

Signaling with Costly Acquisition of Signals

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Abstract

In this paper we identify a novel reason why signaling may fail to separate types, which is specific to cases where the receiver has to sustain a cost to acquire the signal sent by the sender. If the receiver chooses not to sustain the acquisition cost, then all sender's types find it optimal to pool on the minimum signal; also, if the different types of the sender pool on the least costly signal, then the receiver finds it optimal not to sustain the acquisition cost. This kind of coordination failure makes the resulting pooling equilibrium extremely robust, even when costs of signal acquisition are very small or the sender can invite acquisition of the signal through further costly signaling. Also, pooling is shown to be robust to all refinements based on out-of-equilibrium beliefs. These results provide a new source of interest in pooling equilibria.

JEL classification code: D82, D83.

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1 Introduction

Signaling is a pervasive phenomenon in economic interactions, emerging in many situations where there are information asymmetries. Many signaling models have been developed and studied, making the class of signaling games a quite prominent one in economics (see [Riley, 2001](#), for a comprehensive survey).

An important characteristic of signaling games is that they typically show many Bayes-Nash equilibria with rather distinct features: equilibria in which sender's types pool together by sending the same signal and equilibria where sender's types separate from each other by sending different signals.

In applied research, signaling models are often used with the focus on the best separating equilibrium, also called the Riley equilibrium, i.e., the equilibrium where all sender's types separate but they incur the minimum necessary signaling cost to do so. This is in good part due to an important stream of literature that has shown the prominence of the Riley equilibrium when agents are supposed to possess a sufficient degree of forward induction (see [Sobel, 2009](#), for an instructive survey and Subsection 6.1 where we provide more details on this point).

Quite surprisingly, however, not much attention has been given to the possibility that the acquisition of the signal by the receiver might be a costly activity. Is the assumption of freely acquisition of signals innocuous? In this paper we show that is definitely not so. Indeed, even a very small cost of signal acquisition can make a great difference in terms of the robustness (and reasonability) of equilibria. In particular, we show that in the presence of costs to acquire the signal the pooling of sender's types becomes at least as prominent as their separation as an equilibrium outcome. This happens because a coordination failure emerges: (i) if the receiver chooses not to sustain the acquisition cost, then all sender's types find it optimal to pool on the minimal signal, and (ii) if the different types of the sender pool on the least costly signal, then the receiver finds it optimal not to sustain the acquisition cost. So, a complementarity naturally arises between the receiver's incentive to *costly* acquire the signal and the sender's incentive to engage in the *costly* signaling activity. Our results suggest that new attention should be given to pooling outcomes. This could have far-reaching implications, especially in the light of the widespread reliance on separating equilibria in applied models.

This paper is part of a broader project that studies the consequences of introducing frictions in signaling games. In particular two general classes of frictions are considered. The first class comprises exogenous frictions: the signal sent by the sender is subject to a

friction that reduces its informativeness for the receiver. The analysis of this case is developed in [Bilancini and Boncinelli \(2014\)](#). The second class comprises instead endogenous frictions: the signal sent by the sender is subject to a friction whose intensity depends on the choices of either the sender, the receiver, or both. In the present paper we explore this case focusing on endogenous frictions which existence depends on the receiver's decisions.

The paper is organized as follows. In Section 2 we review the literature on costly acquisition of information. In Section 3 we introduce signaling games with costly acquisition of signals by means of an example that is a variant of the classical Spence's signaling model. In Section 4 we define a general class of signaling games with costly acquisition of signals. In Section 5 we show that the existence of acquisition costs may lead to a coordination failure that sustains a pooling equilibrium, which is also shown to be the unique pooling equilibrium; further, we contrast the equilibria that emerge in this class of games with those that emerge in the related class of standard signaling games (i.e., with no acquisition costs). In Section 6 we explore the robustness of the coordination failure along three dimensions: equilibrium refinements acting on out-of-equilibrium beliefs (Subsection 6.1), equilibrium selection by means of further signaling that can act as an “invitation” to acquire the main signal (Subsection 6.2), and smooth acquisition costs (Subsection 6.3).

2 Literature on costly acquisition of information

The idea that the acquisition of information is a strategic choice which comes at a cost is receiving increasing attention in economics. Several models with this feature have been investigated but just a few of them are closely related to our model. In fact, most of these models do not consider a sender-receiver setup, and none of them considers a typical signaling framework.

[Gabaix et al. \(2006\)](#) test the directed cognition model – which assumes that agents use partially myopic option-value calculations to select their next cognitive operation – by studying information acquisition in two different experiments. [Caplin and Dean \(2014\)](#) develop a revealed preference test for the costly acquisition of information, encompassing models of rational inattention, sequential signal processing, and search. [Liu \(2011\)](#) studies the dynamic behavior of firms and customers in markets with costly acquisition of information on past transactions. [Shi \(2012\)](#) analyzes optimal auction design in a setting where values are private and there are several potential buyers who can each costly acquire information about others' valuations prior to participation in the auction. [Oliveros \(2013\)](#) studies the role of abstention in an election where voters can costly acquire information and the cost increases

in the precision of the information acquired.

A paper more closely related to ours is [Dewatripont and Tirole \(2005\)](#) which develops a theory of costly communication where both the sender and the receiver have to sustain a cost in order to communicate. The model can be seen as a standard cheap talk model where the precision of the message depends on the cost sustained by the sender and where the receiver has to sustain a cost – which may depend on the message precision – to acquire the message. Due to such costs, a form of coordination failure arises – that is similar to the one emerging in our model – which gives rise to a robust babbling equilibrium where the message sent by the sender contains no information and the receiver does not acquire it. The main difference with our model is that [Dewatripont and Tirole \(2005\)](#) do not consider costly signaling, but different modes of cheap communication.

[Tirole \(2009\)](#) develops a model of limited cognition and examines its consequences for contractual design. This paper formalizes the idea that the acquisition of information is a costly activity because of cognitive limitation. This same idea is applied to persuasion in [Bilancini and Boncinelli \(2013\)](#) where the receiver has to sustain a cognitive cost to fully and precisely elaborate information. In this model the sender tries to persuade the receiver to accept an offer by sending a costly signal - the reference cue - which refers the offer to a category of offers whose average quality is known by the receiver; the actual quality is a sender's private information, but the sender can pay a cost to acquire it – the elaboration cost. Pooling equilibria emerge also in this setup and turn out to be robust. However, they are not due to a coordination failure: a pooling equilibrium – i.e., an equilibrium where the bad offer and the good offer are proposed with the same reference cue – is sustained by the fact that the receiver accepts or not the offer independently of the observed reference cue, no matter what information she decides to acquire. An important difference between the model by [Bilancini and Boncinelli \(2013\)](#) and the one developed in the present paper is that in the former the acquisition cost is paid to acquire hard information on the state of the world, while in the present paper the cost is paid to acquire the soft information embodied by the signal.

3 A motivating example

Consider the following simple variant of the classical model by [Spence \(1973\)](#). There is one employer E that wants to hire a worker W . There are two types of workers, distinguished by their productivity $\theta \in \{1, 2\}$ which is a worker's private information, and E has a prior $p > 0$ that W is highly productive, i.e., that $\theta = 2$. Technology and market conditions are

such that E 's net profits are given by $\theta - w$ if a worker is hired, with w the wage paid to the hired worker and θ his productivity, while otherwise profits are 0.

Moreover, W can acquire education by incurring a cost that is type-dependent. In particular, suppose that W comes from a foreign country and that he has to move to E 's country in order to be hired. Suppose also that W can only acquire education in the foreign country, and that the only available alternatives are a good school G and a bad school B , which are not previously known to E . For the prospective worker of type θ , the cost of attending G is $2/\theta$ and the cost of attending school B is $1/\theta$. So, attending school G is more costly than attending school B , and it is relatively more so for the low type $\theta = 1$. This provides W with a costly signal $x \in \{G, B\}$ that potentially allows W 's types to separate.

So far, there is no substantial difference from Spence's model. However, what if E , in order to assess the quality of the schooling signal x sent by W , has to actively acquire the information on what school W has attended in the country he comes from, and what attendance costs have been paid? These information can well not come for free and, we stress, this can make the difference. We observe that the costs of acquiring such information can be interpreted as due to the material and/or the cognitive effort which is necessary to retrieve and elaborate the relevant data on x . On the material side, E might have to search and collect information on G and B , and maybe also pay to translate documents that would be otherwise unaccessible. On the cognitive side, E might have to make an effort to elaborate the collected information in order to establish that one school is G with costs 2θ and the other is B with costs θ and to assign the signal s to either G or B – otherwise the schools would be undistinguishable to E . To model this situation suppose that E has to pay a cost $c > 0$ to acquire the signal x sent by W . In particular, if E does not incur the cost c , then E cannot condition his actions on s .

