Liquidity Flows in Interbank Networks*

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Abstract

In this paper we model interbank liquidity networks as flow networks. The aim is to compare the ability of different network structures to cope with the liquidity risk faced by the banks. In particular, we analyze the efficiency of three network structures – star-shaped, complete and incomplete – in transferring liquidity among banks. It turns out that the star-shaped interbank networks achieve the full coverage of liquidity risk with the smallest amount of interbank deposits held by each bank. This implies that star-shaped networks generate the minimum counterparty risk for the banks. Moreover, the star-shaped network is more resilient to systemic risk: the default of one or more banks is less likely to cause the default of the entire system in a star-shaped network than in a complete network. These results provides a rationale of the consistent empirical evidence that interbank network are sparse networks.

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1 Introduction

One of the main functions provided by banks is liquidity transformation. Banks are therefore characterized by a maturity mismatch between long-term assets and short-term liabilities. A necessary consequence is that banks are exposed to a substantial amount of liquidity risk. As a form of coinsurance, banks share such a risk by holding gross liquid positions: each bank deposits a sum in other banks and receives deposits from other banks. These cross-holdings of interbank deposits form an interbank liquidity network.

On the one hand, this network of interbank deposits serves the purpose of re-allocating liquidity from banks that have a liquidity surplus to banks that face liquidity deficits. On the other hand, the interbank network becomes a channel of contagion in case of defaults. By choosing the amount of interbank deposits, banks face a trade-off. The larger the interbank deposits and the larger the possible liquidity transfers (hence the larger the insurance against liquidity risk). However, the larger the interbank deposits and the larger the exposure to counterparty and systemic risk, that is the risk of direct and indirect contagion, respectively. It is then relevant to identify the network shape that allows the largest liquidity transfer with the smallest interbank exposures.

We address this issue in a novel way by modelling an interbank network as a *flow network* and applying some of its properties.[1] A flow network is a weighted directed graph endowed with source nodes and sink nodes. In our model, the source nodes are attached to the banks that experience a liquidity surplus, while the sink nodes are attached to the banks that face a liquidity deficit. We model an interbank liquidity transfer as a flow going from the source nodes to the sink nodes; a flow that is driven by interbank deposits withdrawals.

Under the assumption that interbank deposit withdrawals are coordinated by a social planner, we evaluate and compare the performance of different interbank network structures in providing full coverage of liquidity risk (i.e., the complete transfer of liquidity from surplus banks to deficit banks). We consider a complete, a star-shaped (also known as 'money centre') and an incomplete interbank network. In the complete network every bank is connected to all other banks; in the star-shaped network a bank is at the center

[1] See Ahuja, Magnanti, and Orlin [1] for an exhaustive textbook treatment of flow networks.

and it is connected to all peripheral banks and the latter are connected only with the bank at the center; in the incomplete network each bank is connected only to part of the other banks. It turns out that each structure attains full insurance against liquidity risk but with different amount of interbank deposits. We show that the star-shaped network achieves the full coverage of liquidity risk with the minimum amount of interbank deposits.

The intuition of this result is the following. The cross-holding of interbank deposits between pair of banks that are in the same liquidity condition – i.e., either both in deficit or both in surplus – does not affect the ability of the network to transfer liquidity from surplus banks to deficit bank (also known as the *carrying capacity* of a network). Interbank deposits between banks that are both in need of liquidity, or both have an excess of liquidity, are somehow redundant. Complete and incomplete networks have a carrying capacity equal to the total of interbank deposits that deficit banks hold in surplus banks. Therefore, the interbank deposits that deficit banks hold in other deficit banks represent an 'excess' of interbank exposure (with respect to the coverage of liquidity risk). In a star-shaped network instead the carrying capacity is equal to the total of interbank deposits that deficit banks hold in the bank at the center. In a star-shaped network the central node acts as a hub that channels liquidity flows from banks in surplus towards banks in deficit without excessive interbank exposure. The star-shaped interbank network achieves the largest possible liquidity transfer for any given size of the interbank deposits.

The previous result has two consequences concerning the exposure of a network to financial contagion. First, the star-shaped network guarantees the liquidity risk coverage with the minimum expected losses due to the default of a debtor (i.e., it minimizes counterparty risk). In this respect, a star-shaped network performs better than a complete network which, in turn, performs better than an incomplete interbank network. Second, we show that the star-shaped network is less exposed to systemic risk than the complete network. Ceteris paribus, the minimum shock capable of inducing the insolvency of all banks in a star-shaped network is strictly larger than the corresponding shock in the case of a complete network.[2] We finally analyze the performance of the three network structures when the withdrawal decisions of banks are decentralized and independent. Also when the withdrawals decisions are not coordinated by the planner, the star-shaped network is the most efficient in guaranteeing the full coverage of liquidity risk with the minimum exposure

[2] The comparison on systemic risk cannot be extended to incomplete networks because there are no general results about contagion thresholds for such a class of networks (see footnote 9 for more details).

to contagion.

