define a criterion, namely a metric in the space of opportunity sets, to compare individual endowed with different sets of opportunity.[4] In order to answer such a question, it is worth noticing first that the univariate case allows a natural *total* ordering (namely, it is always possible to say that one person is richer (or poorer) than another one, or those two have the same quantity of income), of personal attributes (e.g. income, wealth, consumption expenditure etc.). On the contrary, any opportunity-sets profile is a multidimensional distribution that typically admits only *partial* rankings (namely, one individual could be better than another one in one dimension but worse in others), of individual opportunity sets as natural and non-controversial. So, we propose a criterion such that all the sets above the poverty threshold are each other indifferent (equivalence class); as for the sets below the poverty thresholds, they are ranked by set inclusion. To be precise, our threshold T mimics the poverty line

[1]In an extension of this paper, we consider that case in which all opportunities are treated as ordinal variables.

[2]For an "intermediate" solution, based on a variable minimal number of dimensions of deprivation see Alkire and Foster (2008).

[3]We remark here that the (i) distinction between essential and non essential alternatives plays a crucial role in our axiomatic construction; and that (ii) the selection of the relevant dimensions is important in any empirical analysis of poverty. However, we believe that the issue of selecting the relevant essential alternatives lies substantially beyond the scope of the present paper: we assume that appropriate judgments on this have been made, and we concern ourselves with the remaining theoretical challenges.

[4]There is an extensive literature devoted to the problem of ranking opportunity sets (see on this the excellent survey by Barberà, Bossert and Pattanaik (2004)).

of the unidimensional case, but induces a (natural) *partial order* on the elements of a opportunity-sets profile. Indeed, given such a set T of basic opportunities, we distinguish an indifference class of people that are rich, because their opportunity sets have T as a proper subset, and a class of poors who lack of at least one essential opportunity and who can be compared and therefore ranked each other in terms of set-inclusion, the mildest and less-controversial criterion in the literature on ranking opportunity sets (see Barbera, Bossert and Pattanaik (2004)).

As for the aggregation step, we collect all the information reflecting the various aspects of poverty of opportunities to yield a global measure of poverty and propose a characterization of two fundamental orderings: the *Head-Count* (HC) and the *Opportunity-Gap* (OG) poverty rankings. Such rankings are the natural counterparts of the most widely used income poverty measures, namely the *head count ratio* and *the income poverty gap* in our richer setting of distributions of opportunity sets. Indeed, the head-count ranking is produced by counting the number of population units whose endowments fail to meet the minimum standard T. Although the HC-poverty ranking does not take into account how much severe the poverty could be, if our threshold T may be seen as a 'basic right' to live a life worthy to live, then the Head-Count may be quite acceptable as a measure of the number of people deprived of that right. The same position may be taken if T is interpreted as a set of 'functionings' necessary for "doing something and being someone" (see Sen (1985)).

On the other hand, if we want to evaluate how much severe is poverty, then we use the opportunity-gap ranking that is produced by counting the number of extra-'total amount of opportunities' (or functionings) each population unit should be actually endowed with in order to achieve the minimum standard, and by *summing* them. The opportunity-gap poverty ranking aggregates by summing the information about the individual gaps and therefore it tells us how poor are the poors. This is admittedly a quite crude 'metric' of 'poverty intensity' that relies on the *cardinality* total preorder, which has been widely studied in the literature on rankings of opportunity sets. To be sure, the latter criterion has been also the target of sustained criticism. However, we argue that our version of the OG-poverty ranking may make much sense as a first approximation to a sound assessment of the aggregate 'intensity of poverty', whenever combined with suitable definitions of the opportunity space and the poverty threshold.

In addition to those two basic characterization, we axiomatically characterize two lexicographic orderings based on the HC and OG rankings and a third one based on a linear combination of the head-count and gap criteria.

Relation to the literature. The present paper is linked to two different branches of literature. First, it is related to the literature on the measurement of multidimensional poverty (see, among others, Alkire and Foster (2008), Chakravarty, Mukherjee and Ranade (1998), Bourguignon and Chakravarty (1998, 2002), Tsui (2002)). However, the approach we propose is different and possibly more general with respect to such a literature. Our abstract setting for modeling the different dimensions of individual deprivation relies on a *finite* domain as opposed to the domain considered in the literature on multidimensional poverty indices that is the Cartesian product of (multivariate)

Euclidean spaces. Moreover, it is a well established result that any multivariate distribution, real-valued or otherwise, typically admits only *partial* rankings (e.g. dominance orderings) of the latter as natural and non-controversial. On the contrary, the literature on multidimensional poverty measurement is concerned with synthetic measures of the degree of poverty among individuals and, in so doing, it reduces all variables we want to compare to scalars. Such an information loosing exercise is in fact disputable in a multivariate context and is the opposite in spirit to what we are going to develop here. Indeed, we propose to characterize poverty rankings that rely on individual poverty preorders rather than controversial total orderings and on some suitable minimalist requirements.

On the other hand, our paper is linked to the literature which focuses on the ranking problems for different distributions of opportunity sets. This problem has been first addressed by Kranich (1996) and Ok (1997), who however focused only on inequality rankings. There is now an extensive literature concerned with the measurement of inequality of opportunity: see, for example, Arlegi and Nieto (1999), Bossert, Fleurbaey, and Van de gaer (1999), Herrero (1997), Herrero, Iturbe-Ormaetxe, and Nieto (1998), Kranich (1996, 1997), Ok (1997), Ok and Kranich (1998), and Savaglio and Vannucci (2007). A survey of this literature may be found in Barberà, Bossert and Pattanaik. (2004). On the other hand, the issue of ranking different distributions of opportunities in terms of the poverty they exhibit has never been addressed before. The present paper, as mentioned above, tries to fill this gap.

The paper is organized as follow. The next section introduces the analytical setting and defines formally the basic problem studied in this paper. Section 3 introduces and discusses a first set of axioms and contains the main results of the paper: the characterization of the Head-Count and the Opportunity-Gap poverty rankings. Section 4 provides and discusses an additional set of axioms aimed at characterizing composite rankings based on the HC and OG. Section 5 concludes with a brief discussion of the results and of directions for future research, while an appendix collects all proofs.

2. The framework

We start by identifying a universal non-empty set of opportunities, denoted by X. We assume that each element in X is desirable in some universal sense. Moreover, following the existing literature, we assume that opportunities are *non-rival*, so that a given attribute is potentially available to everyone simultaneously, and *excludable*, so that providing an opportunity to some individuals does not necessarily imply that everyone has this opportunity.[5]

Let $N = \{1, ..., n\}$ denote the finite set of relevant population units and $\mathcal{P}[X]$ the set of all *finite* subsets of X. Elements of $\mathcal{P}[X]$ are referred to as *opportunity sets*, and mappings $\mathbf{Y} =$

[5]Moreover, our attributes are entities that are mutually non-exclusive. Indeed, a non-negligible part of the literature on opportunity sets does rely on an interpretation of opportunities as objects coming jointly for an agent (notably Ok (1997), Ok and Kranich (1998), Savaglio and Vannucci (2007)), while other prominent contributions do admit both a mutually-exclusive and non-mutually-exclusive interpretation of opportunities (see e.g. Kranich (1996), Herrero *et alii* (1998), Alcalde-Unzu and Ballester (2010), Barberà, Bossert, Pattanaik (2004, section 5).

$(Y_1, ..., Y_n) \in \mathcal{P}[X]^N$ as profiles of opportunity sets, or simply *opportunity profiles*. Hence, each individual in a society is endowed with an *opportunity set* and a society is represented by an *opportunity profile*.

As said in the Introduction, we are interested in ranking such attribute profiles in terms of poverty.[6] Then, in order to proceed with our analysis, we first need to *identify* who is poor in our framework. We therefore define a *poverty threshold* (or poverty line) as a set $T \in \mathcal{P}[X]$, which identifies a collection of *essential opportunities*: an individual is said to be poor or, equivalently, to be below the poverty threshold, if her opportunity set does not contain all the essential attributes i.e. all the elements of T.

It should be emphasized here that a distinctive feature of our approach is that *complete* individual poverty rankings (let alone individual poverty indices as e.g. in Bossert, Chakravarty, D'Ambrosio (2009)), are *not* included among the basic data of our model. Rather, we stick to the quite uncontroversial *set-inclusion partial order* of opportunity sets (a weak dominance criterion) as supplemented with an agreed upon multidimensional poverty threshold. As a result, we end up with a *partial poverty preorder of (individual) opportunity sets* and rely on it as a common basis for individual poverty comparisons. The following example should clarify this point.

Example 1. Let $X = \{x_1, x_2, ..., x_k\}$ be the set of all attributes, $N = \{1, 2, 3\}$ the relevant population and $T = \{x_1, x_3\}$. Then, at attribute profile

$$\mathbf{Y} = (Y_1 = \{x_1, x_2, x_4, x_5, x_6\}, Y_2 = \{x_3, x_4\}, Y_3 = \{x_1, x_2, x_3\}),$$

population units 1 and 2 are poor, while 3 is non-poor because $Y_1 \not\supseteq T_1$, $Y_2 \not\supseteq T$ and $Y_3 \supset T$. However, neither $Y_1 \supseteq Y_2$ or $Y_2 \supseteq Y_1$.[7]

The criterion we adopt to compare individuals endowed with different attribute sets is indeed the following: *all the opportunity sets above the poverty threshold are mutually indifferent, while the sets below the poverty thresholds are ranked by set inclusion.* Therefore, the universe of the non–poor is represented by a unique indifference class and the very mild condition of set inclusion is proposed as the reference ranking rule within the poor-subpopulation.