Consider now the following situation: W chooses G independently of her type, i.e., $x(1) = x(2) = B$, and E decides not to acquire the signal x . It is easy to check that this is an equilibrium in the present example: both types of W strictly lose by switching to the more costly G , and E strictly loses by acquiring the signal because it costs c and provides no new information. We observe that such an equilibrium is very similar to the pooling equilibrium with no-signaling that emerges in Spence's model. However, we stress that the presence of acquisition costs makes this pooling equilibrium much more robust than the no-signaling pooling equilibrium in Spence's model. Since E does not acquire any signal, W cannot use out-of-equilibrium signals to communicate with E , and the reason is that W would not even notice that such signals have been sent. In particular, even if W 's high type deviates from $x(2) = B$ to $x(2) = G$, there is no way to let E know – or even imagine – about such

a deviation. So, arguments based on the reasonability of out-of-equilibrium beliefs cannot refine away this pooling equilibrium.

There is another important difference. In this example the no-signaling pooling equilibrium is the *only* pooling equilibrium, again in contrast with Spence's model where there are multiple pooling equilibria. To see why, consider the case where both types of W pool on G , i.e., $x(1) = x(2) = G$. Given this behavior by W , E finds it strictly profitable *not to incur* the acquisition cost, as acquiring the signal provides no new information. But this implies that G cannot be sustained in equilibrium, since each of W 's type would strictly gain by switching from G to B .

4 The model

We now introduce the more general game of *signaling with costly acquisition of signals* (SCAS). There is one sender S and one receiver R (sometimes referred to as “he” and “she”, respectively). The sender S observes his own type $t \in T$, with T a finite set, and then chooses a signal $x \in X = \mathbb{R}_+$. The receiver R can exert costly effort and acquire the signal x , or save on effort and observe nothing. We denote with $s \in \{s1, s2\}$ such a choice, where $s1$ means that x is not acquired and $s2$ means that x is acquired and effort is exerted.¹ In any case, then R has to take an action $y \in Y = \mathbb{R}$. The prior beliefs by R on T are given by $\{\beta_t^0\}_{t \in T}$ where β_t^0 denotes the probability that S is of type t .

Utility for S is $U : T \times X \times Y \rightarrow \mathbb{R}$, and utility for R is $V : T \times X \times Y \times \{s1, s2\} \rightarrow \mathbb{R}$. The following assumptions on utility functions hold:

- A1. *continuity*: U and V are continuous over x and y ;
- A2. *monotonicity in action*: U is strictly increasing in y ;
- A3. *costly signaling*: U is strictly decreasing in x ;
- A4. *single-crossing property*: $U(t, x, y) \leq U(t, x', y')$, with $x' > x$, implies that $U(t', x, y) < U(t', x', y)$ for all $t' > t$ and $y, y' \in Y$;
- A5. *costly acquisition of signal*: $V(t, x, y, s1) > V(t, x, y, s2)$ for all $t \in T$, $x \in X$, $y \in Y$;

¹This labeling owes to the classification of elaboration processes as “System 1”, or S1, which is fast, cheap and intuitive, and “System 2”, or S2, which is slow, costly and analytical (see, e.g., Kahneman, 2003). We stress this interpretation based on cognitive effort because we think that it applies to many relevant cases of signal acquisition. Other interpretations are possible where the cost of acquiring the signal is entirely due to non-psychological factors.

A6. *constant cost of acquisition*: $V(t, x, y, s1) - V(t, x, y, s2) = c$ for all $t \in T$, $x \in X$, $y \in Y$;

In the light of A6, we define function $v : T \times X \times Y \rightarrow \mathbb{R}$ such that $v(t, x, y) + c = V(t, x, y, s2)$.

A strategy for S is a function $\mu : T \rightarrow X$; we denote with \mathcal{M} the set of all possible μ . A strategy for R is a pair (s, α) where $s \in \{s1, s2\}$ and $\alpha : X \times \{s1, s2\} \rightarrow Y$ is a function such that $\alpha(x, s1) = \alpha(x', s1)$ for all $x, x' \in X$, i.e., R 's action is unconditional on x whenever $s = s1$ is chosen; we denote with \mathcal{A} the set of all such functions.

For given μ and (s, α) , R has posterior beliefs that crucially depend on her choice of s . If R chooses $s = s2$ then she derives, by means of the Bayes rule, posterior beliefs $\{\beta_t(x|\mu, s2)\}_{t \in T}$ where $\beta_t(x|\mu, s2)$ denotes the probability that S is of type t , conditional on the observation of x . If, instead, R chooses $s = s1$ then she can only rely on her priors – no new information is acquired – so that posteriors are trivially identical to priors: $\beta_t(x|\mu, s1) = \beta_t(x'|\mu, s1) = \beta_t^0$, for all $t \in T$ and all $x, x' \in X$.

We introduce the following additional assumption:

A7. *uniqueness of best action under $s1$* : $\rho^{s1}(\mu) = \arg \max_{y \in Y} \sum_{t \in T} \beta_t^0 V(t, \mu(t), y, s1)$ is single valued.

Assumption A7 resembles an assumption that is typically made in standard signaling models: the single-valuedness of the receiver's best reply. We stress, however, that A7 does not ensure this much. In fact, it is consistent with the case where R is indifferent between choosing $\rho^{s1}(\mu)$ with no acquisition of the signal and some other action (or actions) with the acquisition of the signal. It should also be stressed that, because of assumption A6, assumption A7 implies that the best action is given by $\rho^{s1}(\mu)$ whenever posteriors are identical to priors, independently of the choice of s .

An *equilibrium* of a SCAS game is a profile of strategies $(\mu, (s, \alpha))$ such that:²

E1. $\mu(t) \in \operatorname{argmax}_{x \in X} U(t, x, \alpha(x, s))$, for all $t \in T$;

E2. $(s, \alpha) \in \operatorname{argmax}_{s \in \{s1, s2\}, \alpha \in \mathcal{A}} \sum_{t \in T} \beta_t^0 \left(\sum_{k \in T} \beta_k(\mu(t)|\mu, s) V(z, \mu(t), \alpha(\mu(t), s), s) \right)$;

E3. the beliefs $\{\beta_t(x|\mu, s)\}_{t \in T}$ are calculated by means of Bayes rule whenever possible.

²We note that, as for standard signaling games with two players and just one move for the sender, a Bayes-Nash equilibrium is also Sequential ([Kreps and Wilson, 1982](#)).

The meaning of E1 is straightforward: S must be best-relying to R . Similarly, the meaning of E2 is that R must be best-relying to S given her beliefs, but this might not be immediately clear from the stated condition. To see why it is so note that E2 is equivalent to the following set of conditions:

- if $s = s1$, then α is such that $\alpha(x, s1) = \tilde{y} = \rho^{s1}(\mu)$ for all $x \in X$ and $\sum_{t \in T} \beta_t^0 v(t, \mu(t), \tilde{y}) \geq \sum_{t \in T} \beta_t^0 \left(\sum_{k \in T} \beta_k(\mu(t)|\mu, s2) v(z, \mu(t), \alpha(\mu(t), s2)) - c \right)$ for all $\alpha \in \mathcal{A}$;
- if $s = s2$, then $\alpha = \tilde{\alpha} \in \operatorname{argmax}_{\alpha \in \mathcal{A}} \sum_{k \in T} \beta_k(\mu(t)|\mu, s2) v(z, \mu(t), \alpha(\mu(t), s2))$ for all $t \in T$ and $\sum_{t \in T} \beta_t^0 v(t, \mu(t), y) \leq \sum_{t \in T} \beta_t^0 \left(\sum_{k \in T} \beta_k(\mu(t)|\mu, s2) v(z, \mu(t), \tilde{\alpha}(\mu(t), s2)) - c \right)$ for all $y \in Y$;

Condition E3 is also straightforward. We observe that, in the present setup, posterior beliefs along the equilibrium path are the following:

- if $s = s2$ then, for all $t \in T$ and for all x such that $\mu(t') = x$ for some $t' \in T$:

$$\beta_t(x|\mu, s) = \begin{cases} \frac{\beta_t^0}{\sum_{k:\mu(k)=x} \beta_k^0} & \text{if } \mu(t) = x \\ 0 & \text{if } \mu(t) \neq x; \end{cases}$$

- if $s = s1$ then $\beta_t(x|\mu, s) = \beta_t^0$ for all $t \in T$ and for all $x \in X$.