To motivate our analysis, we refer to recent empirical studies that document the structure of existing interbank networks. This strand of empirical research has been spurred by the crucial role that interbank markets played in the 2007/2008 financial crisis. The picture that emerges is consistent across different studies and it sustains our result. Based on transaction data from the Fedwire system, Soromäki et al. [24] and Beck and Atalay [8] find that the actual interbank lending networks formed by US commercial banks is quite sparse. It consists of a core of highly connected banks, while the remaining peripheral banks connect to the core banks. An almost identical feature is found in banking networks in other countries like the UK, Canada, Japan, Germany and Austria (see, respectively, Bank of England [7], Embree and Roberts [18], Inaoka et al. [21], Craig and von Peter [14], Boss et al. [9]). Our model provides a rationale for these findings since the star-shaped is a special case of core-periphery structures, with one node in the core.

Several papers have analyzed empirically the relationship between interbank network structure and the exposure to contagion. Degryse and Nguyen [15] investigate the evolution of contagion risk for the Belgian banking system. They find that a change from a complete structure (where all banks have symmetric links) toward a money-centers structure (where money centers are symmetrically linked to otherwise disconnected banks) has decreased the risk and impact of contagion. Mistrulli [23] focuses on the Italian interbank network and, analyzing its evolution through time, finds that complete connection among banks is not always less conducive to contagion than other structures. He shows that less connected networks could be more resilient to contagion. The evidence provided by these studies is supportive of our results.

The remainder of the paper is organized as follows. In the rest of the introduction, we discuss the related literature. In section 2 we formalize the interbank network as a liquidity flow network. Section 3 analyzes the efficient network (section 3.1) and its implication for counterparty risk (section 3.2) and contagion risk (section 3.3). Section 4 shows that, when withdrawal decisions are decentralized, the same results of the efficient network obtain. Sections 5 concludes. The Appendix contains the proofs.

1.1 Related literature

Our paper stems from the micro-banking literature that investigates the relationship between interbank deposit structures and systemic risk. This literature focuses on simple network structures and ad-hoc liquidity shock occurrences in order to obtain analytical results, which clearly depend on the assumptions of each model. Allen and Gale [5] show that the banking system is more fragile when the interbank market is incomplete (cycle-shaped) than when the interbank market is complete. Brusco and Castiglionesi [11] and Freixas et al. [20] instead show that an incomplete cycle-shaped interbank market is more resilient than a complete interbank market.

Similar to the micro-banking literature, the present paper considers the interbank network as a way to eliminate aggregate liquidity risk and it analyzes how different network structures are able to cope with idiosyncratic risk. That is, how efficiently interbank networks channel liquidity from banks that have excessive liquidity holdings to banks that are in need of liquidity. Different from the micro-banking literature, we consider networks with an arbitrary numbers of banks and, more importantly, a wider realization of liquidity shocks. While it is common to assume "alternate" liquidity shocks in the banking literature (i.e., adjacent banks have different liquidity shocks so they always trade in the interbank market), we do not restrict the occurrence of liquidity shocks (i.e., adjacent banks can have the same shock). The present paper also consider star-shaped and incomplete networks beside the complete and cycle-shaped networks usually studied in the micro-banking literature.[3] This has important implications for the banking literature since one of our results establishes that incomplete networks are not able to guarantee the reallocation of liquidity in the interbank market. This implies that cycle-shaped networks, which are a particular case of incomplete ones, cannot represent an efficient device to reallocate liquidity when a more general liquidity shock structure is taken into account.

Our analysis is also inspired by the work by Eisenberg and Noe [17]. They represent the network of payment system as a lattice, and study the flows of payment that clear such network of financial obligations. Given the operating cash flows of the agents in the system and a generic network of obligations, they show that the clearing payments vector is unique under a mildly restrictive condition.[4] Since the clearing payments vector cannot

[3]Freixas et al. [20] actually analyze also a three-banks example of a money center system, arguing that too-central-to-fail policies could be rationalized by avoiding contagious defaults to the peripheral banks. Unlike the present paper, they cannot compare the money center system with other structures of interbank networks.

[4]Eisenberg and Noe [17] use Tarski's fixed-point theorem to establish that the clearing payments vector has a lower and an upper bound. To guarantee uniqueness, they introduce a regularity condition that requires that in the set of agents involved in a contagion process there is at least one agent with strictly

be characterized in analytical form for generic networks, Eisenberg and Noe [17] provide a computational characterization of such a vector.[5] Our work also studies flows of payments in networks of obligations, but our method and focus are different. We use flow network theory (as opposed to lattice theory) to characterize, in an analytical fashion, the maximum flow of interbank payments. We focus our analysis on three specific classes of networks, in order to compare their efficiency in providing the full coverage of the liquidity risk.

The present paper is also related to the growing theoretical literature that model interbank relationship as networks (see Allen and Babus [3] for a survey). Leitner [22] shows how the threat of contagion may be part of an optimal network. The possibility that the failure of a bank can spread to the entire network makes ex-ante optimal to establish links among banks to obtain mutual insurance and prevent the collapse of the network. Babus [6] shows that this form of insurance between banks emerges endogenously in a network formation game. Castiglionesi and Navarro [12] instead rationalize the formation of the interbank network structure as the trade off between liquidity coinsurance and counterparty risk. Allen et al. [4] analyze the interaction between financial connections due to overlapping portfolio exposure and systemic risk.