[6]Notice that our general model can also be related to behaviorally oriented notions of opportunity sets by the following interpretation. Let X be a possibly multidimensional space of relevant, observable functionings, N^* a population, $\mathbf{x} \in X^{N^*}$ the profile of achieved functionings within the population under consideration, and $\boldsymbol{\pi} = \{\pi_1, ..., \pi_n\} \in \Pi(N)$ a partition of the population into a finite set $N = \{1, ..., n\}$ of types according to a fixed set of verifiable criteria. Then, the opportunity set of type $i \in N$ at $(\mathbf{x}, \boldsymbol{\pi})$ is $X_i = \{x \in X : \text{there exists } j \in \pi_i \text{ such that } x_j = x\}$.

[7]It is worth noticing that in the paper we treat the essential opportunities as exogenous, hence we follow the so-called *absolute* approach to poverty in the identification step. However, we observe here that our threshold T could also be taken to be contingent on suitable profiles of opportunity sets. In particular, the threshold T may be defined as the *median* of individual opportunity sets of a given distribution (or perhaps more to the point as the *median* of the interval of opportunity sets ranging from the smallest to the *median* opportunity set of the original distribution). Moreover, the threshold T may be regarded as the *median* of a set of proposals advanced by members of a panel of experts. That is so because the set of possible thresholds (i.e. the set of subsets of X) is in fact a distributive lattice with respect to set-inclusion and therefore the *median* of any subset of possible thresholds is well-defined.

Formally, our starting point is a (partial) individual poverty preorder $\succcurlyeq_T^*$ on $\mathcal{P}[X]$ induced by the poverty threshold T and defined as follows: for any $Y, Z \in \mathcal{P}[X]$,

$$Y \succcurlyeq_T^* Z \text{ if and only if } [Z \cap T \supseteq Y \cap T \text{ or } Z \supseteq T].$$

The notation $\mathbf{Y}_{|T}$ will be employed in the rest of this paper to denote attribute profile $(Y_i \cap T)_{i \in N}$.

Remark 1. Clearly, partial preorder $\succcurlyeq_T^*$ has both a bottom indifference class comprising precisely the empty set, and a top indifference class including all supersets of T. Moreover, it is easily checked that -by construction- $\succcurlyeq_T^*$ is *graded* namely all the maximal chains joining an arbitrary pair of opportunity sets have the same *length*: it follows that $\succcurlyeq_T^*$ admits a *rank function* (actually, a *poverty* rank function) namely an integer-valued function $r : \mathcal{P}[X] \to \mathbb{Z}_+$ that 'preserves' $\succcurlyeq_T^*$ and such that $r(Y) = r(Z) + 1$ whenever Y is an *upper cover* (or immediate successor) of Z according to $\succcurlyeq_T^*$.[8] In particular, it is also easy (and left to the reader) to check that the poverty rank of an opportunity set Y according to $\succcurlyeq_T^*$ is precisely the number of opportunities included in the threshold T that are missing in Y, namely it amounts to our Opportunity-Gap ranking as informally described in the Introduction, and formally defined below. Thus the poverty rank function induced by $\succcurlyeq_T^*$ does indeed provide us with a meaningful numerical index of individual poverty: but notice that it is an auxiliary *derivative* notion that can be defined in a natural way on the basis of our general assumptions, not a primitive notion requiring further independent stipulations of its own. □

Finally, we define a *poverty ranking* of opportunity profiles under threshold T as a preorder $\succcurlyeq_T$ on $\mathcal{P}[X]^N$ such that for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$ whenever $Z_i \succcurlyeq_T^* Y_i$ for each $i \in N$.

In the present work, we generalize two of the most widely used income poverty measures, namely the *head count ratio* and the *income poverty gap* in a context of opportunity profiles:

Definition 1. *The head-count (HC) poverty ranking of opportunity profiles under threshold T is the total preorder $\succcurlyeq_T^h$ on $\mathcal{P}[X]^N$ defined as follows: for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$,*

$$\mathbf{Y} \succcurlyeq_T^h \mathbf{Z} \textit{ if and only if } h_T(\mathbf{Y}) \geq h_T(\mathbf{Z}),$$

where for each $\mathbf{W} \in \mathcal{P}[X]^N$, $h_T(\mathbf{W}) = \# H_T(\mathbf{W})$ and $H_T(\mathbf{W}) = \{i \in N : W_i \not\supseteq T\}$.

The head-count poverty ordering ranks two distributions on the basis of the number of individuals that are below the poverty threshold T. Hence, it captures the incidence of poverty. Although such a measure gives useful information on the poverty in a distribution, the head-count does not take into account the depth or the severity of the deprivation suffered by the poor. In order to capture this aspect of the aggregate poverty, we also propose the *opportunity-gap (OG) poverty ranking* which measures the aggregate intensity of poverty.

Definition 2. *The opportunity-gap (OG) poverty ranking of opportunity profiles under threshold T is the total preorder $\succcurlyeq_T^g$ on $\mathcal{P}[X]^N$ defined as follows: for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$,*

$$\mathbf{Y} \succcurlyeq_T^g \mathbf{Z} \textit{ if and only if } g_T(\mathbf{Y}) \geq g_T(\mathbf{Z}),$$

[8] See e.g. Anderson (1987) for further details concerning the rank function of a graded poset.

where for each $\mathbf{W} \in \mathcal{P}[X]^N$*,* $g_T(\mathbf{W}) = \sum_{i \in H_T(\mathbf{W})} \# \{x : x \in T \setminus W_i\}$.

Thus, for each poor individual, the intensity of poverty, the "individual poverty gap", is measured by the number of essential alternatives she does not have access to. That is, for each poor individual i, with opportunity set W_i, the "individual poverty gap" $g_T(W_i)$ is given by the following refined cardinality difference[9] with respect to the threshold set T: $g_T(W_i) = |\#(T) - \#(W_i \cap T)|$.

Notice that the aggregate poverty gap $g_T(\mathbf{W})$ also records the number of population units that are 'poor' with respect to *some* essential opportunity. Therefore, computing $g_T(\mathbf{W})$ amounts to counting the number of poors with respect to *each* dimension or essential opportunity in T, and then adding those numbers across dimensions. In that respect, the aggregate poverty gap may also be regarded as an alternative version of the head-count of poors.

In the following, we propose some desirable properties that a poverty ranking should satisfy.

3. The basic characterizations

The axiomatic structure to be presented below will lead us to a characterization of the foregoing poverty criteria.

3.1. The axioms.

Let us start by introducing some basic properties for a poverty ranking $\succcurlyeq_T$ of $\mathcal{P}[X]^N$:

Anonymity (AN): For any $\mathbf{Y} \in \mathcal{P}[X]^N$ and any permutation π of N such that $\pi\mathbf{Y} = (Y_{\pi(1)}, ..., Y_{\pi(n)})$, we have that $\mathbf{Y} \sim_T \pi\mathbf{Y}$.

Irrelevance of Inessential Attributes (IIA): For any $\mathbf{Y} \in \mathcal{P}[X]^N$, $i \in N$, and $x \in Y_i \setminus T$: $\mathbf{Y} \sim_T (\mathbf{Y}_{-i}, Y_i \setminus \{x\})$.

Irrelevance of Insufficient Additions to the Poor (IIAP): For any $\mathbf{Y} \in \mathcal{P}[X]^N$, $x \in X$ and $i \in N$, if $T \nsubseteq Y_i \cup \{x\}$ then $\mathbf{Y} \sim_T (\mathbf{Y}_{-i}, Y_i \cup \{x\})$.

Dominance at Essential Profiles (DEP): For any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ such that both $Y_i \in \{T, \emptyset\}$ and $Z_i \in \{T, \emptyset\}$ for all $i \in N$:

$$\mathbf{Y} \succ_T \mathbf{Z} \text{ if and only if } \# \{i \in N : Y_i = \emptyset\} > \# \{i \in N : Z_i = \emptyset\}.$$

The first three axioms are *invariance properties*, in the sense that they require our poverty rankings to ignore certain aspects of the opportunity distributions and to focus on others. The first, *Anonymity,* is an axiom that requires a symmetric treatment of individuals, thereby preventing the relevant ranking from taking into account information concerning the identities of individuals. *Irrelevance of Inessential Attributes* says that if the opportunity set of an individual i is reduced by the subtraction of an opportunity which is not essential, then the new profile of opportunity sets exhibits the same degree of poverty as the original profile. This axiom is reminiscent of the *focus axiom*, used in the income poverty paradigm (see also Bourguignon and Chakravarty (2002) for the multidimensional case), which requires invariance with respect to reduction in the incomes of the non-poor; however, instead of distinguishing between the poor and the non-poor, in the

[9] The cardinality difference relation was introduced and axiomatically characterized by Kranich (1996).

current scenario the basic distinction is between essential and non-essential opportunities. *Irrelevance of Insufficient Additions to the Poor* says that if the opportunity set of a poor individual i is augmented by an option while leaving her without some essential opportunities, then the new profile of opportunity sets exhibits the same degree of poverty as the original profile.[10]

While the previous invariance properties are useful in identifying the information that our poverty rankings should use, the last axiom is a *dominance property*, which identifies classes of transformation that have a certain effect on the poverty rankings, thereby restricting the set of poverty criteria. *Dominance at Essential Profiles* indeed considers a particular case in which two 'degenerate' profiles are composed of either empty sets or sets coinciding with the poverty threshold T. In this special case, a profile exhibits more poverty than another one if the number of people endowed with the empty set in the former is higher than the number of individuals endowed with the empty set in the latter.