Lastly, in order to better contrast our results with the results on standard signaling games we find it useful to define the standard signaling game that can be obtained from a SCAS game by forcing R to play $s = s2$ and setting $c = 0$. We call this a game of *signaling with free acquisition of signals* (SFAS). Note that a SFAS game with utilities U and v is actually the standard signaling game – i.e., with no costs to acquire the signal – that can be obtained from a SCAS game with utilities U and v and any acquisition cost $c > 0$. In the light of this, we denote with $\Gamma(U, v, c)$ a given SCAS game, and with $\Gamma(U, v, 0)$, its associated SFAS game.

5 Equilibria

The set of Bayes-Nash equilibria of a SCAS game is in general different from the set of Bayes-Nash equilibria of a typical signaling game. This difference is mostly due to the inexistence of

pooling equilibria where sender's types pool on non-zero signals. To make this claim precise we provide a number of results characterizing the set of equilibria of a generic SCAS game, and we compare it to the set of equilibria of the associated SFAS game.

5.1 Coordination failure leads to pool on the null signal

Our first result states that when coordination between the sender and the receiver fails, the emerging equilibrium is one where all sender's types pull on the null signal. More precisely, it turns out that in a SCAS game the pooling equilibrium where all sender's types pool on the signal $x = 0$ is the *unique pooling equilibrium*. The following proposition formalizes:

PROPOSITION 1. *A SCAS game $\Gamma(U, v, c)$ has unique pooling equilibrium, for any $c > 0$. At this equilibrium, the strategy profile is $(\mu^P, (s^P, \alpha^P))$ where $\mu^P(t) = 0$ for all $t \in T$, $s^P = s1$, and $\alpha^P(x, s1) = \alpha^P(x, s2) = \rho^{s1}(\mu) = y^*$ for all $x \in X$.*

Proof. We first show that the profile $(\mu^P, (s^P, \alpha^P))$ is an equilibrium. Preliminarily, note that by A7 (uniqueness of best-response action under $s1$) R 's expected utility $\sum_{t \in T} \beta_t^0 v(t, \mu^P(t), y) = \sum_{t \in T} \beta_t^0 V(t, \mu^P(t), y, s1)$ admits a maximum over Y , denoted by y^* , and, hence, the profile $(\mu^P, (s^P, \alpha^P))$ exists.

Consider R deviating from (s^P, α^P) . Since $\alpha^P(x, s1) = y^* = \rho^{s1}(\mu)$ for all $x \in X$, no strictly profitable deviation from α^P exists as long as R maintains $s1$. Consider a deviation to (s', α') with $s' = s2$ and some $\alpha' \in \mathcal{A}$. We observe that, since $\mu^P(t) = 0$ for all $t \in T$, R expects to obtain no additional information by playing $s2$ instead of $s1$, and therefore her posterior beliefs must be equal to her priors $\{\beta_t^0\}_{t \in T}$. This implies that y^* is a best response action also when $s' = s2$ and signal $\mu^P(t) = 0$, for all $t \in T$, is observed. Since by A5 and A6 (respectively, positivity and constancy of the acquisition cost) the only effect on utility of playing $s2$ instead of $s1$ is to sustain the constant cost $c > 0$, it follows that R 's expected utility is lower under deviation $(s2, \alpha')$, for all $\alpha' \in \mathcal{A}$, than under (s^P, α^P) .

Consider S deviating from μ^P . In particular, consider R deviating to μ' such that $\mu'(t') > 0$ for some $t' \in T$. Recall that $\alpha^P(x, s1) = y^*$ for all $x \in X$, i.e., the action chosen by R is y^* independently of the actual value of $\mu'(t)$, $t \in T$. This, together with assumption A3 (U strictly decreasing in x) implies that S 's expected utility can cannot be greater under any $\mu' \in \mathcal{M}$ than under μ^P .

We now show that no pooling equilibrium other than $(\mu^P, (s^P, \alpha^P))$ exists. Consider the profile $(\mu^{P'}, (s^{P'}, \alpha^{P'}))$ where $\mu^{P'}(t) = x^{P'} > 0$ for all $t \in T$. Note that, exactly because $\mu^{P'}(t) = x^{P'}$ for all $t \in T$, along the equilibrium path R never learns anything and so R takes the same action $y^{P'} = \alpha^{P'}(\mu^{P'}(t), s^{P'})$ for all $t \in T$. If $s^{P'} = s2$, then – by assumptions

A5 and A6 – R 's gets an expected utility of $\sum_{t \in T} \beta_t^0 v(t, x^{P'}, y^{P'}) - c$ which is strictly lower than $\sum_{t \in T} \beta_t^0 v(t, x^{P'}, y^{P'})$, i.e., the expected utility that R obtains by playing $s1$ together with any $\alpha^{P''} \in \mathcal{A}$ such that $\alpha^{P''}(x, s1) = y^{P'}$ for all $x \in X$. So, in order for $(s^{P'}, \alpha^{P'})$ to be a best reply for R to $\mu^{P'}$, it must be that $s^{P'} = s1$ and, hence, $\alpha^{P'}(x, s1)$ must be constant over X and, in particular, such that $\alpha^{P'}(x, s1) = \rho^{s1}(\mu^{P'})$ for all $x \in X$. But if this is the case, then we claim that S has a profitable deviation. In particular, consider S deviating to μ^P . Since R always responds with $\rho^{s1}(\mu^{P'})$ and by A3 (U strictly decreasing in x), it follows that S 's expected utility is strictly greater under μ^P than under $\mu^{P'}$. Finally, we observe that, by E2 (R best replies given her beliefs at all information sets), R has to best reply to μ^P also at the information set activated by playing $s2$. Since $\mu^{P'}(t) = x^{P'}$ for all $t \in T$, at such information sets R 's posterior beliefs are identical to priors, implying that α^P must be such that $\alpha^P(x, s2) = \rho^{s1}(\mu^{P'})$ for all $x \in X$. \square

5.2 Sufficiently low acquisition costs allow separation

Proposition 1 establishes that the presence of a positive cost to acquire the signal – no matter how small – induces a strong reduction in the number and variety of pooling equilibria, actually leading to a unique outcome where all sender's types pool on the null signal. Our second result shows that such a strong reduction does not take place for separating and semi-separating equilibria. More precisely, if a SFAS game has separating or semi-separating equilibrium, then also all associated SCAS games with acquisition costs sufficiently low possess the same equilibrium. The following proposition formalizes:

PROPOSITION 2. *Let $(\mu^S, (s^S, \alpha^S))$ be an equilibrium profile of the SFAS game $\Gamma(U, v, 0)$ where $\mu^S(t) \neq \mu^S(t')$ for some $t, t' \in T$. Then, there exists $\bar{c}((\mu^S, (s^S, \alpha^S))) > 0$ such that $(\mu^S, (s^S, \alpha^S))$ is an equilibrium of all SCAS games $\Gamma(U, v, c)$ with $c \leq \bar{c}((\mu^S, (s^S, \alpha^S)))$.*

Proof. Let $(\mu^S, (s^S, \alpha^S))$ be an equilibrium of $\Gamma(U, v, 0)$ with supporting beliefs $\{\beta_t^S(x|\mu, s)\}_{t \in T}$. Since $\Gamma(U, v, 0)$ is a SFAS game, it must be that $s^S = s2$. Moreover, since $(\mu^S, (s^S, \alpha^S))$ is an equilibrium of $\Gamma(U, v, 0)$, μ^S must be a best-reply to (s^S, α^S) for S .

Consider now the SCAS game $\Gamma(U, v, c)$. Note that $(\mu^S, (s^S, \alpha^S))$ is a profile which is feasible also in $\Gamma(U, v, c)$, for any $c > 0$. Note also that, by construction, S has the same set of strategies and faces the same payoffs in game $\Gamma(U, v, 0)$ and game $\Gamma(U, v, c)$, for any $c > 0$. Hence, if μ^S is a best reply to (s^S, α^S) for S in $\Gamma(U, v, 0)$ then it is also a best reply to (s^S, α^S) for S in $\Gamma(U, v, c)$, for any $c > 0$.