An alternative approach has resorted to numerical simulations to shed some light on the dynamics of contagion processes in generic and complex financial networks. In this literature the analysis relies on numerical simulations of default contagion either on randomly generated networks (see Alenton et al. [2] and Cifuentes et al. [13]) or on national interbank systems (see Upper [25]). Our approach is alternative to the previous ones since, to the best of our knowledge, the present paper is the first to apply flow theory to study the efficiency of interbank networks.

2 The Interbank Network

Let $N := (\Omega, \Lambda)$ be an *interbank network*, i.e., a connected, directed and weighted graph. The node ω_i $(i = 1, 2, ..., n)$ in Ω represents a bank and the links in $\Lambda \subseteq \Omega^2$ represent the interbank deposits that connect the members of Ω among themselves. The short term liabilities of a bank ω_i in Ω comprise customers (households) deposits, $h_{i,}$ and interbank

positive operating cash flow.

[5] The authors characterise the clearing payment vector as the solution of a linear programming problem and of an algorithm that numerically computes such a vector.

deposits, d_i. For simplicity, we assume that a bank in Ω has no long-term liability but its own equity e_i. On the asset side, a bank ω_i holds long-term assets, a_i, which are liabilities of agents that do not belong to Ω, and short-term assets c_i, which are deposits made by bank ω_i in other banks of the network. The budget identity of a bank is: $a_i + c_i = h_i + d_i + e_i$. A link $l_{ij} \in \Lambda$ represents the interbank obligations, and its direction goes from the debtor node ω_i to the creditor node ω_j. The weight of the link $l_{ij} \in \Lambda$ is equal to the amount of money c_{ji} that bank ω_j has deposited in bank ω_i.

To analyze the flows of liquidity that can be carried by an interbank network N, we need to model a liquidity shock. In this way, we transform the network N into a flow network. We assume that the liquidity shock consists of a reallocations of customer deposits across banks, while the aggregate liquidity in the network remains constant. Formally, a *liquidity shock* is an ordered vector of scalars $\boldsymbol{\delta} = [\delta_1, \delta_2, ..., \delta_n]$, where $\sum_{\Omega} \delta_i = 0$. We consider *symmetric* liquidity shocks: that is, half of the banks faces an increase in customer deposits equal to the scalar $\overline{\delta}$, while the other half faces a decrease in customer deposits equal to its opposite, $-\ \overline{\delta}$. Notice this is a conventional way of representing the liquidity risk due to fluctuations in customer deposits (see, among others, Allen and Gale [5] and Brusco and Castiglionesi [11]). To each node ω_i in Ω that experiences an increase of customer deposits, we attach a *source node* s_i and a link l_{si}, that connects the source node to the bank. Correspondingly, to each node ω_i in Ω that faces a decrease of such deposits, we attach a *sink node* t_i and a link l_{it}, that connects the bank to the sink node.

A liquidity shock that hits an interbank network N is then defined as a four-tuple $\Delta = \{S, T, \Lambda^+, \Lambda^-\}$ composed by: i) the set of source nodes $S = \{s_i | \forall i \in \Omega \text{ s.t. } \delta_i > 0\}$; ii) the set of sink nodes $T = \{t_i | \forall i \in \Omega \text{ s.t. } \delta_i < 0\}$; and iii) the sets of links $\Lambda^+ = \{l_{si}\}$ and $\Lambda^- = \{l_{it}\}$ that connect sources and sinks to the corresponding banks. Adding a liquidity shock Δ to an interbank liquidity network N we obtain an *interbank liquidity flow* network L, which is an n-tuple $L = \{N, \Delta, \Gamma\} = \{\Omega, \Lambda, S, T, \Lambda^+, \Lambda^-, \Gamma\}$, where Γ is a *capacity function* that associates to the links in Λ a capacity equal to the value of the corresponding interbank deposits, and to the links in Λ^+ and Λ^- a capacity equal to the value of the corresponding variations of customers' deposits.

An *interbank liquidity flow* in L is a value assignment to the links in Λ, Λ^+ and Λ^- such that: i) no link carries a flow larger than its own capacity (*capacity constraint*); ii) the divergence of a node, i.e., the difference between its inflow and its outflow, is null for all nodes in Ω (*flow conservation property*). A flow that complies with both these requirements is said to be *feasible*. In other words, a flow in a flow network is feasible if it comes out of the sources, crosses the network and ends entirely in the sinks, without exceeding the capacity of the links that carry it. The value of the largest feasible flow that can cross a flow network is called the *carrying capacity* of the network. As will be clearer below, the carrying capacity depends on the structure of the network.