Our first proposition shows that these axioms are necessary and sufficient conditions for the characterization of the HC-poverty ranking $\succcurlyeq_T^h$:

Proposition 1. *Let $\succcurlyeq_T$ be a poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$. Then $\succcurlyeq_T$ is the HC-ranking $\succcurlyeq_T^h$ if and only if $\succcurlyeq_T$ satisfies* AN, IIA, IIAP *and* DEP. *Moreover, such a characterization is tight.*

We now introduce two further axioms:

Strict Monotonicity with respect to Essential Deletions (SMED): For any $\mathbf{Y} \in \mathcal{P}[X]^N$, $i \in N$, and $x \in Y_i \cap T$: $(\mathbf{Y}_{-i}, Y_i \setminus \{x\}) \succ_T \mathbf{Y}$.

Independence of Balanced Essential Deletions (IBED): For any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, $i \in N$, $y \in T \setminus Y_i$and $z \in T \setminus Z_i$:

$$\mathbf{Y} \succcurlyeq_T \mathbf{Z} \text{ if and only if } (\mathbf{Y}_{-i}, Y_i \cup \{y\}) \succcurlyeq_T (\mathbf{Z}_{-i}, Z_i \cup \{z\}).$$

Strict Monotonicity with respect to Essential Deletions is another *dominance property* which says that if the opportunity set of an individual i is reduced by the subtraction of an essential element, then the new profile exhibits a higher degree of poverty than the original profile. This axiom is a direct translation in our context of the Monotonicity axiom used within the income inequality framework (see Foster (2006)). Once again, the difference relies on the fact that in the current scenario the crucial distinction is between essential and non-essential opportunities rather than between poor and non-poor individuals.

Finally, we propose a standard independence axiom, *Independence of Balanced Essential Deletions,* which concerns deletions of an essential element from the opportunity set of an individual i in two profiles $\mathbf{Y}, \mathbf{Z}$. Such balanced deletions are required to preserve the ranking of the two profiles: hence IBED amounts to imposing equal weights on the essential opportunities, which is

[10]To illustrate the significance of IIAP, consider, for instance, a situation with one essential opportunity, namely the right to have an education and an individual that has no access to it. According to our definition, this person is poor. Therefore, the possible non-essential opportunity to free access to all libraries of her town does not increase her freedom of choice (because she is not able to read), hence her possibility 'to be someone or to do something'.

natural as all of them are *equally* all-important. It is quite obvious that by fixing an arbitrary and more detailed preference structure and making reference to it, one might say that a certain essential opportunity is more valuable than another. But this more specialized preferential structure would not be justified in our most parsimonious, minimalist setting. Why should we consider two essential opportunities as for instance the right to vote and freedom of speech (or, say, access to minimal education and minimal health care) as non-trivially, i.e. asymmetrically ranked, given the strict complementarity between them as embodied in the very definition of threshold T?[11]

The first two and the last two axioms of this section are necessary and sufficient to characterize our opportunity-gap criterion:

Proposition 2. *Let $\succcurlyeq_T$ be a poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$. Then $\succcurlyeq_T$ is the OG-ranking $\succcurlyeq_T^g$ if and only if $\succcurlyeq_T$ satisfies* AN, IIA, SMED *and* IBED. *Moreover, such a characterization is tight.*

Thus, we provide two simple characterizations of the most basic poverty rankings of opportunity profiles.[12] We would like to stress that, to the best of our knowledge, those results have no counterpart in the standard literature on poverty indices of income distributions, though the head-count and poverty-gap are among the most widely used criteria in the theoretical and empirical literature on poverty.

4. Composite rankings

In this section, we propose and characterize two lexicographic orderings based on the HC and OG rankings and a third one based on a linear combination of them.

The first composite criterion, the (HG)- *lexicographic poverty ranking,* combines in a lexicographic order the HC and the OG rankings, with priority given to the HC criterion.

Definition 3. *The (HG)-lexicographic poverty ranking of opportunity profiles under threshold T is the preorder $\succcurlyeq_T^{hg}$ on $\mathcal{P}[X]^N$ defined as follow: for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$,*

$$\mathbf{Y} \succcurlyeq_T^{hg} \mathbf{Z} \textit{ if and only if either } \mathbf{Y} \succ_T^h \mathbf{Z} \textit{ or } (\mathbf{Y} \sim_T^h \mathbf{Z} \textit{ and } g_T(\mathbf{Y}) \geq g_T(\mathbf{Z})).$$

The (GH)-*lexicographic poverty ranking* also combines in a lexicographic order the HC and the OG rankings, but with priority given to the OG criterion.

[11] It is worth emphasizing here that our talk about *complementarity* refers to the following simple point concerning preorder $\succcurlyeq_T^*$: movements along distinct *comparable* indifference classes of that preorder (say, from opportunity set Y to opportunity set Z of a lower poverty rank) require that whatever opportunities in T are included in Y, the opportunity set of higher rank (i.e. the poorer opportunity set) are retained in opportunity set Z. No *substitution* between opportunities of Y and Z can be contemplated because such a substitution would invariably render Y and Z mutually *incomparable* -as opposed to indifferent- with respect to $\succcurlyeq_T^*$. That 'complementarity-talk' of is to be taken in a largely intuitive if meaningful way. We think it could be articulated in a suitably formulated general setting but we do not wish to develop it here.

[12] It can be easly checked that HC-poverty ranking does not satisfy IBED and SMED, while OG-poverty ranking does not satisfy IIAP, but it satisfies DEP.

Definition 4. *The (GH)- lexicographic poverty ranking of opportunity profiles under threshold T is the preorder $\succcurlyeq_T^{gh}$ on $\mathcal{P}[X]^N$ defined as follow: for any $\mathbf{Y},\mathbf{Z}\in\mathcal{P}[X]^N$,*

$$\mathbf{Y}\succcurlyeq_T^{gh}\mathbf{Z} \text{ if and only if either } \mathbf{Y}\succ_T^{g}\mathbf{Z} \text{ or } (\mathbf{Y}\sim_T^{g}\mathbf{Z} \text{ and } h_T(\mathbf{Y})\geq h_T(\mathbf{Z})).$$

Finally, the (HG)-*weighted poverty ranking linearly* combines the HC and the OG criteria.

Definition 5. *A (HG)-weighted poverty ranking of opportunity profiles under threshold T is a preorder $\succcurlyeq_T^{w}$ on $\mathcal{P}[X]^N$ defined as follow: there exist $w=(w_1,\ w_2)\in\mathbb{R}^2_{++}$ such that, for any $\mathbf{Y},\mathbf{Z}\in\mathcal{P}[X]^N$,*

$$\mathbf{Y}\succcurlyeq_T^{w}\mathbf{Z} \text{ if and only if } w_1h_T(\mathbf{Y})+w_2g_T(\mathbf{Y})\geq w_1h_T(\mathbf{Z})+w_2g_T(\mathbf{Z}).$$

We remark that when the poverty ranking must be equal to $\succcurlyeq_T^{hg}$ or to $\succcurlyeq_T^{gh}$, this means that one of those two orderings must have *priority* over the other in the aggregation process, while the ranking $\succcurlyeq_T^{w}$ allows for a trade-off between the number of poor and the overall amount of poverty of an opportunity profile.

4.1. **More axioms.** In order to characterize such composite rankings, we have to introduce some additional axioms:

Qualified Independence of Balanced Essential Deletions (Q-IBED): For any $\mathbf{Y},\mathbf{Z}\in\mathcal{P}[X]^N$, for any $y,z\in X$ and for any $i\in N$, such that $Y_i\subset T$, $Z_i\subset T$, $y\in Y_i\cap T$ and $z\in Z_i\cap T$:

$$\mathbf{Y}\succcurlyeq_T\mathbf{Z} \text{ if and only if } (\mathbf{Y}_{-i},Y_i\setminus\{y\})\succcurlyeq_T(\mathbf{Z}_{-i},Z_i\setminus\{z\}).$$

The *Q-IBED* axiom concerns deletions of an essential element from the opportunity set of a poor individual i in two profiles $\mathbf{Y},\mathbf{Z}$. Such balanced deletions are required to preserve the mutual ranking of the opportunity profiles under consideration. This axiom is clearly implied by the *IBED* axiom introduced before.