Instead, R has a set of strategies and a payoff structure in $\Gamma(U, v, c)$ that are different from those of game $\Gamma(U, v, 0)$. In particular, R 's strategy set in $\Gamma(U, v, 0)$ is $\{s1\} \times \mathcal{A}$ which is

a restriction of $\{s1, s2\} \times \mathcal{A}$, R 's strategy set in $\Gamma(U, v, c)$; R 's payoff structure in $\Gamma(U, v, c)$ is that faced in $\Gamma(U, v, 0)$ with the addition, in the light of A5 and A6 (respectively, positivity and constancy of the acquisition cost), of $-c$ in case R chooses $s = s2$. So, R 's expected utility in game $\Gamma(U, v, c)$ under profile $(\mu^S, (s^S, \alpha^S))$ and beliefs $\{\beta_t^S(x|\mu, s)\}_{t \in T}$ is given by:

$$\sum_{t \in T} \beta_t^0 \left(\sum_{k \in T} \beta_k^S(\mu^S(t)|\mu^S, s2) v(z, \mu^S(t), \alpha^S(\mu(t), s2)) - c \right) \quad (1)$$

Consider a deviation by R from (s^S, α^S) to $(s1, \alpha')$, with $\alpha'(x, s1) = \rho^{s1}(\mu^S)$ for all $x \in X$. Note that, by definition of ρ^{s1} , this is the best deviation entailing $s = s1$ that is available to R . Note also that, because R observes no signal under $s1$, R 's posterior beliefs are equal to priors $\{\beta_t^0\}_{t \in T}$, so that R 's expected utility for deviating to $(s1, \alpha')$ is given by:

$$\sum_{t \in T} \beta_t^0 v(t, \mu^S(t), \rho^{s1}(\mu^S)) \quad (2)$$

Let us set:

$$\bar{c}((\mu^S, (s^S, \alpha^S))) = \sum_{t \in T} \beta_t^0 \left(\sum_{k \in T} \beta_k^S(\mu^S(t)|\mu^S, s2) v(z, \mu^S(t), \alpha^S(\mu(t), s2)) \right) - \sum_{t \in T} \beta_t^0 v(t, \mu^S(t), \rho^{s1}(\mu^S)) \quad (3)$$

Note that, if $\bar{c}((\mu^S, (s^S, \alpha^S))) = 0$, then the statement of the Proposition holds trivially. Suppose $\bar{c}((\mu^S, (s^S, \alpha^S))) > 0$. By construction, for any $c \leq \bar{c}$, expected utility (1) is not lower than expected utility (2). This implies that for any $c \leq \bar{c}((\mu^S, (s^S, \alpha^S)))$ the strategy (s^S, α^S) is a best reply to μ^S for S in game $\Gamma(U, v, c)$. This, in turn, implies that, for any $c \leq \bar{c}((\mu^S, (s^S, \alpha^S)))$, $(\mu^S, (s^S, \alpha^S))$ is an equilibrium of the SCAS game $\Gamma(U, v, c)$. \square

The main insight of Proposition 2 is that any separating or semi-separating equilibrium of a SFAS game is also an equilibrium of the SCAS games with same utility functions U and v and with sufficiently small acquisition costs. From this it is straightforward to conclude that if acquisition costs are low enough, then a SCAS game has all separating and semi-separating equilibria of the associated SFAS game. The following corollary formalizes:

COROLLARY 1. *For any SFAS game $\Gamma(U, v, 0)$, there exists $\bar{c}^{\min} > 0$ such that if $(\mu^S, (s^S, \alpha^S))$ is an equilibrium of $\Gamma(U, v, 0)$ and $\mu^S(t) \neq \mu^S(t')$ for some $t, t' \in T$, then $(\mu^S, (s^S, \alpha^S))$ is an equilibrium of the SCAS game $\Gamma(U, v, c)$ for all positive $c \leq \bar{c}^{\min}$.*

Proof. We observe that, since T is finite, the set $M = \{\mu \in \mathcal{M} \mid \exists t, t' \in T, \mu(t) \neq \mu(t')\}$ is also finite. So, we can construct a lower bound for \bar{c} as follows:

$$\bar{c}^{min} = \min_{\mu \in M} \left[\sum_{t \in T} \beta_t^0 \left(\max_{\alpha \in \mathcal{A}} \sum_{k \in T} \beta_k(\mu(t) | \mu, s_2) v(z, \mu(t), \alpha(\mu(t), s_2)) \right) - \sum_{t \in T} \beta_t^0 v(t, \mu, \rho^{s_1}(\mu)) \right] \quad (4)$$

Since $\bar{c}^{min} \leq \bar{c}((\mu, (s, \alpha)))$ for all $(\mu, (s, \alpha))$ such that $\mu \in M$ and (s, α) is a best reply to μ under beliefs $\{\beta_t(s, \alpha)\}_{t \in T}$, a fortiori $\bar{c}^{min} \leq \bar{c}((\mu, (s, \alpha)))$ holds for all $(\mu, (s, \alpha))$ that form a separating or a semi-separating equilibrium of the SFAS game $\Gamma(U, v, 0)$. Then, by applying Proposition 2 for all such equilibria, we get the result. \square

5.3 Surplus maximization

So far, we have shown that in a SCAS game the presence of costs to acquire the signal leads to the existence of a unique pooling equilibrium – where all senders’ type pool on the null signal – while separating and semi-separating equilibria may exist in the same number and quality of those of the associated SFAS game, but only if acquisition costs are not too large. Furthermore, as it is shown in Section 6, the no-signal pooling equilibrium turns out to be robust to many standard refinements of equilibria. This suggests that in a SCAS game the no-signal pooling equilibrium is typically more focal than in the associated SFAS game, and this is more likely to be the case the larger the acquisition costs. Is this good or bad news?

From a welfare perspective, the no-signal pooling equilibrium can lead to either better or worse outcome with respect to an alternative separating or semi-separating equilibrium. Of course, this is true for standard signaling games as well, but in the presence of costs to acquire the signal the no-signal pooling equilibrium is more likely to be the second best, and the likelihood of this increases in the cost size. The next proposition makes this point clear:

PROPOSITION 3. *For any SFAS game $\Gamma(U, v, 0)$, there exists $\bar{c}^P > 0$ such that in all SCAS games $\Gamma(U, v, c)$ with $c \geq \bar{c}^P$ the unique pooling equilibrium $(\mu^P, (s^P, \alpha^P))$ entails a total surplus which is not lower than in any other equilibrium.*

TO BE DONE. \square

COROLLARY 2. *If $(\mu^P, (s^P, \alpha^P))$ is a pooling equilibrium of the SFAS game $\Gamma(U, v, 0)$ and entails a total surplus which is not lower than in any other equilibrium, then $(\mu^P, (s^P, \alpha^P))$ also entails this for all SCAS games $\Gamma(U, v, c)$, $c > 0$. Moreover, the converse does not hold.*

TO BE DONE. \square

6 On the robustness on the coordination failure

6.1 Refinements acting on out-of-equilibrium beliefs

Many refinements of Bayes-Nash equilibria have been proposed, especially for signaling games. Most of them follow the idea that out-of-equilibrium beliefs should not be totally free, but need to satisfy some criterion of reasonableness. All such refinements imply sequentiality ([Kreps and Wilson, 1982](#)).

[Cho and Kreps \(1987\)](#) introduce the Intuitive Criterion which requires that out-of-equilibrium beliefs place zero weight on types that can never gain from deviating from the considered equilibrium.

[Banks and Sobel \(1987\)](#) introduce Divinity which requires that out-of-equilibrium beliefs place relatively more weight on types that gain more from deviating from the considered equilibrium. They also introduce Universal Divinity which requires that beliefs survives Divinity for all possible priors. The surviving beliefs do not depend on the priors – while those surviving Divinity in general do.

Motivated by [Banks and Sobel \(1987\)](#), [Cho and Kreps \(1987\)](#) have also introduced D1, which requires that out-of-equilibrium beliefs are supported on types that have the most to gain from deviating from the considered equilibrium, and D2, which requires to place zero weight on types that always have some other type gaining strictly from deviating from the considered equilibrium. In general Divinity turns out to be weakening of D1, while Universal Divinity to be a strengthening of D2.

[Cho and Sobel \(1990\)](#) demonstrate that, for monotonic signaling games, the set of D1 and Universal Divinity are equivalent to Strategic Stability ([Kohlberg and Mertens, 1986](#)); moreover, if the single-crossing property is satisfied, then D1 yields a unique equilibrium.

The perfect sequential equilibrium by [Grossman and Perry \(1986\)](#) is more tricky. It selects equilibria that survive backward induction in a game where nodes are not only identified by paths of play but also by beliefs at such nodes – they call a strategy of this game a metastrategy. The concept of perfect sequential equilibrium selects a set of equilibria – possibly empty – that is a subset of that obtained with the intuitive criterion.