Finding the carrying capacity of a network is a fundamental problem in the theory of flow networks – known as the *maximum flow problem*. A solution to this problem is given by the *minimum cut-maximum flow* theorem provided by Ford and Fulkerson [19]. This theorem states that the carrying capacity of a network is equal to the capacity of the *cut* which has the smallest *capacity* among all possible cuts of the network.[6] In other words, the cut of the smallest capacity is the bottleneck of a network and sets the upper bound to the magnitude of the flows that such a network can transfer from sources to sinks. The maximum feasible flow of a network is achievable by a network administrator, with a proper value assignment to the flows carried by each link in Λ.[7]

In the present analysis, the convenience of the Ford-Fulkerson theorem is that the maximum liquidity transfer implementable by an interbank liquidity network is achieved by a social planner who acts as a network administrator. In the next Section (Lemmas 1, 2 and 3) we apply the Ford-Fulkerson theorem to determine the value of the carrying capacity of an interbank network.

3 The Efficient Network

The efficiency of an interbank network in providing coverage from liquidity risk, depends on the banks' choice about deposits placements and withdrawals. Ex ante, before the occurrence of the liquidity shock, the banks decide how much to deposit in other banks, and in which banks to place such deposits. These choices determine the shape of the network N, i.e., the set of existing links Λ and the capacities of such links. We take such decisions as given, and we focus the attention on three network structures: the star-shaped, complete and incomplete regular networks.

[6] A *cut* is a partiton $\{U, \overline{U}\}$ of the set of nodes of a flow network such that $S \subseteq U$ and $T \subseteq \overline{U}$, i.e. all source nodes are in U and all sink nodes are in $\overline{U}$. The *capacity* of a cut is the sum of the capacities of its *forward links*, which are the links going from U into $\overline{U}$.

[7] To obtain the maximum flow, the network administrator must ensure that all forward links that cross the minimum cut are filled to capacity, while all backward links that cross such a cut must carry no flow.

The second relevant decision that banks have to made after the shock occurs, is how much to withdraw and from whom. This decision determines the actual liquidity transfer that, starting from the surplus nodes, reaches the deficit nodes. The efficient network is derived under the assumption that the withdrawals decisions of the banks in Ω are coordinated by a network administrator (who acts as a central planner).[8] The objective of the planner is to achieve the complete coverage of liquidity risk with the minimum amount of interbank deposits c hold by banks.

We also make the following assumptions to simplify the analysis and the comparison of complete, incomplete and star-shaped networks:

1. In the complete and incomplete networks, all banks are identical. In the star-shaped network, all peripheral banks are identical and the composition of the balance sheet of the central node is assumed to be proportional to the one of a peripheral node, i.e. they have the same balance sheet ratios.

2. All deposits in a network have the same size: c_{ij} is the same for all l_{ji} in Λ.

Assumption 1 is not strictly necessary. All the results presented below hold as long as all banks in Ω are 'proportional' to one another, that is they have the same ratios between pairs of their balance sheet items. Assumption 2 is coherent with assumption 1 and the assumption of symmetric shock $(\bar{\delta}, -\bar{\delta})$: if identical banks face the same liquidity risk, it is plausible that they deposit in other banks the same amount of deposits.

To evaluate and compare the performance of the three different interbank flow networks, we proceed in two steps. For each network considered, we first establish the conditions that ensure that the complete coverage of liquidity risk is feasible. In other words, we identify sufficient and necessary conditions for the existence of interbank liquidity transfers which are large enough to cover the liquidity shortages generated by the liquidity shock. Then, given those conditions, we establish for each network structure the minimum interbank deposit that supports the optimal re-allocation of liquidity.

3.1 Full Coverage of Liquidity Risk

An interbank network can provide full coverage of liquidity risk only if its carrying capacity is large enough to reallocate deposits across banks, satisfying the liquidity needs generated

[8] In Section 4 we analyze decentralized banks withdrawal decisions.

by the changes in customers' deposits. To be able to fully cover the liquidity risk posed by symmetric shocks $(\overline{\delta}, -\overline{\delta})$, an interbank network L must be able to carry a flow larger than or equal to $(n/2)\overline{\delta}$, which is the total liquidity need of the banks in deficit.

It is important to highlight that the coverage of liquidity risk in our analysis corresponds to the first-best allocation characterized by Allen and Gale [5]. To apply flow network theory to their environment, it is sufficient to associate to the banks with an high liquidity need a sink node and to the banks with a low liquidity need a source node.[9] Like in Allen and Gale, where the first best allocation is obtained with different sizes of interbank deposits, depending on the network considered, full coverage of liquidity risk is here guaranteed with different interbank exposures depending on the network at hand. We generalize the analysis by Allen and Gale considering networks with an arbitrary number of banks, and alternative structures like the star-shaped.

1. Star-shaped Networks. A *star-shaped* interbank network consists of a central node, $\omega_c \in \Omega$, that places a deposit in each of the remaining $n-1$ peripheral banks which, in turn, place their deposits in ω_c and exchange no deposits among themselves. Let $L^s = \left\{\Omega, \Lambda^s, S, T, \Lambda^+, \Lambda^-, \overline{\Gamma}\right\}$ be a star-shaped interbank liquidity flow network that complies with the above assumptions, i.e. $\Lambda^s = \{l_{ic}, l_{ci} | i \in \Omega \backslash \omega_c\}$, and the capacity function $\overline{\Gamma}$ assigns i) a capacity equal to $\overline{\delta}$ to all the links in Λ^+ and Λ^-, and ii) a capacity equal to c to all the links in Λ^s. That is, c is the amount deposited by each peripheral bank in ω_c and is the amount that the central node deposits in each of the peripheral ones. Moreover, let Ω^+ be the set of banks that experience a positive change in customers deposits, and Ω^- be the set of banks that experience a negative change in customers deposits.