Conditional Dominance (CD): Let $\succcurlyeq_T$ be a poverty ranking with threshold T. Suppose there exist a positive integer k and $f_1,...,f_k\in\mathbb{R}^{\mathcal{P}[X]^N}$, such that for all $\mathbf{Y}$,$\mathbf{Z}\in\mathcal{P}[X]^N$,

$$f_i(\mathbf{Y})\geq f_i(\mathbf{Z}),\quad i=1,...,k \text{ entails } \mathbf{Y}\succcurlyeq_T\mathbf{Z}.$$

Then, for all $\mathbf{Y},\mathbf{Z}\in\mathcal{P}[X]^N$,

$$(f_1(\mathbf{Y}),...,f_k(\mathbf{Y}))\neq(f_1(\mathbf{Z}),...,f_k(\mathbf{Z})) \text{ and } f_i(\mathbf{Y})\geq f_i(\mathbf{Z}),\ i=1,...,k \text{ entails } \mathbf{Y}\succ_T\mathbf{Z}.$$

Suppose that a finite list $f_1,...,f_k$ of numerical poverty criteria are available such that any two opportunity profiles having the same list of scores under those poverty criteria must be 'equally poor' i.e. belong to the same equivalence class of the poverty ranking. Then, the *Conditional Dominance* axiom dictates that opportunity profile $\mathbf{Y}$ is strictly poorer than opportunity profile $\mathbf{Z}$ whenever the poverty-score-vector of $\mathbf{Y}$ weakly dominates the poverty-score-vector of $\mathbf{Z}$.[13]

Non-Compensation (NC): Let $\succcurlyeq_T$ be a poverty ranking with threshold T. Suppose there exist a positive integer k and $f_1,...,f_k\in\mathbb{R}^{\mathcal{P}[X]^N}$, such that:

[13]We thank Marc Fleurbaey and Nicolas Gravel for having pointed out a flaw in an earlier version of the CD axiom.

(i) for all $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$: if $f_i(\mathbf{Y}) = f_i(\mathbf{Z})$, $i = 1, ..., k$, then $\mathbf{Y} \sim_T \mathbf{Z}$,

(ii) there exist $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ and $i^* \in \{1, ..., k\}$, such that:

$$f_{i^*}(\mathbf{Y}) > f_{i^*}(\mathbf{Z}) \quad \text{and} \quad f_j(\mathbf{Z}) > f_j(\mathbf{Y}) \quad \text{for any} \quad j \in \{1, ..., k\}, \quad j \neq i^*, \text{ and } \quad \mathbf{Y} \succ_T \mathbf{Z}.$$

Then for all $\mathbf{U}, \mathbf{V} \in \mathcal{P}[X]^N$: $\mathbf{U} \succ_T \mathbf{V}$ whenever $f_{i^*}(\mathbf{U}) > f_{i^*}(\mathbf{V})$.

Again, given a finite list of real-valued poverty criteria $f_1, ..., f_k$ that ensure "poverty equivalence" for any two opportunity profiles with the same poverty-score-vector, condition *NC* prevents trade-offs among distinct criteria if one of them is strong enough to prevail against all the other for at least one pair of opportunity profiles. Indeed, NC embodies a basic feature of lexicographic rankings (see e.g. Fishburn (1975) where a similar condition is introduced in order to characterize lexicographic orderings over products of ordered sets).

The next two axioms propose two different and alternative dominance conditions, based on two basic transformations. Consider a profile $\mathbf{Y}$ and two individuals, i and j, which are just below the poverty threshold: that is, they miss just one essential option, say x and y respectively. Now consider two different transformations of profile $\mathbf{Y}$: (i) transfer of opportunity x from j to i; (ii) deletion of another essential element from the set available to j. By the joint effect of this double transformation we get a new profile $\mathbf{Z}$ where the number of individuals below the poverty thresholds is decreased, since i is not poor at $\mathbf{Z}$ while j is still poor (indeed, poorer than before). However the aggregate number of attributes in T that are missing from Z_i or Z_j is increased. What is then the net effect on our poverty ranking? The answer will depend on the specific weight we give to the number of poor in our society *vis-à-vis* to the aggregate severity of poverty. The two axioms we propose give different and opposite answers: according to the *Local Head-Count-Priority*, poverty decreases; according to *Local Gap-Priority*, poverty increases. Formally,

Local Head-Count Priority (HP): Let $\succcurlyeq_T$ be a poverty ranking with threshold T, such that $\#T \geq 3$. For any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, if [there exist $i, j \in N$ and $x, y \in T$, with $x \neq y$, such that for any $l \neq i, j$, $Y_l = Z_l$, $Y_i = T \setminus \{x\}$, $Y_j = T \setminus \{y\}$, $Z_i = T$, and $Z_j = \emptyset$], then $\mathbf{Y} \succ_T \mathbf{Z}$.

Local Gap-Priority (GP): Let $\succcurlyeq_T$ be a poverty ranking with threshold T, such that $\#T \geq 3$. For any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, if [there exist $i, j \in N$ and $x, y \in T$, with $x \neq y$, such that for any $l \neq i, j$, $Y_l = Z_l$, $Y_i = T \setminus \{x\}$, $Y_j = T \setminus \{y\}$, $Z_i = T$, and $Z_j = \emptyset$, then $\mathbf{Z} \succ_T \mathbf{Y}$.

The foregoing pair of axioms are key conditions to characterize the lexicographic combinations of the head-count and opportunity-gap criteria.

The last axiom is a quite standard and technical axiom from social choice theory, that is generally used for the characterization of *utilitarian* social welfare functions.

Cardinal Unit-Comparability (CUC): Let $\succcurlyeq_T$ be a poverty ranking with threshold T. Suppose there exist a positive integer k and $f_1, ..., f_k \in \mathbb{R}^{\mathcal{P}[X]^N}$, such that for all $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$: if $f_i(\mathbf{Y}) = f_i(\mathbf{Z})$, $i = 1, ..., k$ entails $\mathbf{Y} \sim_T \mathbf{Z}$. Now, posit

$$\Phi = \left\{ \begin{array}{c} \varphi = (\varphi_1, ..., \varphi_k) : \varphi_i \in \mathbb{R}^{\mathbb{R}}, i = 1, ..., k \text{ such that there exist} \\ \alpha > 0, \beta_i \in \mathbb{R} \text{ with } \varphi_i(x) = \alpha x + \beta_i \text{ for any } x \in \mathbb{R} \end{array} \right\}.$$

Then, for all $\mathbf{Y}, \mathbf{Z}, \mathbf{V}, \mathbf{U} \in \mathcal{P}[X]^N$, $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$,

$$\begin{aligned}(f_1(\mathbf{U}), ..., f_k(\mathbf{U})) &= ((\varphi_1 \circ f_1)(\mathbf{Y}), ..., (\varphi_k \circ f_k)(\mathbf{Y})) \text{ and} \\ (f_1(\mathbf{V}), ..., f_k(\mathbf{V})) &= ((\varphi_1 \circ f_1)(\mathbf{Z}), ..., (\varphi_k \circ f_k)(\mathbf{Z})) \text{ with} \\ \varphi &= (\varphi_1, ..., \varphi_k) \in \Phi\end{aligned}$$

entail $\mathbf{U} \succcurlyeq_T \mathbf{V}$.

CUC induces an information environment where the admissible transformations of the finite list of real-valued poverty criteria $f_1, ..., f_k$ that ensure "poverty equivalence" for any two opportunity profiles with the same poverty-score-vector are increasing affine functions and, in addition, the scaling unit must be the same for all individuals. This assumption allows for interpersonal comparisons of differences in the values of the relevant criteria. However, parameter levels cannot be compared interpersonally because the intercepts of the affine transformations may differ arbitrarily across individuals.

4.2. **More results.** The next characterizations rely on a simple Lemma showing that if a ranking of opportunity profiles satisfies AN, IIA, DEP and Q-$IBED$ then two opportunity profiles must be indifferent whenever they exhibit the same number of poor and the same aggregate poverty gap. In other words, this Lemma shows that any such a ranking is specified by just two criteria or parameters (namely h_T and g_T).

Lemma 1. *Let $\succcurlyeq_T$ be a poverty ranking on $\mathcal{P}[X]^N$ and a total preorder which satisfies AN, IIA, DEP and Q-$IBED$. Then, for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, $(h_T(\mathbf{Y}), g_T(\mathbf{Y})) = (h_T(\mathbf{Z}), g_T(\mathbf{Z}))$ entails $\mathbf{Y} \sim_T \mathbf{Z}$.*

We are now able to characterize our composite rankings. The first proposition characterizes the (HG)- *lexicographic poverty ranking* $\succcurlyeq_T^{hg}$ which uses the opportunity-gap criterion as a refinement of the head-count criterion.

Proposition 3. *Let $\succcurlyeq_T$ be a poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$, such that $\#T \geq 3$, and a total preorder. Then, $\succcurlyeq_T = \succcurlyeq_T^{hg}$ if and only if $\succcurlyeq_T$ satisfies* AN, IIA, DEP, Q-IBED, CD, NC, *and* HP. *Moreover, such a characterization is tight.*

The next proposition characterizes the (GH)- *lexicographic poverty ranking* $\succcurlyeq_T^{gh}$ which employs the head-count criterion in order to refine the opportunity-gap criterion.