The undefeated equilibrium by [Mailath et al. \(1993\)](#) rests on totally different grounds and restrict beliefs according to payoff comparisons at distinct sequential equilibria. More precisely, a first sequential equilibrium is defeated by a second sequential equilibrium if there exists a non-negligible set of types that prefer to deviate from what they do in the first equilibrium to what they do in the second equilibrium and, at the same time, the beliefs of the non-deviating types in the first equilibrium are not consistent with such a deviation for

this set of types. A sequential equilibrium is undefeated if no other equilibrium defeats it.

These refinements relate to each other in a non trivial way,³ but all rely on the possibility that a deviation by the sender triggers a path of play along which the receiver gets some piece of information that is unexpected along the equilibrium path. However, in the pooling equilibrium of a SCAS game the receiver does not acquire the signal, so that this possibility does not exist. Intuitively, this is why refinements based on out-of-equilibrium beliefs does not have a bite in such case.

To formalize this point let us introduce the following definitions. For given strategy profile $(\mu, (s, \alpha))$ and priors $\{\beta_t^0\}_{t \in T}$, R has a set of beliefs $\{\beta_t(x|\mu, s)\}_{t \in T}$ associated with each of her information sets where an action in Y has to be chosen. Denote with $X^e(\mu, (s, \alpha)) \subseteq X$ the set of signals that R can observe on information sets along the equilibrium path, i.e., at information sets that contain decision nodes along the equilibrium path. Denote with $X^o(\mu, (s, \alpha)) = X \setminus X^e(\mu, (s, \alpha))$ the set of signals that R can observe only at information sets off the equilibrium path, i.e., at information sets that do not contain decision nodes lying on the equilibrium path.

Moreover, denote with $X^{o1}(\mu, (s, \alpha)) \subseteq X^o(\mu, (s, \alpha))$ the set of signals off the equilibrium path that R cannot observe as a consequence of S deviating from μ because a deviation by R is required. Also, denote with $X^{o2}(\mu, (s, \alpha)) = X^o(\mu, (s, \alpha)) \setminus X^{o1}(\mu, (s, \alpha))$ the set of signals off the equilibrium path that R can potentially observe as a consequence of S deviating from μ . In a SCAS game, we call *receiver-triggered out-of-equilibrium beliefs* the beliefs held by R which are activated by signals in $X^{o1}(\mu, (s, \alpha))$, and we call *sender-triggered out-of-equilibrium beliefs* the beliefs held by R which are activated by signals in $X^{o2}(\mu, (s, \alpha))$.

A refinement that rules away equilibria by restricting admissible beliefs to a subset of those possibly activated by signals in $X^o(\mu, (s, \alpha))$ can be regarded as a *refinement acting on out-of-equilibrium beliefs*. All equilibrium refinements discussed above are evidently refinements acting on out-of-equilibrium beliefs. Note that, although such refinements require R to observe an unexpected signal, they can potentially act on beliefs activated by all $x \in X^o(\mu, (s, \alpha))$, i.e., they act not only on sender-triggered out-of-equilibrium beliefs, but also on receiver-triggered ones.

The following proposition establishes that the pooling equilibrium identified by Proposition 1 is robust to any refinement acting on out-of-equilibrium beliefs:

³In particular, an equilibrium satisfying D1 must also satisfy D2 which in turn requires to satisfy the Intuitive Criterion; an equilibrium satisfying Universal Divinity must also satisfy Divinity which in turn requires to satisfy the Intuitive Criterion; an Undefeated equilibrium need only be Sequential, while a Perfect Sequential Equilibrium must satisfy the Intuitive Criterion.

PROPOSITION 4. *In the SCAS game, the profile $(\mu^P, (s^P, \alpha^P))$ where $\mu^P(t) = 0$ for all $t \in T$, $s^P = s_1$, and $\alpha^P(x, s_1) = \alpha^P(x, s_2) = \rho^{s_1}(\mu^P)$ is an equilibrium that survives any possible equilibrium refinement acting on out-of-equilibrium beliefs.*

Proof. The considered equilibrium profile $(\mu^P, (s^P, \alpha^P))$ prescribes that R plays $s^P = s_1$, so it follows that $X^e(\mu^P, (s^P, \alpha^P)) = X^{o2}(\mu^P, (s^P, \alpha^P)) = \emptyset$ and $X^{o1}(\mu^P, (s^P, \alpha^P)) = X$, because R can observe a signal $x \in X$ only if she deviates from her equilibrium strategy. In particular, no sender-triggered out-of-equilibrium belief exists because, since $s^P = s_1$, any $x \in X$ that is chosen by S leads to the same R 's information set, which is on the equilibrium path. Hence, at this information set, R must have constant beliefs which are identical to the priors $\{\beta_t^0\}_{t \in T}$ and which cannot be refined away by refinements acting on out-of-equilibrium beliefs.

So, refinements acting on out-of-equilibrium beliefs can rule out only beliefs associated with informations sets that become active when a signal in $x \in X^{o1}(\mu^P, (s^P, \alpha^P)) = X$ is observed. However, none of these receiver-triggered out-of-equilibrium beliefs is necessary to sustain the considered equilibrium. To see why, note that R always uses the priors $\{\beta_t^0\}_{t \in T}$ and S 's strategy μ^P to evaluate whether to deviate or not from (s^P, α^P) ; so, there is no single deviation by S that can induce R to deviate from (α^P, s^P) ; since μ^P and (α^P, s^P) are mutual best response, it follows that S has no strictly profitable deviation from μ^P , no matter what are the receiver-triggered out-of-equilibrium beliefs held by R . Hence, no refinement acting on out-of-equilibrium beliefs can refine away the considered equilibrium $(\mu^P, (s^P, \alpha^P))$. \square

Let us remark that refinements acting on out-of-equilibrium beliefs do have a bite in a SCAS game. Actually, they do refine away a lot of separating equilibria, in a manner similar to what they do in a signaling game without costly acquisition of signals. Indeed, in any separating equilibrium with profile $(\mu^S, (s^S, \alpha^S))$, the receiver R must be playing $s^S = s_2$, so that signals $x \in X^{o2}(\mu^S, (s^S, \alpha^S)) = X \setminus X^e(\mu^S, (s^S, \alpha^S)) \neq \emptyset$ lead to sender-triggered out-of-equilibrium beliefs which crucially sustain separation by punishing S 's deviations from μ^S – as it happens in standard signaling games.

6.2 Inviting to acquire the signal through further signaling

It seems natural to ask whether the prominence of separation is restored if S has the possibility to communicate to R that he is actually sending an informative signal – i.e., a signal that separates (at least partly) types – and that therefore the signal is worth acquisition.

One can think of many situations where indeed the sender can send, together with the main signal x , an accompanying costly signal, say z , that acts as an invitation for the receiver

to engage in the costly acquisition of x . The analysis that shows that, in fact, not much can be restored by the use of z .

To have an intuition of why it is so, note that in order for the accompanying signal z to help separation, types must separate on z . Indeed, if separation is attained on x and the receiver acquires x , then all types would strictly prefer to save on costs and pool on a null z . So, suppose that separation is effectively attained on z . Then, the receiver strictly prefers not to incur the cost of acquiring x – since its acquisition would add no useful information – with the result that the only communication that takes place is that through z . However, in order for this kind of separation to be more robust than a pooling equilibrium, it is *necessary* that the sender’s utility function satisfies an equivalent of the single-crossing property on types and z – e.g., by satisfying the single-crossing on x and z – and this is not guaranteed in general. But even if such a necessary condition holds, to restore the prominence of separation the receiver must be able to acquire z for free. In fact, if the receiver has to incur a positive cost to acquire z – no matter how small – then the coordination failure is still possible and the very same arguments discussed in Sub-section 6.1 apply also in this setup.

To provide a more formal discussion of these ideas, we construct a variant of the SCAS game that we call SCAS *game with invitation signal* (SCAS-IS). Consider a SCAS game and add a preliminary stage where S can send an additional costly signal $z \in Z = \mathbb{R}_+$. Note that, if z can be used by S to signal to R in a credible way that the signal x is informative, then z can indeed be used by S to invite R to acquire the signal x . This kind of credibility is assumed by means of a single-crossing property over x and z , for all $t \in T$, which captures the fact that the types who are investing resources to signal through x has relatively smaller costs to for sending z (see assumption B5).

As happens for the signal x , also the invitation signal z can be acquired or not by R . The choice by R to acquire z is denoted with $r \in \{r1, r2\}$, where $r = r1$ means that R does not acquires z while $r = r2$ means that R acquires z .