Lemma 1 *The carrying capacity of a star-shaped interbank liquidity flow network L^s is equal to $(n/2)\overline{\delta}$ if $c \geq \overline{\delta}$. Conversely, if $c < \overline{\delta}$, then the carrying capacity of L^s is equal to $(n/2)c$.*

Thus, for all $c \geq \overline{\delta}$, the carrying capacity of a network L^s is equal to the value of a complete liquidity transfer. In a star-shaped network the full coverage of liquidity risk is

[9] Following the notation of Allen and Gale [5], let us indicate with ω_H (ω_L) the high (low) liquidity need faced by a bank and with $\gamma = (\omega_H + \omega_L)/2$ the expected liquidity need. The flow from the source nodes would be $\overline{\delta} = \gamma - \omega_L$ and the flow to the sink node would be $-\overline{\delta} = -(\omega_H - \gamma)$. Notice that $\omega_H - \gamma = \gamma - \omega_L$. In Allen and Gale banks use the network to insure against customer deposits fluctuations from the expected value γ. In our model, such fluctuations have expected value equal to zero

attainable as long as the interbank deposits of a peripheral bank are larger than or equal to the possible liquidity need $\overline{\delta}$. This upper bound is achieved by the social planner that coordinates banks' withdrawals. We have the following

Proposition 1 *In a star-shaped interbank liquidity flow network L^s, full coverage of liquidity risk is achieved with $c \geq \overline{\delta}$ if interbank deposits withdrawals are coordinated.*

2. Complete Networks. In a *complete* interbank network, each bank places a deposit in every other bank: $\Lambda^c = \{l_{ij} | i \neq j; i, j = 1, ..., n\}$. Let $L^c = \left\{\Omega, \Lambda^c, S, T, \Lambda^+, \Lambda^-, \overline{\Gamma}\right\}$ be a complete interbank liquidity flow network where the capacity function $\overline{\Gamma}$ assigns i) a capacity equal to $\overline{\delta}$ to all the links in Λ^+ and Λ^-, and ii) the same capacity c_{ij} to all the links in Λ^s, where $\sum_j c_{ij} = (n-1)c_{ij} = c$. As above, let Ω^+ be the set of surplus banks and Ω^- be the set of deficit banks.

Lemma 2 *The carrying capacity of a complete interbank liquidity flow network L^c is equal to $(n/2)\overline{\delta}$ if $c \geq \frac{n-1}{n}2\overline{\delta}$. Conversely, if $c < \frac{n-1}{n}2\overline{\delta}$, then the carrying capacity of L^c is equal to $\frac{n^2}{4(n-1)}c$ which, for $c < \frac{n-1}{n}2\overline{\delta}$, is smaller than $(n/2)\overline{\delta}$.*

Therefore the carrying capacity of L^c is equal to $\min[(n/2)\overline{\delta}, \frac{n^2}{4(n-1)}c]$. A complete interbank network is able to support the complete liquidity transfer if and only if the total interbank deposits c hold by each bank is at least $\frac{n-1}{n}2\overline{\delta}$. In a complete network, this upper bound to the liquidity transfers is achieved by the central planner. We have the following

Proposition 2 *In a complete interbank liquidity flow network L^c, full coverage of liquidity risk is achieved with $c \geq \frac{n-1}{n}2\overline{\delta}$ if interbank deposits withdrawals are coordinated.*

3. Incomplete Regular Networks. Generic incomplete interbank networks, i.e. without restrictions on their shape, are difficult to analyze. For the sake of tractability, we focus the attention on incomplete networks which are *regular* and *bilateral*. That is, networks where i) all nodes have the same degree, and the indegree and outdegree of each node are equal;[10] ii) deposits are pairwise mutual. In order to ensure that an incomplete network is capable of reallocating liquidity across banks, it is necessary that the network is connected for any realization of the shock. This connectivity is guaranteed if each bank

[10]The indegree (outdegree) of a node is the number of its incoming (outgoing) links.

places deposits in at least $n/2$ banks. If this condition is not met, a shock may divide the network into two disconnected sub-networks with no carrying capacity.