Proposition 4. *Let $\succcurlyeq_T$ be a poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$ such that $\#T \geq 3$, and a total preorder. Then, $\succcurlyeq_T = \succcurlyeq_T^{gh}$ if and only if $\succcurlyeq_T$ satisfies* AN, IIA, DEP, Q-IBED, CD, NC, *and* GP. *Moreover, such a characterization is tight.*

Our final proposition characterizes the class of (HG)- *weighted poverty rankings.*

Proposition 5. *Let $\succcurlyeq_T$ be a poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$ and a total preorder, and suppose $n > \#T \geq 2$. Then, $\succcurlyeq_T = \succcurlyeq_T^w$ for some $w = (w_1, w_2) \in (\mathbb{R} \setminus \{0\})^2$ if and only if $\succcurlyeq_T$ satisfies* AN, IIA, DEP, Q-IBED, CD *and* CUC. *Moreover, such a characterization is tight.*[14]

Thus, our characterizations of $\succcurlyeq_T^{hg}$, $\succcurlyeq_T^{gh}$, and $\succcurlyeq_T^w$ rely on common core properties. Within the class of rankings that satisfies those properties, the NC axiom identifies the lexicographic combinations of the head-count and opportunity-gap criteria, while CUC axiom is strong enough to characterize the class of rankings induced by their linear combination. The rankings characterized in Propositions 3, 4 and 5 are complete and they might therefore provide useful tools for the analysis and assessment of poverty whenever a policy maker is focussing on a poverty reduction strategy.

5. Final remarks

(1) It is worth noticing here that another entirely isomorphic interpretation of our model is actually available. Indeed, an individual opportunity sets may be equivalently regarded as a binary (or Boolean) vector of a multidimensional binary (or Boolean) achievement space. Hence, an opportunity profile can be represented by a $\{0,1\}^{n\times k}$-dimensional matrix, where n is the population size and k is the number of the considered opportunities (namely $\#X = k$) that an individual can (hence, $\{1\}$) or cannot (then, $\{0\}$) achieve. Now, since the range of characteristic functions, namely the support $\{0,1\}^{n\times k}$ of the aforementioned structure, can be regarded as the subset of the (larger set of matrices defined on the) $n \times k$-dimensional Euclidean space adopted in the most relevant works in the literature on multidimensional poverty (see among others Tsui (2002)), it might be argued that our model is definitively a *special case* of the latter.

Thus, it is worth stressing here that our Boolean setting is *not* a substructure of that Euclidean space because $\{0,1\}^{n\times k}$ is not closed with respect to all of the relevant Euclidean vector-space operations and therefore it cannot be properly embedded in $\mathbb{R}^{n\times k}$ as a linear space.

As a further distinguishing characteristic of our model, we must emphasize the fact that it departs from the very rich setting of the rest of literature on multidimensional poverty adopting a more parsimonious and possibly less controversial approach. Indeed, it can be argued that only certain *partial* rankings (e.g. dominance rankings) of the many aspects of personal deprivations can be safely assumed to be natural and non-controversial. In other words, distinct individuals may be poor (and non-poor, respectively)) in different dimensions to the effect of making highly ambiguous and disputable any attempt to rank them in terms of poverty. That quite elementary consideration prompts us to start from a (threshold induced) partial preorder of individual poverty. On the contrary to repeat, the literature on multidimensional poverty measurement is typically concerned with synthetic measures of the degree of poverty of each individual. The latter are evaluated by functions inducing a total preorder of the personal endowments, but that move amounts to reducing to a scalar all variables pertaining to individual poverty. A reason why we

[14]It is worth noticing that the structure of our proof of Proposition 5 replicates to a large extent the style of proof of Theorem 4.4 in Dutta and Sen (1996) as subsequently amended by Alcalde-Unzu and Ballester (2005).

choose to rely on a single poverty (threshold induced) *partial* preorder of individual achievements, rather than on controversial if implicit total preorders of individual poverty.

(2) The need for complementing the traditional evaluation of income poverty by a full-fledged analysis of the deprivation suffered in many dimensions of individual and social life has been forcefully defended by many scholars in the last decades. Such an extension of the scope of poverty measurement may substantially improve our understanding of poverty in any given population and may well have far-reaching policy implications. To keep the analysis as general as possible, in this paper the different dimensions have been treated in an abstract way: we have defined an opportunity set as any finite subset in some arbitrary opportunity space and we have attempted to outline an axiomatic theory for the measurement of poverty in such a multidimensional framework. To the best of our knowledge, there have been no previous attempts to characterize poverty rankings of opportunity profiles within that setting.

We have characterized two fundamental rankings, the head-count and the opportunity-gap poverty rankings, which generalize to a multidimensional environment two of the best known and most widely used poverty indices, namely the *head count ratio* and *the income poverty gap*. In addition, we have axiomatically characterized two lexicographic rankings based on the HC and AG rankings and a third one based on a linear combination of them.

We are of course aware of the critique of the head-count and poverty-gap measures, as formulated by Sen within the income poverty framework, and based on their inability to take into account the inequality among the poor. That critique has led to the characterization of richer families of income poverty indices (see Clark, Hemming and Ulph (1981) and Foster, Greer and Thorbecke (1984)). It would be interesting to study such an extension in our setting.

Moreover, we have only considered comparisons of opportunity profiles for a fixed population. A possible extension of our analysis would be to compare the attribute profiles with different numbers of individuals. This would make it possible to rank opportunity profiles for different countries, different demographic groups, and for different time periods.

Finally, the recent availability of individual data on different dimensions of poverty makes it possible an empirical application based on the rankings characterized in this paper. All these topics will be the object of future research.

6. Appendix: Proofs

Proof of Proposition 1. It is straightforward to check that $\succcurlyeq_T^h$ is a poverty ranking and does indeed satisfy AN, IIA, DEP and IIAP.

Conversely, suppose $\succcurlyeq_T$ is a poverty ranking that satisfies AN, IIAP, IIA, and DEP. Now, consider $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ such that $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$. Then, by repeated application of IIA and transitivity, $\mathbf{Y}_{|T} \succcurlyeq_T \mathbf{Z}_{|T}$. Next, observe that $(T^{N\setminus H_T(\mathbf{Y})}, \emptyset^{H_T(\mathbf{Y})}) \sim_T \mathbf{Y}_{|T} \succcurlyeq_T \mathbf{Z}_{|T} \sim_T (T^{N\setminus H_T(\mathbf{Z})}, \emptyset^{H_T(\mathbf{Z})})$, by repeated application of IIAP. Let us now suppose that $h_T(\mathbf{Z}) > h_T(\mathbf{Y})$: then, by AN and DEP,

$\mathbf{Z} \succ_T \mathbf{Y}$, a contradiction. Hence, $h_T(\mathbf{Y}) \geq h_T(\mathbf{Z})$, i.e. $\mathbf{Y} \succcurlyeq^h_T \mathbf{Z}$. To prove the reverse inclusion, suppose that $\mathbf{Y} \succcurlyeq^h_T \mathbf{Z}$, i.e. $h_T(\mathbf{Y}) \geq h_T(\mathbf{Z})$. Then, consider $(T^{N\setminus H_T(\mathbf{Y})}, \emptyset^{H_T(\mathbf{Y})})$, $(T^{N\setminus H_T(\mathbf{Z})}, \emptyset^{H_T(\mathbf{Z})})$ and a permutation π of N such that $\pi(H_T(\mathbf{Z})) \subseteq \pi(H_T(\mathbf{Y}))$.

By IIA, $\mathbf{Y} \sim_T (T^{N\setminus H_T(\mathbf{Y})}, \emptyset^{H_T(\mathbf{Y})})$ and $\mathbf{Z} \sim_T (T^{N\setminus H_T(\mathbf{Z})}, \emptyset^{H_T(\mathbf{Z})})$. By AN, $(T^{N\setminus H_T(\mathbf{Y})}, \emptyset^{H_T(\mathbf{Y})}) \sim_T (T^{\pi(N\setminus H_T(\mathbf{Y}))}, \emptyset^{\pi(H_T(\mathbf{Y}))})$ and $(T^{N\setminus H_T(\mathbf{Z})}, \emptyset^{H_T(\mathbf{Z})}) \sim_T (T^{\pi(N\setminus H_T(\mathbf{Z}))}, \emptyset^{\pi(H_T(\mathbf{Z}))})$.

Clearly, if $\pi(H_T(\mathbf{Z})) = \pi(H_T(\mathbf{Y}))$, then $(T^{\pi(N\setminus H_T(\mathbf{Y}))}, \emptyset^{\pi(H_T(\mathbf{Y}))}) = (T^{\pi(N\setminus H_T(\mathbf{Z}))}, \emptyset^{\pi(H_T(\mathbf{Z}))})$, hence, by transitivity of $\succcurlyeq_T$, $\mathbf{Y} \sim_T \mathbf{Z}$. Let us then suppose that $\pi(H_T(\mathbf{Z})) \subset \pi(H_T(\mathbf{Y}))$. By DEP, it follows that:

$$(T^{\pi(N\setminus H_T(\mathbf{Y}))}, \emptyset^{\pi(H_T(\mathbf{Y}))}) \succ_T (T^{\pi(N\setminus H_T(\mathbf{Z}))}, \emptyset^{\pi(H_T(\mathbf{Z}))}),$$

hence, in particular, $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$.