In a SCAS-IS game, utility for S is $U : T \times X \times Z \times Y \rightarrow \mathbb{R}$, and utility for R is $V : T \times X \times Y \times \{s1, s2\} \times \{r1, r2\} \rightarrow \mathbb{R}$. The following assumptions on utility functions hold:

- B1. *continuity*: U and V are continuous over x , z , and y ;
- B2. *monotonicity in action*: U is strictly increasing in y ;
- B3. *costly signaling*: U is strictly decreasing in x and z ;
- B4. *single-crossing property on (t, x)* : $U(t, x, z, y) \leq U(t, x', z, y')$, with $x' > x$, implies that $U(t', x, z, y) < U(t', x', z, y)$ for all $t' > t$, $y, y' \in Y$, and $z \in Z$;

- B5. *single-crossing property on (x, z) :* $U(t, x, z, y) \leq U(t, x, z', y')$, with $z' > z$, implies that $U(t, x, z, y) < U(t, x', z', y)$ for all $x' > x$, $y, y' \in Y$, and $t \in T$;
- B6. *costly acquisition of signal x :* $V(t, x, y, s1, r) > V(t, x, y, s2, r)$ for all $t \in T$, $x \in X$, $y \in Y$, $r \in \{r1, r2\}$;
- B7. *constant cost of acquiring x :* $V(t, x, y, s1, r) - V(t, x, y, s2, r) = c_x$ for all $t \in T$, $x \in X$, $y \in Y$, $r \in \{r1, r2\}$;
- B8. *costly acquisition of signal z :* $V(t, x, y, s, r1) > V(t, x, y, s, r2)$ for all $t \in T$, $x \in X$, $y \in Y$, $s \in \{s1, s2\}$;
- B9. *constant cost of acquiring z :* $V(t, x, y, s, r1) - V(t, x, y, s, r2) = c_z$ for all $t \in T$, $x \in X$, $y \in Y$, $s \in \{s1, s2\}$;

In the light of B6-B9, we have that $v(t, x, y) + c_x + c_z = V(t, x, y, s2, r2)$, $v(t, x, y) + c_x = V(t, x, y, s2, r1)$, and $v(t, x, y) + c_z = V(t, x, y, s1, r2)$.

In a SCAS-IS game, a strategy for S is a pair (ζ, μ) where $\zeta : T \rightarrow Z$ describes a type's choice of z while $\mu \in \mathcal{M}$ describes, as in a SCAS game, a type's choice of x ; we denote with \mathcal{Z} the set of all possible ζ . A strategy for R is a triple (r, σ, α) where:

- $r \in \{r1, r2\}$ describes R 's choice to acquire the invitation signal z ;
- $\sigma : Z \times \{r1, r2\} \rightarrow \{s1, s2\}$ describes R 's choice to acquire the signal x conditional on z , with σ satisfying $\sigma(z, r1) = \sigma(z', r1)$ for all $z, z' \in Z$, i.e., with the choice of s being unconditional on z if $r = r1$; we denote with Σ the set of all such functions;
- $\alpha : X \times \{s1, s2\} \times \{r1, r2\} \times Z \rightarrow Y$ describes (with a slight abuse of notation) R 's choice of y , with α satisfying:
 - $\alpha(x, s1, r2, z) = \alpha(x', s1, r2, z)$ for all $x, x' \in X$,
 - $\alpha(x, s1, r1, z) = \alpha(x', s1, r1, z')$ for all $x, x' \in X$ and $z, z' \in Z$,
 - $\alpha(x, s2, r1, z) = \alpha(x', s2, r1, z')$ for all $z, z' \in Z$,

i.e., with the choice of y being unconditional on z if $r = r1$ and unconditional on x if $s = s1$; we denote with \mathcal{A}^{IS} the set of all such functions.

For given (ζ, μ) and (r, σ, α) , R has posterior beliefs that depend on both her choice of r and her choice of s . If R chooses $s = s2$ and $r = r2$ then she derives, by means of the Bayes rule, posterior beliefs $\{\beta_t(z, x | ((\zeta, \mu), s2, r2))\}_{t \in T}$ where $\beta_t(m, x | (\zeta, \mu), s2, r2)$

denotes the probability that S is of type t , conditional on the observation of z and x . If R chooses $s = s1$ and $r = r2$ then she does not observe x but still observes z , so that her posteriors are given by $\beta_t(z, x | (\zeta, \mu), s1, r2) = \beta_t(z, x' | (\zeta, \mu), s1, r2)$, for all $t \in T$ and all $x, x' \in X$. If R chooses $s = s2$ and $r = r1$ then she observes x but does not observe z , so that her posteriors are given by $\beta_t(z, x | (\zeta, \mu), s2, r1) = \beta_t(z', x | (\zeta, \mu), s2, r1)$, for all $t \in T$ and all $z, z' \in Z$. Finally, if R chooses $s = s1$ and $r = r1$ then she can only rely on her priors – no new information is acquired – so that posteriors are trivially identical to priors: $\beta_t(z, x | (\zeta, \mu), s1, r1) = \beta_t(z', x' | (\zeta, \mu), s1, r1) = \beta_t^0$, for all $t \in T$ and all $x, x' \in X$ and $z, z' \in Z$.

We also introduce an equivalent of assumption A7 for the current setup, which accommodates the fact that R obtains no information if she refuses to acquire both z and x :

B10. *uniqueness of best action under $s1$ and $r1$:*

$$\rho^{s1, r1}(\mu) = \arg \max_{y \in Y} \sum_{t \in T} \beta_t^0 V(t, \mu(t), y, s1, r1) \text{ is single valued.}$$

Note that, since z does not affect R 's utility, function $\rho^{s1, r1}(\mu)$ is the same of function $\rho^{s1}(\mu)$ of a SCAS game.

To define the equilibrium of a SCAS-IS game in a compact and readable form, let us introduce some further notation:

$$\begin{aligned} \mathbb{E}V[s, \alpha | r1, (\zeta, \mu)] &= \sum_{t \in T} \beta_t^0 \left(\sum_{k \in T} \beta_k(\zeta(t), \mu(t) | (\zeta, \mu), s, r1) V(\mu(t), \alpha(\mu(t), s, r, \zeta(t)), s, r1) \right); \\ \mathbb{E}V[\sigma, \alpha | r2, (\zeta, \mu)] &= \sum_{t \in T} \beta_t^0 \left(\sum_{k \in T} \beta_k(\zeta(t), \mu(t) | (\zeta, \mu), \sigma(\zeta(t), r2), r2) \cdot \right. \\ &\quad \left. \cdot V(\mu(t), \alpha(\mu(t), \sigma(\zeta(t), r2), r2, \zeta(t)), \sigma(\zeta(t), r2), r2)) \right); \end{aligned}$$

where $\mathbb{E}V[s, \alpha | r1, (\zeta, \mu)]$ is R 's expected utility against (ζ, μ) when choosing $r = r1$, $\sigma(z) = s$ for all $z \in Z$, and some $\alpha \in \mathcal{A}^{IS}$, while $\mathbb{E}V[\sigma, \alpha | r2, (\zeta, \mu)]$ is R 's expected utility against (ζ, μ) when choosing $r = r2$, some $\sigma \in \Sigma$, and some $\alpha \in \mathcal{A}^{IS}$.

The *equilibrium* of a SCAS-IS game is a profile of strategies $((\zeta, \mu), (r, \sigma, \alpha))$ such that:

$$D1. (\zeta(t), \mu(t)) \in \arg \max_{z \in Z, x \in X} U(t, x, z, \alpha(x, s, r, z)), \text{ for all } t \in T;$$

$$D2. r = r1 \text{ implies that } \sigma(z) = \tilde{s} \text{ for all } z \in Z, (\tilde{s}, \alpha) \in \arg \max_{s \in \{s1, s2\}, \alpha \in \mathcal{A}^{IS}} \mathbb{E}V[s, \alpha | r1, (\zeta, \mu)], \text{ and} \\ \mathbb{E}V[\tilde{s}, \alpha | r1, (\zeta, \mu)] \geq \mathbb{E}V[\sigma, \alpha | r2, (\zeta, \mu)];$$

$$r = r2 \text{ implies that } (\sigma, \alpha) \in \arg \max_{\sigma \in \Sigma, \alpha \in \mathcal{A}^{IS}} \mathbb{E}V[\sigma, \alpha | r2, (\zeta, \mu)] \text{ and } \mathbb{E}V[\tilde{s}, \alpha | r1, (\zeta, \mu)] \geq \mathbb{E}V[\sigma, \alpha | r2, (\zeta, \mu)];$$

D3. the beliefs $\{\beta_t(z, x | (\zeta, \mu), s, r)\}_{t \in T}$ are calculated by means of Bayes rule whenever possible.