Let $L^r = \{\Omega, \Lambda^r, S, T, \Lambda^+, \Lambda^-, \overline{\Gamma}\}$ be an *incomplete regular* interbank liquidity flow network where the capacity function $\overline{\Gamma}$ assigns i) a capacity equal to $\overline{\delta}$ to all the links in Λ^+ and Λ^-, and ii) the same capacity c_{ij} to all the links in Λ^r. Let k be the indegree (and the outdegree) of all nodes in Ω, then $kc_{ij} = c$ is the amount of interbank deposits held by each bank. We have the following

Lemma 3 *The carrying capacity of an incomplete regular interbank liquidity flow network L^r with degree k (with $k = n/2, ...n-1$) is equal to $(n/2)\overline{\delta}$ if and only if $c_{ij} \geq \frac{1}{k+1-n/2}\overline{\delta}$, hence $c \geq \frac{k}{k+1-n/2}\ \overline{\delta}$. Conversely, if $c_{ij} < \frac{1}{k+1-n/2}\overline{\delta}$, then the carrying capacity of L^r is equal to $\frac{n}{2}(k+1-n/2)c_{ij}$ which is smaller than $(n/2)\overline{\delta}$.*

Thus, in an incomplete regular network with degree equal to $k \geq n/2$ and a link capacity equal to $c_{ij} \geq \frac{1}{k+1-n/2}\overline{\delta}$, a complete coverage of liquidity risk is possible. Under these conditions, a central planner can achieve the full coverage of liquidity risk. We have the following

Proposition 3 *In an incomplete regular interbank liquidity flow network L^r with degree $k \geq n/2$, full coverage of liquidity risk is achieved with $c \geq \frac{k}{k+1-n/2}\ \overline{\delta}$ if interbank deposits withdrawals are coordinated.*

We are now in a position to compare the amount of interbank deposits that banks have to hold to cover the liquidity risk in the three different interbank networks. If $n = 2$ the amount of interbank deposits hold in the three networks is the same and equal to the liquidity shock $\overline{\delta}$. This is quite intuitive since in a two-banks network the structure of the network does not play any role. If $n \geq 3$ the amount of interbank deposits hold in the star-shaped network is strictly less than the interbank deposit in the complete network. In the incomplete network the total interbank deposits c that each bank holds depends on the value of the degree k. Since c is decreasing in k, the minimum interbank deposits in the incomplete regular network is obtained when $k = n-1$, which implies $c = \frac{n-1}{n}2\overline{\delta}$. In an incomplete regular network the amount of interbank deposits hold by each bank it is at best equal to the amount in the complete network, otherwise it is higher.

While the star-shaped network allows each bank to hold interbank deposits equal to the liquidity shock $\overline{\delta}$, the complete and the incomplete regular networks induce banks to hold an amount of interbank deposits that exceeds the value $\overline{\delta}$. And this is so even if the interbank withdrawals are coordinated by a network administrator. Notice that, for n large enough, a bank in the star-shaped network holds an amount of interbank deposits that is roughly half of the amount that a bank holds in the complete network. The saving on interbank holdings is quite sizable when the number of banks in the network becomes large.

Why the complete and, *a fortiori,* the incomplete network force banks to hold too much interbank deposits? We already noticed that interbank deposits between two banks with the same liquidity shock (in particular, banks in deficit) are redundant since they do not increase the carrying capacity of the interbank network. Formally, in all the three network structures the cut (Ω^+, Ω^-) is the one with the smallest capacity, that is the one that sets the upper bound to the value of the feasible flows. Note that no link between pairs of banks in Ω^+ or in Ω^- crosses such a cut. For this reason, in L^c and L^r, the cross-holding of deposits among banks in Ω^- (as well as the cross-holding of deposits among banks in Ω^+) increases the amount of interbank exposures in the network without increasing the systemic capability of transferring liquidity from surplus banks to deficit banks. In the star-shaped network, instead, there is no cross-holding of deposits among deficit banks or among surplus banks, thus there are no excessive interbank exposures. In a star-shaped network each bank can hold an amount of interbank deposits equal to the shock $\overline{\delta}$ without spare interbank exposure.

We summarize the previous analysis in the following

Proposition 4 *Assume $n \geq 3$, then the star-shaped network achieves the full coverage of liquidity risk with the minimum amount of interbank deposits.*

We have implicitly assumed so far that reducing the size of interbank deposits is valuable since it reduces the risk of contagion without further defining such risk. We are going to analyze this issue in the next two sections considering two forms of financial contagion: i) counterparty risk, i.e. the risk that a debtor defaults on its obligations, and ii) systemic risk, i.e. the risk that the network is affected by a domino effect capable of propagating the losses originated from a shock.

3.2 Counterparty risk

The counterparty risk faced by a bank is the risk of suffering a loss due to the default of a debtor. Naturally, such a loss grows with the value of the credit granted to the defaulting debtor. In an interbank network L, the default of a bank $\omega_i \in \Omega$ inflicts a loss to the banks that placed a deposit in ω_i. Taking the defaults as uncorrelated events, the exposure to counterparty risk of a member of L grows linearly with its total intra-network exposures.[11]

Let the defaults of the nodes in Ω be represented by binomial random variables (solvent/insolvent) which are identically and independently distributed. Let p be the probability of default of a node, and $\lambda^e \in [0, 1]$ be the share of the value of a financial claim that is expected to be lost upon default of the debtor (i.e., the expected loss-given-default). Then, the loss that a node $\omega_i \in \Omega$ expects to incur because of the default of one or more of its debtors in $P(\omega_i) = \{\omega_j | l_{ji} \in \Lambda\}$ is equal to $p\lambda^e \sum_j c_{ij}$, for $j \in P(\omega_i)$. Since $\sum_j c_{ij}$ is smaller in L^s than in L^c, and smaller in L^c than in L^r, we have that the star-shaped network minimizes the exposure of banks to counterparty risk, followed - in order - by the complete network and the incomplete regular network.