In the supplementary appendix, we show that the characterization provided above is tight. □

Proof of Proposition 2. It is easily checked that $\succcurlyeq^g_T$ is a poverty ranking and does satisfy AN, IIA, SMED and IBED.

Conversely, suppose $\succcurlyeq_T$ is a poverty ranking that satisfies AN, IIA, SMED and IBED. Then, consider $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ such that $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$. Again, by repeated application of IIA and transitivity, $\mathbf{Y}_{|T} \succcurlyeq_T \mathbf{Z}_{|T}$. Now, suppose that $g_T(\mathbf{Z}) > g_T(\mathbf{Y})$. Then, by repeated application of IBED, $\mathbf{Z}'_{|T} \sim_T \mathbf{Y}_{|T}$ for some $\mathbf{Z}'$ such that $Z'_i \subseteq Z_i$ for each $i \in N$, and $g_T(\mathbf{Z}') = g_T(\mathbf{Y})$. It follows that, by repeated application of SMED, $\mathbf{Z}_{|T} \succ_T \mathbf{Z}'_{|T}$, hence by transitivity, $\mathbf{Z}_{|T} \succ_T \mathbf{Y}_{|T}$. Thus, by repeated application of IIA and transitivity again, $\mathbf{Z} \succ_T \mathbf{Y}$, a contradiction.

On the other hand, suppose that $\mathbf{Y} \succcurlyeq^g_T \mathbf{Z}$, i.e. $g_T(\mathbf{Y}) \geq g_T(\mathbf{Z})$, and consider $\mathbf{T} = (T, ..., T) \in \mathcal{P}[X]^N$. Of course, $\mathbf{T} \sim_T \mathbf{T}$, by reflexivity. Then, by AN and repeated application of IBED to $\mathbf{T} \sim_T \mathbf{T}$, it follows that $\mathbf{Y}' \succcurlyeq_T \mathbf{Z}$ for some $\mathbf{Y}'$ such that $Y'_i \setminus T = Y_i \setminus T$ and $Y_i \subseteq Y'_i$ for each $i \in N$, and $g_T(\mathbf{Y}') = g_T(\mathbf{Z})$. If, in particular, $g_T(\mathbf{Y}') = g_T(\mathbf{Y})$ then $\mathbf{Y}' = \mathbf{Y}$, hence $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$, and we are done. Otherwise, there exist $i \in N$ and $x \in T \cap (Y'_i \setminus Y_i)$, hence $\mathbf{Y} \succ_T \mathbf{Z}$ by transitivity and repeated application of SMED. In any case, $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$ as required.

In the supplementary appendix, we show that the characterization provided above is tight. □

Proof of Lemma 1. Let us suppose $h_T(\mathbf{Y}) = h_T(\mathbf{Z})$, $g_T(\mathbf{Y}) = g_T(\mathbf{Z})$. Also, notice that for any $\mathbf{U} \in \mathcal{P}[X]^N$, $h_T(\mathbf{U}) = h_T(\mathbf{U}_{|T})$ and $g_T(\mathbf{U}) = g_T(\mathbf{U}_{|T})$ by definition of h_T and g_T respectively. Therefore, $h_T(\mathbf{Y}_{|T}) = h_T(\mathbf{Z}_{|T}) = m$ and $g_T(\mathbf{Y}_{|T}) = g_T(\mathbf{Z}_{|T}) = k$ for some m, k non-negative (observe that $m = 0$ if and only if $k = 0$). Next, posit $\widetilde{\mathbf{V}} = \left(\widetilde{V}\right)_{i=1,\ldots,n}$ with $\widetilde{V}_i = T$ if $V_i \supseteq T$, and $\widetilde{V}_i = \emptyset$ if $V_i \not\supseteq T$ and note that $h_T(\mathbf{V}_{|T}) = h_T\left(\widetilde{\mathbf{V}}\right)$ since $\widetilde{\mathbf{V}}$ does not alter the set of poor population units in $\mathbf{V}_{|T}$. Next, $\mathbf{Y}_{|T} \succcurlyeq_T \mathbf{Z}_{|T}$ if and only if $\widetilde{\mathbf{Y}} \succcurlyeq_T \widetilde{\mathbf{Z}}$ by AN and a repeated application of Q-$IBED$ ($(m|T| - k)$ times). Moreover, since $h_T\left(\widetilde{\mathbf{Y}}\right) = h_T\left(\widetilde{\mathbf{Z}}\right)$, it follows by DEP that neither $\widetilde{\mathbf{Y}} \succ_T \widetilde{\mathbf{Z}}$ nor $\widetilde{\mathbf{Z}} \succ_T \widetilde{\mathbf{Y}}$. Therefore, $\widetilde{\mathbf{Y}} \sim_T \widetilde{\mathbf{Z}}$ because $\succcurlyeq_T$ is a total preorder. Finally, $\mathbf{Y} \sim_{\mathbf{T}} \mathbf{Y}_{|T}$ and $\mathbf{Z} \sim_{\mathbf{T}} \mathbf{Z}_{|T}$ by repeated applications of IIA. It follows, by transitivity, that $\mathbf{Y} \sim_{\mathbf{T}} \mathbf{Z}$. □

Proof of Proposition 3. Again, it is easily checked that $\succcurlyeq^{hg}_T$ does indeed satisfy AN, IIA, DEP, Q-IBED, CD, NC and HP.

On the other hand, let $\succcurlyeq_T$ be a poverty ranking and a total preorder that satisfies AN, IIA, DEP, Q-IBED, CD, NC and HP.

Let $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, such that $\mathbf{Y} \succcurlyeq_T^{hg} \mathbf{Z}$, then one of the following cases obtains:

a) $h_T(\mathbf{Y}) > h_T(\mathbf{Z})$ and $g_T(\mathbf{Y}) = g_T(\mathbf{Z})$

b) $h_T(\mathbf{Y}) > h_T(\mathbf{Z})$ and $g_T(\mathbf{Y}) > g_T(\mathbf{Z})$

c) $h_T(\mathbf{Y}) > h_T(\mathbf{Z})$ and $g_T(\mathbf{Z}) > g_T(\mathbf{Y})$

d) $h_T(\mathbf{Y}) = h_T(\mathbf{Z})$ and $g_T(\mathbf{Y}) > g_T(\mathbf{Z})$

e) $h_T(\mathbf{Y}) = h_T(\mathbf{Z})$ and $g_T(\mathbf{Y}) = g_T(\mathbf{Z})$

Under case a) b), d) $\mathbf{Y} \succ_T \mathbf{Z}$ by *CD*. Under case c), $\mathbf{Y} \succ_T \mathbf{Z}$ by Lemma 1 and *NC* and *HP*. In e) by Lemma 1 $\mathbf{Y} \sim_T \mathbf{Z}$. Hence, in any case, $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$.

Conversely, let $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, such that $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$, then the following cases should be distinguished:

1) $h_T(\mathbf{Y}) > h_T(\mathbf{Z})$

2) $h_T(\mathbf{Z}) > h_T(\mathbf{Y})$

3) $h_T(\mathbf{Y}) = h_T(\mathbf{Z})$ and $g_T(\mathbf{Y}) > g_T(\mathbf{Z})$

4) $h_T(\mathbf{Y}) = h_T(\mathbf{Z})$ and $g_T(\mathbf{Z}) > g_T(\mathbf{Y})$

5) $h_T(\mathbf{Y}) = h_T(\mathbf{Z})$ and $g_T(\mathbf{Z}) = g_T(\mathbf{Y})$.

Under case 1), 3), $\mathbf{Y} \succ_T^{hg} \mathbf{Z}$ by definition. Under case 2), two subcases should be distinguished, namely either $g_T(\mathbf{Z}) \geq g_T(\mathbf{Y})$ or $g_T(\mathbf{Y}) > g_T(\mathbf{Z})$. If $g_T(\mathbf{Z}) \geq g_T(\mathbf{Y})$ then by *CD* $\mathbf{Z} \succ_T \mathbf{Y}$, a contradiction. If, on the contrary, $g_T(\mathbf{Y}) > g_T(\mathbf{Z})$ then, by Lemma 1 and *NC* and *HP*, $\mathbf{Z} \succ_T \mathbf{Y}$ a contradiction again. Moreover, under case 4) by *CD* $\mathbf{Z} \succ_T \mathbf{Y}$, a contradiction. Finally, under case 5), we have that $\mathbf{Y} \sim_T^{hg} \mathbf{Z}$ by definition. Hence, the desired result follows.

In the supplementary appendix, we show that the characterization provided above is also tight. □

Proof of Proposition 4. The proof replicates almost verbatim the previous one. We reproduce it in the supplementary appendix for the sake of completeness. □

Proof of Propostion 5. Checking that $\succcurlyeq_T^w$ is a poverty ranking which satisfies *AN*, *IIA*, *DEP, Q-IBED, CD,* and *CUC* is straightforward. Then, we only need to prove the 'if' part.