The meaning of D1 is that S must be best-relying to R , while the meaning of D2 is that R must be best-relying to S given her beliefs – this is better seen by noting that, for $r = r1$, condition D2 is substantially identical to E2 of SCAS game. Condition D3 is straightforward. For the sake of completeness we observe that, in a SCAS-IS game, posterior beliefs along the equilibrium path are the following:

- if $r = r2$ and $s = s2$ then, for all $t \in T$ and for all (z, x) such that $(\zeta(t'), \mu(t')) = (z, x)$ for some $t' \in T$:

$$\beta_t(z, x | ((\zeta, \mu), s, r) = \begin{cases} \frac{\beta_t^0}{\sum_{k: (\zeta(k), \mu(k)) = (z, x)} \beta_k^0} & \text{if } (\zeta(t), \mu(t)) = (z, x) \\ 0 & \text{if } (\zeta(t), \mu(t)) \neq (z, x); \end{cases}$$

- if $r = r1$ and $s = s2$ then, for all $t \in T$ and for all x such that $\mu(t') = x$ for some $t' \in T$:

$$\beta_t(z, x | ((\zeta, \mu), s, r) = \begin{cases} \frac{\beta_t^0}{\sum_{k: \mu(k) = x} \beta_k^0} & \text{if } \mu(t) = x \\ 0 & \text{if } \mu(t) \neq x; \end{cases}$$

- if $r = r2$ and $s = s1$ then, for all $t \in T$ and for all z such that $\zeta(t') = z$ for some $t' \in T$:

$$\beta_t(z, x | ((\zeta, \mu), s, r) = \begin{cases} \frac{\beta_t^0}{\sum_{k: \zeta(k) = z} \beta_k^0} & \text{if } \zeta(t) = z \\ 0 & \text{if } \zeta(t) \neq z; \end{cases}$$

- if $r = r1$ and $s = s1$ then $\beta_t(z, x | (\zeta, \mu), s, r) = \beta_t^0$ for all $t \in T$ and for all $x \in X$.

For comparability purposes, we observe for any given SCAS game $\Gamma(U, v, c)$ we have, besides the associated SFAS game $\Gamma(U, v, 0)$, also an associated SCAS-IS game, that we denoted with $\Gamma(\hat{U}, v, c_x, c_z)$ where $c_x = c$ and $\hat{U}(t, x, z, y) = U(t, x, y) + \gamma(z)$ for some appropriate function γ .

We now turn our attention to the relevant beliefs in SCAS-IS game. For a given strategy profile $((\zeta, \mu), (r, \sigma, \alpha))$ and priors $\{\beta_t^0\}_{t \in T}$, R has a set of beliefs $\{\beta_t(z, x | (\zeta, \mu), s, r)\}_{t \in T}$ associated with each of her information sets where an action in Y has to be chosen. Let $\mathbb{S} = Z \times X \cup Z \times \emptyset \cup \emptyset \times X$ be the set of potentially observable pairs of signals – we consider the union with $Z \times \emptyset \cup \emptyset \times X$ to encompass the case where only either z or x is

acquired by R . Denote with $(Z, X)^e((\zeta, \mu), (r, \sigma, \alpha)) \subseteq \mathbb{S}$ the set of pairs of signals that R can observe on information sets along the equilibrium path, i.e., at information sets that contain decision nodes along the equilibrium path. Denote with $(Z, X)^o((\zeta, \mu), (r, \sigma, \alpha)) = \mathbb{S} \setminus (Z, X)^e((\zeta, \mu), (r, \sigma, \alpha))$ the set of pairs of signals that R can observe only at information sets off the equilibrium path, i.e., at information sets that do not contain decision nodes lying on the equilibrium path.

Moreover, denote with $(Z, X)^{o1}((\zeta, \mu), (r, \sigma, \alpha)) \subseteq (Z, X)^o((\zeta, \mu), (r, \sigma, \alpha))$ the set of pairs of signals off the equilibrium path that R cannot observe as a consequence of S deviating from (ζ, μ) because a deviation by R is required. Also, denote with $(Z, X)^{o2}((\zeta, \mu), (r, \sigma, \alpha)) = (Z, X)^o((\zeta, \mu), (r, \sigma, \alpha)) \setminus (Z, X)^{o1}((\zeta, \mu), (r, \sigma, \alpha))$ the set of pairs of signals off the equilibrium path that R can potentially observe as a consequence of S deviating from (ζ, μ) . Similarly for what done for a SCAS game, we call *receiver-triggered out-of-equilibrium beliefs* the beliefs held by R which are activated by signals in $(Z, X)^{o1}((\zeta, \mu), (r, \sigma, \alpha))$, and we call *sender-triggered out-of-equilibrium beliefs* the beliefs held by R which are activated by signals in $(Z, X)^{o2}((\zeta, \mu), (r, \sigma, \alpha))$. So, in a SCAS-IS game a *refinement acting on out-of-equilibrium beliefs* is a refinement that rules away equilibria by restricting admissible beliefs to a subset of those possibly activated by signals in $(Z, X)^o((\zeta, \mu), (r, \sigma, \alpha))$. Note also that – as it happens in SCAS game – although such refinements require that R observes an unexpected signal, they can potentially act on beliefs activated by all $(z, x) \in (Z, X)^o((\zeta, \mu), (r, \sigma, \alpha))$, i.e., on both sender-triggered and receiver-triggered out-of-equilibrium.

PROPOSITION 5. *In the SCAS-IS game, the profile $((\zeta^P, \mu^P), (r^P, \sigma^P, \alpha^P))$ where $\zeta^P(t) = 0$ and $\mu^P(t) = 0$ for all $t \in T$, $r^P = r1$, $\sigma^P(z, r1) = s1$, and $\alpha^P(x, s, r, z) = \rho^{s1, r1}(\mu^P)$ for $s \in \{s1, s2\}$ and $r \in \{r1, r2\}$ is the unique equilibrium where all types pool on the same pair of signals and it survives any possible equilibrium refinement acting on out-of-equilibrium beliefs.*

Proof. We first show that $((\zeta^P, \mu^P), (r^P, \sigma^P, \alpha^P))$ is an equilibrium. Preliminarily, note that by B10 (uniqueness of best-response action under $s1$ and $r1$) R 's expected utility $\sum_{t \in T} \beta_t^0 V(t, \mu^P(t), y, s1, r1)$ admits a maximum over Y , denoted by y^* , and, hence, the profile $(\mu^P, (s^P, \alpha^P))$ exists.

Consider R deviating from $(r^P, \sigma^P, \alpha^P)$. Since $\alpha^P(x, s1, r1, z) = y^* = \rho^{s1, r1}(\mu)$ for all $x \in X$ and $z \in Z$, no strictly profitable deviation from α^P exists as long as R maintains $\sigma(z) = s1$ for all $z \in Z$ and $r = r1$. Consider a deviation to (r', σ', α') with $r' = r2$, and some $\sigma' \in \Sigma$ and $\alpha' \in \mathcal{A}^{IS}$. We observe that, since $\zeta^P(t) = 0$ and $\mu^P(t) = 0$ for all $t \in T$, R expects to obtain no additional information by playing $r2$ instead of $r1$ and σ' instead

of σ^P , and therefore her posterior beliefs must be equal to her priors $\{\beta_t^0\}_{t \in T}$. This implies that y^* is a best response action also when $r2$ and/or $s2$ are played, and signals $\zeta^P(t) = 0$ and/or $\mu^P(t) = 0$, for all $t \in T$, are observed. By B6 and B7 (respectively, positivity and constancy of the acquisition cost of x) the only effect on utility of playing $s2$ instead of $s1$ is to sustain the constant cost $c_x > 0$; moreover, by B8 and B9 (respectively, positivity and constancy of the acquisition cost of z) the only effect on utility of playing $r2$ instead of $r1$ is to sustain the constant cost $c_z > 0$; so, it follows that R 's expected utility is lower under deviation (r', σ', α') , for all $\sigma \in \Sigma$ and all $\alpha' \in \mathcal{A}^{IS}$, than under $(r^P, \sigma^P, \alpha^P)$.