3.3 Systemic risk

Systemic risk is broadly defined as the risk that, in a network of agents connected to one another by financial obligations, the initial default of one or more agents can induce the default of otherwise solvent agents and possibly lead to a systemic crisis involving the entire network. To compare the exposure to systemic risk of the three network structures, we represent them as *financial flow networks*.[12] We turn an interbank network $N = (\Omega, \Lambda)$ into a flow network adding

1. A set $A = \{a^k\}$ of *source nodes*, i.e., nodes with no incoming links, that represent the external assets held by the members of Ω.

2. Two *sinks* (i.e. terminal nodes with no outgoing links) Q and H, where Q represents

[11] There is evidence that defaults in financial systems are correlated. This evidence has challenged traditional conterparty risk analysis, and there are attempts to embed correlated defaults in the evaluation of counterparty risk (see, inter alia, Brigo and Pallavicini [10]). Such analysis would fall beyond the limits of this paper and we content ourselves with considering counterparty risk with uncorrelated defaults. We do however take into account the clustering of defaults in the analysis of systemic risk in Section 3.3.

[12] For a more general analysis of domino effects in financial flow networks see Eboli [16].

the shareholders who own the equity of the agents in Ω, and H represents the households who hold debt claims, in the form of deposits and bonds, against the agents in Ω.

3. Three sets of links: $\Lambda^a = \{l_i^k\}$, that connect the external assets a^k to their owners in Ω; ii) $\Lambda^q = \{l_Q^i\}$, that connect the agents in Ω to the sink Q, i.e., to their shareholders; and iii) $\Lambda^h = \{l_H^i\}$ that connect the agents in Ω to the sink H, i.e., to their bondholders and depositors.

Let i) $N^c = \{\Omega, \Lambda^c, A, Q, H, \Lambda^a, \Lambda^q, \Lambda^h\}$, ii) $N^r = \{\Omega, \Lambda^r, A, Q, H, \Lambda^a, \Lambda^q, \Lambda^h\}$ and iii) $N^s = \{\Omega, \Lambda^s, A, Q, H, \Lambda^a, \Lambda^q, \Lambda^h\}$ be, respectively, the financial flow networks corresponding to the complete, incomplete and star-shaped interbank liquidity networks defined above.

The resiliency to systemic risk of the different networks is evaluated assuming that at least one of the banks in Ω goes bankrupt due to an exogenous solvency shock. This shock is defined as a loss of value of some assets a^k that causes the insolvency of one or more banks in the network. This is called the set of *primary* defaults and it is indicated with Φ. To focus on direct balance-sheet contagion, we assume that the solvency shock does not affect the value of the assets a_i of the banks in $\Omega \backslash \Phi$. Default contagion occurs if the losses transmitted by the banks in Φ to their creditors are large enough to cause *secondary* defaults, i.e. the default of one or more banks in $\Omega \backslash \Phi$.

As a measure of the systemic risk, we characterize two thresholds of default contagion. The *first threshold* of contagion τ_1 of a network is the smallest shock that is sufficient to cause secondary defaults. The *final threshold* of contagion τ_2 is the smallest shock that is capable of inducing the failure of *all* nodes in the network. Therefore, ceteris paribus, the higher these two thresholds are and the larger the size of the external shock capable of inducing default contagion is (i.e., the more resilient the network is). Particular relevance has τ_2 since it indicates the threshold of the meltdown of the entire network.

We determine both thresholds for the complete and the star-shaped interbank networks, while for the incomplete regular network we must content ourselves with considering the first threshold of contagion.[13] We assume that the total stock of equity, $E = \sum_{i \in \Omega} e_i$, and the total external debt, $H = \sum_{i \in \Omega} h_i$, are the same in the star-shaped and in the complete

[13] Final thresholds of contagion of incomplete regular networks can be characterized only for 'cycle-shaped' networks (see Eboli [16]). However, cycle-shaped networks cannot provide full coverage of liquidity risk like any incomplete regular networks with degree $k < n/2$ (see also Proposition 3). For this reason we do not consider cycle-shaped networks.

networks.[14] We start to analyze the complete network since it is easier to compute the two thresholds of contagion. Indeed, they coincide independently on the composition of the set of primary defaults. We have the following

Proposition 5 *In a complete network N^c the first threshold of contagion τ_1^c and the final threshold of contagion τ_2^c coincide and are equal to*

$$\tau_1^c = \tau_2^c = ne_i + e_i \frac{h_i}{d_{ij}}, \tag{1}$$

where $d_{ij} = c_{ji}$ is the amount deposited by bank j in bank i.