First, notice that for any $\mathbf{Y} \in \mathcal{P}[X]^N$, $h_T(\mathbf{Y}), g_T(\mathbf{Y}) \in \mathbb{Z}_+$, $h_T(\mathbf{Y}) \leq n$, and $h_T(\mathbf{Y}) \leqslant g_T(\mathbf{Y}) \leqslant n \cdot t$, where $t = \#T$. Now, take any poverty ranking $\succcurlyeq_T$ that is a total preorder and satisfies *AN*, *IIA*, *DEP, Q-IBED, CD,* and *CUC.*

We distinguish two basic cases, namely:

Case I: There exist $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, such that $A = (h_T(\mathbf{Y}), g_T(\mathbf{Y})) \neq (h_T(\mathbf{Z}), g_T(\mathbf{Z})) = B$ and $\mathbf{Y} \sim_T \mathbf{Z}$.

Then, observe that all points lying on line joining A and B are $\sim_T$ indifferent. Indeed, $A \sim_T B$ by hypothesis. Then, $A - A \sim_T B - A$, i.e. $O \sim_T B - A$ by CUC. Hence, for any $\lambda > 0$, $O \sim_T \lambda(B-A)$ by CUC , which, in turn, entails $A \sim_T \lambda(B-A)+A$. Similarly, $O \sim_T B-A$ implies that $-(B-A) \sim_T O$. Then, for any $\lambda > 0$, $\lambda(-(B-A)) \sim_T O$, entails $A + \lambda(-(B-A)) \sim_T A$. Let us denote $w_1 x + w_2 y = k$, with $w_1, w_2 \in \mathbb{R}_+ \setminus \{0\}$ and $k \in \mathbb{R}$ the real line joining $\mathbf{Y}$ and $\mathbf{Z}$.

Moreover, observe that by CUC, $E = (h_T(\mathbf{Y}) + \delta_1, g_T(\mathbf{Y}) + \delta_2) \sim_T (h_T(\mathbf{Z}) + \delta_1, g_T(\mathbf{Z}) + \delta_2) = D$ for any $\delta_1, \delta_2 \in \mathbb{R}$. Therefore, all proper indifference curves are parallel to each other. Of course, there might exist a finite number of isolated points. But, then for each one of them, one can draw a line through it which is parallel to the other indifference curves. Finally, notice that by CD $\mathbf{U} \succ_T \mathbf{V}$ whenever $w_1 h_T(\mathbf{U}) + w_2 g_T(\mathbf{U}) = k_1$, $w_1 h_T(\mathbf{V}) + w_2 g_T(\mathbf{V}) = k_2$ and $k_1 > k_2$. Therefore, $\succcurlyeq_T = \succcurlyeq_T^w$ by definition of $\succcurlyeq_T^w$.

Case II: There is no pair $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ such that $(h_T(\mathbf{Y}), g_T(\mathbf{Y})) \neq (h_T(\mathbf{Z}), g_T(\mathbf{Z}))$ and $\mathbf{Y} \sim_T \mathbf{Z}$, hence for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ such that $(h_T(\mathbf{Y}), g_T(\mathbf{Y})) \neq (h_T(\mathbf{Z}), g_T(\mathbf{Z}))$ either $\mathbf{Y} \succ_T \mathbf{Z}$ or $\mathbf{Z} \succ_T \mathbf{Y}$.

Of course, if $(h_T(\mathbf{Y}), g_T(\mathbf{Y})) = (h_T(\mathbf{Z}), g_T(\mathbf{Z}))$ then $w_1 h_T(\mathbf{Y}) + w_2 g_T(\mathbf{Y}) = w_1 h_T(\mathbf{Z}) + w_2 g_T(\mathbf{Z})$ for any $w_1, w_2 \in \mathbb{R}_+$, and, by Lemma 1, $\mathbf{Y} \sim_T \mathbf{Z}$.

Moreover, if $(h_T(\mathbf{Y}), g_T(\mathbf{Y})) \geqslant (h_T(\mathbf{Z}), g_T(\mathbf{Z}))$ and $(h_T(\mathbf{Y}), g_T(\mathbf{Y})) \neq (h_T(\mathbf{Z}), g_T(\mathbf{Z}))$, then $w_1 h_T(\mathbf{Y}) + w_2 g_T(\mathbf{Y}) > w_1 h_T(\mathbf{Z}) + w_2 g_T(\mathbf{Z})$ for any $w_1, w_2 \in \mathbb{R}_+ \setminus \{0\}$, and, by CD, $\mathbf{Y} \succ_T \mathbf{Z}$.

Therefore, it suffices to check pairs $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ such that either:

i) $h_T(\mathbf{Y}) > h_T(\mathbf{Z})$ and $g_T(\mathbf{Z})) > g_T(\mathbf{Y})$ (also denoted, relying on an obvious choice of a Cartesian coordinate system in the real plane, as $\mathbf{Z} \in NW(\mathbf{Y})$ i.e. '$\mathbf{Z}$ is North-West of $\mathbf{Y}$') or

ii) $h_T(\mathbf{Z}) > h_T(\mathbf{Y})$ and $g_T(\mathbf{Y})) > g_T(\mathbf{Z})$ (also denoted as $\mathbf{Y} \in NW(\mathbf{Z})$ i.e. '$\mathbf{Y}$ is North-West of $\mathbf{Z}$').

First, consider any $\succcurlyeq_T$ such that for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, $\mathbf{Y} \succ_T \mathbf{Z}$ whenever $\mathbf{Y} \in NW(\mathbf{Z})$.

It is clearly the case that $w_1 h_T(\mathbf{Y}) + w_2 g_T(\mathbf{Y}) > w_1 h_T(\mathbf{Z}) + w_2 g_T(\mathbf{Z})$ for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ such that $\mathbf{Y} \in NW(\mathbf{Z})$, provided

$w_2(g_T(\mathbf{Y}) - g_T(\mathbf{Z})) > w_1(h_T(\mathbf{Z}) - h_T(\mathbf{Y}))$ or equivalently, since $\frac{h_T(\mathbf{Z}) - h_T(\mathbf{Y})}{g_T(\mathbf{Y}) - g_T(\mathbf{Z})} \leq n-1$, whenever $\frac{w_2}{w_1} > n-1$, i.e. $\frac{w_1}{w_2} < \frac{1}{n-1}$, for all $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ such that $\mathbf{Y} \in NW(\mathbf{Z})$.

Now, consider any $\succcurlyeq_T$ such that for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, $\mathbf{Y} \succ_T \mathbf{Z}$ whenever $\mathbf{Z} \in NW(\mathbf{Y})$.

Clearly, $w_1 h_T(\mathbf{Y}) + w_2 g_T(\mathbf{Y}) > w_1 h_T(\mathbf{Z}) + w_2 g_T(\mathbf{Z})$ for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ such that $\mathbf{Z} \in NW(\mathbf{Y})$, provided $w_1(h_T(\mathbf{Y}) - h_T(\mathbf{Z})) > w_2(g_T(\mathbf{Z}) - g_T(\mathbf{Y}))$, i.e. (since $\frac{g_T(\mathbf{Z}) - g_T(\mathbf{Y})}{h_T(\mathbf{Y}) - h_T(\mathbf{Z})} \leq (n-1) \cdot t - n = n \cdot (t-1) - t$) whenever $\frac{w_1}{w2} > n \cdot (t-1) - t$, for all $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ such that $\mathbf{Z} \in NW(\mathbf{Y})$.

Therefore, it only remains to be considered the case of a total preorder $\succcurlyeq_T$ with the required properties such that there exist $\mathbf{Y}, \mathbf{Z}, \mathbf{Y}', \mathbf{Z}' \in \mathcal{P}[X]^N$, with $\mathbf{Y} \succ_T \mathbf{Z}$ and $\mathbf{Y}' \succ_T \mathbf{Z}'$, while $\mathbf{Y} \in NW(\mathbf{Z})$ and $\mathbf{Z}' \in NW(\mathbf{Y}')$. In this case, posit $m^- = \max \left\{ \frac{g_T(\mathbf{Z}) - g_T(\mathbf{Y})}{h_T(\mathbf{Y}) - h_T(\mathbf{Z})} : \mathbf{Z} \in NW(\mathbf{Y}) \text{ and } \mathbf{Y} \succ_T \mathbf{Z} \right\}$ and $m^+ = \min \left\{ \frac{g_T(\mathbf{Z}) - g_T(\mathbf{Y})}{h_T(\mathbf{Y}) - h_T(\mathbf{Z})} : \mathbf{Z} \in NW(\mathbf{Y}) \text{ and } \mathbf{Z} \succ_T \mathbf{Y} \right\}$ (notice that $m^+$and m^- are both well defined under our special hypothesis on $\succcurlyeq_T$).

Next, we shall prove that it is not the case that $m^+ \not< m^-$ (the proof is along the same lines of that provided in Alcalde-Unzu and Ballester (2005): we reproduce it in the supplementary appendix for the sake of completeness).

Therefore, since we have already shown that $m^- \neq m^+$, it follows that $m^- < m^+$.