Consider S deviating from (ζ^P, μ^P) . In particular, consider R deviating to (ζ', μ') such that either $\zeta'(t') > 0$ for some $t' \in T$ or $\mu'(t') > 0$ for some $t' \in T$, or both. Recall that $\alpha^P(x, s1, r1, z) = y^*$ for all $x \in X$ and all $z \in Z$, i.e., the action chosen by R is y^* independently of the actual value of $\zeta'(t)$ and $\mu'(t)$, $t \in T$. This, together with assumption B3 (U strictly decreasing in x and z) implies that S 's expected utility can cannot be greater under any $(\zeta', \mu') \in \mathcal{Z} \times \mathcal{M}$ than under (ζ^P, μ^P) .

We now show that no pooling equilibrium other than $((\zeta^P, \mu^P), (r^P, \sigma^P, \alpha^P))$ exists. Consider the profile $((\zeta^{P'}, \mu^{P'}), (r^{P'}, \sigma^{P'}, \alpha^{P'}))$ where $\zeta^{P'}(t) = z^{P'} \geq 0$ and $\mu^{P'}(t) = x^{P'} \geq 0$ for all $t \in T$, with $z^{P'} x^{P'} \neq 0$. Note that, exactly because $\zeta^{P'}(t) = z^{P'}$ and $\mu^{P'}(t) = x^{P'}$ for all $t \in T$, along the equilibrium path R never learns anything and so R takes the same action $y^{P'} = \alpha^{P'}(\mu^{P'}(t), s^{P'}, r^{P'}, \zeta^{P'}(t))$ for all $t \in T$. By assumptions B6-B9, R 's expected utility is:

- $\mathbb{E}V[\sigma^{P'}, \alpha^{P'} | r2, (\zeta, \mu)] = \sum_{t \in T} \beta_t^0 v(t, x^{P'}, y^{P'}) - c_x - c_z$, if $\sigma^{P'}(z) = s2$ for all $z \in Z$ and $r^{P'} = r2$;
- $\mathbb{E}V[s1, \alpha^{P'} | r2, (\zeta, \mu)] = \sum_{t \in T} \beta_t^0 v(t, x^{P'}, y^{P'}) - c_z$, if $\sigma^{P'} z = s1$ for all $z \in Z$ and $r^{P'} = r2$;
- $\mathbb{E}V[s2, \alpha^{P'} | r1, (\zeta, \mu)] = \sum_{t \in T} \beta_t^0 v(t, x^{P'}, y^{P'}) - c_x$, if $\sigma^{P'} z = s2$ for all $z \in Z$ and $r^{P'} = r1$;
- $\mathbb{E}V[s1, \alpha^{P'} | r1, (\zeta, \mu)] = \sum_{t \in T} \beta_t^0 v(t, x^{P'}, y^{P'})$, if $\sigma^{P'} z = s1$ for all $z \in Z$ and $r^{P'} = r1$;

These expected utilities imply that R strictly prefers to play, instead of $(r^{P'}, \sigma^{P'}, \alpha^{P'})$, any strategy $(r^{P''}, \sigma^{P''}, \alpha^{P''})$ such that $r^{P''} = r1$, $\sigma^{P''}(z) = s1$ for all $z \in Z$, and $\alpha^{P''} \in \mathcal{A}^{IS}$ such that $\alpha^{P''}(x, s1, r1, z) = y^{P'}$ for all $x \in X$ and $z \in Z$. So, in order for $(r^{P'}, \sigma^{P'}, \alpha^{P'})$ to be a best reply for R to $(\zeta^{P'}, \mu^{P'})$, it must be that $r^{P'} = r1$ and $\sigma(z)^{P'} = s1$ for all $z \in Z$; hence, $\alpha^{P'}(x, s1, r1, z)$ must be constant over X and Z , and in particular it must be such that $\alpha^{P'}(x, s1, r1, z) = \rho^{s1, r1}(\mu^{P'})$ for all $x \in X$ and $z \in Z$. But if this is the case, then

S must have a profitable deviation. In particular, consider S deviating to (ζ^P, μ^P) . Since R always responds with $\rho^{s1,r1}(\mu^P)$ and by B3 (U strictly decreasing in x and z), it follows that S 's expected utility is strictly greater under (ζ^P, μ^P) than under $(\zeta^{P'}, \mu^{P'})$. Finally, we observe that, by D2 (R best replies given her beliefs at all information sets), R has to best reply to (ζ^P, μ^P) also at the information set activated by playing $r2$ and/or $s2$. Since $\zeta^{P'}(t) = z^{P'}$ and $\mu^{P'}(t) = x^{P'}$ for all $t \in T$, at such information sets R 's posterior beliefs are identical to priors, implying that α^P must take value equal to $\rho^{s1,r1}(\mu^P)$ for all $x \in X$ and $z \in Z$.

Finally, we show that $((\zeta^P, \mu^P), (r^P, \sigma^P, \alpha^P))$ survives any possible equilibrium refinement acting on out-of-equilibrium beliefs. Since R plays $r^P = r1$ and $\sigma(z)^P = s1$ for all $z \in Z$, it follows that $(Z, X)^e((\zeta^P, \mu^P), (r^P, \sigma^P, \alpha^P)) = (Z, X)^{o2}((\zeta^P, \mu^P), (r^P, \sigma^P, \alpha^P)) = \emptyset$ and $(Z, X)^{o1}((\zeta^P, \mu^P), (r^P, \sigma^P, \alpha^P)) = \mathbb{S}$, because R can observe a pair in \mathbb{S} only if she deviates from her equilibrium strategy. In particular, no sender-triggered out-of-equilibrium belief exists because, since $\sigma(z)^P = s1$ for all $z \in Z$, any pair in \mathbb{S} that is chosen by S leads to the same R 's information set, which is on the equilibrium path. Hence, at this information set, R must have constant beliefs which are identical to the priors $\{\beta_t^0\}_{t \in T}$ and which cannot be refined away by refinements acting on out-of-equilibrium beliefs.

So, refinements acting on out-of-equilibrium beliefs can rule out only beliefs associated with information sets that become active when a pair $(Z, X)^{o1}((\zeta^P, \mu^P), (r^P, \sigma^P, \alpha^P)) = \mathbb{S}$ is observed. However, none of these receiver-triggered out-of-equilibrium beliefs is necessary to sustain the considered equilibrium. To see why, note that R always uses the priors $\{\beta_t^0\}_{t \in T}$ and S 's strategy (ζ^P, μ^P) to evaluate whether to deviate or not from $(r^P, \sigma^P, \alpha^P)$; so, there is no single deviation by S that can induce R to deviate from $(r^P, \sigma^P, \alpha^P)$; since (ζ^P, μ^P) and $(r^P, \sigma^P, \alpha^P)$ are mutual best response, it follows that S has no strictly profitable deviation from (ζ^P, μ^P) , no matter what are the receiver-triggered out-of-equilibrium beliefs held by R . Hence, no refinement acting on out-of-equilibrium beliefs can refine away the pooling equilibrium $((\zeta^P, \mu^P), (r^P, \sigma^P, \alpha^P))$. \square

6.3 Smooth acquisition costs

One can naturally think of the process of signal acquisition as a smooth one: the greater the cost sustained to acquire the signal, the greater the acquisition of the signal content. So, a natural question to ask is whether the robustness of pooling is retained under such a smooth process. It turns out that the robustness of pooling does not depend at all on the fact that R 's acquisition choice is binary, i.e., either pay the acquisition cost and acquire x or pay nothing and acquire nothing. In the following we provide a formal argument in this regard.

A simple way to model a smooth process of signal acquisition is to consider a stochastic acquisition where the probability of acquiring x is a increasing function of the cost paid, as follows. Consider a SCAS game and add the possibility for R to choose an acquisition effort $e \in [0, 1]$, which replaces the choice of $s \in \{s1, s2\}$. Assume that with probability $1 - e$ no signal is acquired, while with probability e the signal is acquired. So, now R utility function $V : T \times X \times Y \times [0, 1]$. Moreover, assumption A6 is replaced by:

- A6'. *smooth and separable cost of acquisition:* $V(t, x, y, e) - V(t, x, y, 0) = c(e)$ for all $t \in T$, $x \in X$, $y \in Y$, $e \in [0, 1]$, with c twice differentiable;
- A8. *increasingly costly effort:* $c' > 0$ and $c'' > 0$;

[TO BE DONE]

A different way to model smooth process of signal acquisition is to have the signal x always acquired but with some blurring noise whose incidence negatively depends on e . We do not explore this case here as the issue of noisy signaling is studied in detail in a companion paper ([Bilancini and Boncinelli, 2014](#)).

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