Now, take any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, such that $\mathbf{Z} \in NW(\mathbf{Y})$, and $\frac{g_T(\mathbf{Z}) - g_T(\mathbf{Y})}{h_T(\mathbf{Y}) - h_T(\mathbf{Z})} \leq m^-$. Then, $\frac{g_T(\mathbf{Z}) - g_T(\mathbf{Y})}{h_T(\mathbf{Y}) - h_T(\mathbf{Z})} < m^+$, whence, by definition, *not* $\mathbf{Z} \succ_T \mathbf{Y}$. On the other hand, $(h_T(\mathbf{Y}), g_T(\mathbf{Y})) \neq (h_T(\mathbf{Z}), g_T(\mathbf{Z}))$, hence, by assumption, $\mathbf{Y} \nsim_T \mathbf{Z}$. Since $\succcurlyeq_T$ is a total preorder, it follows that

$\mathbf{Y} \succ_T \mathbf{Z}$. Similarly, take any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, such that $\mathbf{Z} \in NW(\mathbf{Y})$, and $m^+ \leq \frac{g_T(\mathbf{Z})-g_T(\mathbf{Y})}{h_T(\mathbf{Y})-h_T(\mathbf{Z})}$. Then, $\frac{g_T(\mathbf{Z})-g_T(\mathbf{Y})}{h_T(\mathbf{Y})-h_T(\mathbf{Z})} > m^-$, whence, by definition, *not* $\mathbf{Y} \succ_T \mathbf{Z}$. On the other hand, $(h_T(\mathbf{Y}), g_T(\mathbf{Y})) \neq (h_T(\mathbf{Z}), g_T(\mathbf{Z}))$, hence, by assumption, $\mathbf{Y} \not\sim_T \mathbf{Z}$. Since $\succcurlyeq_T$ is a total preorder, it follows that $\mathbf{Z} \succ_T \mathbf{Y}$, a contradiction. Therefore, for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, it cannot be the case that $m^- < \frac{g_T(\mathbf{Z})-g_T(\mathbf{Y})}{h_T(\mathbf{Y})-h_T(\mathbf{Z})} < m^+$.

But then, take any $w_1, w_2 \in \mathbb{R}_+ \setminus \{0\}$ such that $m^- < \frac{w_1}{w_2} < m^+$. By our previous observations, for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ (so in particular for any such $\mathbf{Y}, \mathbf{Z}$ with $\mathbf{Y} \in NW(\mathbf{Z})$) either $\frac{g_T(\mathbf{Z})-g_T(\mathbf{Y})}{h_T(\mathbf{Y})-h_T(\mathbf{Z})} \leq m^-$ or $m^+ \leq \frac{g_T(\mathbf{Z})-g_T(\mathbf{Y})}{h_T(\mathbf{Y})-h_T(\mathbf{Z})}$. If $\frac{g_T(\mathbf{Z})-g_T(\mathbf{Y})}{h_T(\mathbf{Y})-h_T(\mathbf{Z})} \leq m^-$, then, as shown above, $\mathbf{Y} \succ_T \mathbf{Z}$. Moreover, $\frac{w_2}{w_1} \cdot \frac{g_T(\mathbf{Z})-g_T(\mathbf{Y})}{h_T(\mathbf{Y})-h_T(\mathbf{Z})} < (m^-)^{-1} \cdot m^- = 1$, i.e. $w_2 \cdot (g_T(\mathbf{Z}) - g_T(\mathbf{Y})) < w_1 \cdot (h_T(\mathbf{Y}) - h_T(\mathbf{Z}))$, whence, by definition, $\mathbf{Y} \succ_T^w \mathbf{Z}$ (with $w = (w_1, w_2)$). Conversely, let $\mathbf{Y} \succ_T \mathbf{Z}$. Similarly, if $m^+ \leq \frac{g_T(\mathbf{Z})-g_T(\mathbf{Y})}{h_T(\mathbf{Y})-h_T(\mathbf{Z})}$, then, as shown above, $\mathbf{Z} \succ_T \mathbf{Y}$ and $\frac{w_2}{w_1} \cdot \frac{g_T(\mathbf{Z})-g_T(\mathbf{Y})}{h_T(\mathbf{Y})-h_T(\mathbf{Z})} > (m^+)^{-1} \cdot m^+ = 1$, i.e. $w_2 \cdot (g_T(\mathbf{Z}) - g_T(\mathbf{Y})) > w_1 \cdot (h_T(\mathbf{Y}) - h_T(\mathbf{Z}))$, whence, by definition, $\mathbf{Z} \succ_T^w \mathbf{Y}$ (with $w = (w_1, w_2)$), and the thesis follows.

In the supplementary appendix, we show that the characterization provided above is also tight.

□

References

[1] Alcalde-Unzu J. and M. Ballester, Some remarks on ranking opportunity sets and Arrow impossibility theorems: correspondence results, *Journal of Economic Theory* 124 (2005), 116 - 123.

[2] Alcalde-Unzu J. and M. Ballester, On ranking opportunity distributions, *Social Choice and Welfare* 34 (2010), 3 - 31.

[3] Alkire, S. and J. Foster, Counting and multidimensional poverty measurement, *Journal of Public Economics*, 95 (2011), 476 - 487.

[4] Anderson, I., Combinatorics of Finite Sets, (1987) - Clarendon Press, Oxford.

[5] Arlegi, R. and Nieto, J., Equality of opportunity: cardinality-based criteria," in: de Swart, H. (ed.), *Logic, Game Theory and Social Choice*, Tilburg University Press, Tilburg, (1999), pp. 458–481.

[6] Barberà S., W.Bossert, P.K. Pattanaik: Ranking sets of objects, in S. Barberà, P.C. Hammond, C. Seidl (eds.): *Handbook of Utility Theory, vol.II, Extensions*, Kluwer Academic Publishers, Dordrecht (2004), pp.893-977.

[7] Bossert, W., Fleurbaey, M., and Van de gaer, D., Responsibility, talent, and compensation: a second best analysis, *Review of Economic Design* 4, (1999), 35 - 55.

[8] Bourguignon, F. and S. Chakravarty, A Family of Multidimensional Poverty Measures, in *Advances in Econometrics, Income Distribution, and Methodology of Science, Essays in Honor of Professor C. Dagum*, (1998), D. J. Slottje (ed.). New York: Springer-Verlag.

[9] Bourguignon, F. and S. Chakravarty, Measurement of Multidimensional Poverty, *Journal of Economic Inequality* 1(2002), 25 - 49.

[10] Chakravarty, S., D. Mukherjee and R. Ranade, On the Family of Subgroup and Factor Decomposable Measures of Multidimensional Poverty, in *Research on Economic Inequality* 8 (1998), Connecticut: JAI Press

[11] Clark, S., R. Hemming and D. Ulph, On indices for the measurement of poverty, *Economic Journal* 91 (1981), 515 - 26.

[12] Dutta B. and A. Sen, Ranking opportunity sets and Arrow impossibility theorems: correspondence results, *Journal of Economic Theory* 71 (1996), 90 - 101.

[13] Fishburn, P., Axioms for lexicographic preferences, *Review of Economic Studies* 42 (1975), 415 - 419.

[14] Foster, J., Inequality Measurement, in *The Elgar Companion to Development Studies* (D. Clark, Ed.), Cheltenham: Edward Elgar, (2006).

[15] Foster J, J. Greer and E. Thorbecke, A Class of Decomposable Poverty Measures, *Econometrica* 52 (1984), 761 - 766.

[16] Herrero, C., Equitable opportunities: an extension, *Economics Letters* 55, (1997), 91–95.

[17] Herrero, C., Iturbe-Ormaetxe, I., and Nieto, J., Ranking opportunity profiles on the basis of the common opportunities, *Mathematical Social Sciences* 35, (1998), 273–289.

[18] Kranich, L., Equitable opportunities: an axiomatic approach, *Journal of Economic Theory* 71 (1996), 131–147.

[19] Kranich, L., Equitable opportunities in economic environments, *Social Choice and Welfare* 14 (1997), 57–64.

[20] Ok, E., On opportunity inequality measurement," *Journal of Economic Theory* 77 (1997), 300–329.

[21] Ok, E. and Kranich, L., The measurement of opportunity inequality: a cardinality-based approach, *Social Choice and Welfare* 15 (1998), 263–287.

[22] Rawls, J., *A Theory of Justice,* (1971) - Harvard University Press, Cambridge.

[23] Roemer, J.E., *Theories of Distributive Justice*, (1996) - Harvard University Press, Cambridge.

[24] Savaglio E., Vannucci S., Filtral preorders and opportunity inequality, *Journal of Economic Theory* 132 (2007), 474 -492.

[25] Sen, A.K., Poverty: an ordinal approach to measurement, *Econometrica* 44 (1976), 219-231.

[26] Sen A. K., *Commodities and Capabilities*, (1985) - Oxford University Press.

[27] Sen A. K., Welfare, preference and freedom, *Journal of Econometrics* 50 (1991), 15 - 29.

[28] Tsui, K. Y., Multidimensional poverty indices, Social Choice and Welfare 19 (2002), 69 - 93.

°DSE, University of Bari - email: v.peragine@uniba.it

* *Corresponding Author*, DEc, University of Pescara - email: ernesto@unich.it

+DEPS, University of Siena - email: stefano.vannucci@unisi.it