Under the assumption that all banks in N^c are equal to one another, we have $E = ne_i$, $H = nh_i$ and $d_i = (n-1)d_{ij}$. Therefore the total intra-network exposures of the banks in N^c is equal to $D^c = \sum_{i\in\Omega} d_i = (n-1)\sum_{i\in\Omega} d_{ij} = (n-1)nd_{ij}$. Hence equation (1) can be rewritten as

$$\tau_1^c = \tau_2^c = E + \frac{EH}{D^c}\frac{n-1}{n}. \tag{2}$$

In the star-shaped network the two thresholds of contagion may coincide or not depending if the central node is in the set of primary defaults or not. We have the following

Proposition 6 *If the central node ω_c is in the set of primary defaults Φ, then in a star-shaped network N^s the first threshold of contagion τ_1^s and the final threshold of contagion τ_2^s coincide and we have either*

$$\tau_1^s = \tau_2^s = (n-1)e_p + e_c + e_p\frac{h_c}{d_p} \text{ for } \Phi = \omega_c, \tag{3}$$

or

$$\tau_1^s = \tau_2^s = \left[(n-1)e_p + e_c + e_p\frac{h_c}{d_p}\right]\left(1+\frac{h_p}{d_p}\right) - \sigma_c\frac{h_p}{d_p} \text{ for } \Phi = \{\omega_c, \omega_p | \text{ for some } p \in \Omega\backslash\omega_c\}, \tag{4}$$

where σ_c is the loss of value of assets a_c borne by the central node.[15]

[14]This assumption is made to compare the resiliency of different networks. In the present analysis this assumption is not restrictive since banks do not face any agency problem. In an environment with moral hazard instead the level of bank capitalization affects the network structure (see Castiglionesi and Navarro [12]).

[15]In order not to violate the definition of first threshold, σ_c is restricted to be smaller than or equal to $e_c + e_p(n-m) + e_p h_c/d_p$. In turn, this implies that the threshold in (4) is strictly larger than the one in (3).

From equations (3) and (4) we have that the resiliency of a star-shaped network depends on the ratio h_c/d_p that, since in the star-shaped network $d_c = (n-1)d_p$, can be written as $h_c(n-1)/d_c$. Thus, equations (3) and (4) depend on the ratio between customer and intra-network deposits of the central node. To isolate this effect and focus the analysis solely on the shape of a network, we have assumed that the balance sheet of the central node is a re-scaling of the one of a peripheral node, i.e. center and peripheral nodes have the same balance sheet ratios, in particular $h_c/d_c = h_p/d_p$ and $e_c/d_c = e_p/d_p$.

In a star-shaped network, the total intra-network exposures of the banks in N^s is $D^s = \sum_{p\in\Omega\backslash\omega_c} d_p + d_c = 2(n-1)d_p$, while the stocks of equity and customer deposits are $E = (n-1)e_p + e_c$ and $H = (n-1)h_p + h_c$, respectively. Under the assumption of equal balance sheet ratios, we have that $e_c = E/2$, $h_c = H/2$, $e_p = E/2(n-1)$ and $h_p = H/2(n-1)$. Thus, equations (3) and (4) can be rewritten, respectively, as:

$$\tau_1^s = \tau_2^s = E + \frac{EH}{D^s}\frac{1}{2} \text{ for } \Phi = \omega_c \tag{5}$$

$$\tau_1^s = \tau_2^s = \left[E + \frac{EH}{D^s}\frac{1}{2}\right]\left(1 + \frac{H}{D^s}\right) - \sigma_c\frac{H}{D^s} \text{ for } \Phi = \{\omega_c, \omega_i| \text{ for some } i \in \Omega\backslash\omega_c\} \tag{6}$$

If the central node ω_c is not in the set of primary defaults, the first and the final thresholds of contagion of the star-shaped network do not coincide. We have the following

Proposition 7 *If $\omega_c \notin \Phi$, then in a star-shaped network the first and final thresholds of contagion are, respectively, equal to*

$$\tau_1^s = me_p + e_c\left(1 + \frac{h_p}{d_p}\right) \tag{7}$$

and

$$\tau_2^s = \left[(n-1)e_p + e_c + e_p\frac{h_c}{d_p}\right]\left(1 + \frac{h_p}{d_p}\right), \tag{8}$$

where m is the minimum number of peripheral defaults which is sufficient to induce the default of the central node, i.e. m is such that $\sum_{p=1}^{m} d_p = e_c$.

Assuming equal balance sheet ratios, equations (7) and (8) can be rewritten, respectively, as

$$\tau_1^s = e_c + me_p + \frac{EH}{D^s}\frac{1}{2} \tag{9}$$

and

$$\tau_2^s = \left[E + \frac{EH}{D^s}\frac{1}{2}\right]\left(1 + \frac{H}{D^s}\right). \tag{10}$$

The relative values of these contagion thresholds, hence the relative exposure to systemic risk of the networks N^c and N^s, depend on the magnitude of the intra-network deposits D^c and D^s. As shown in Section 3.1, the amount of intra-network deposits required to achieve full coverage of liquidity risk in the star-shaped network is smaller than the amount required in the complete network, that is $D^c = \frac{n-1}{n} 2D^s$. It follows that the threshold in (10) is larger than the one in (6) which, in turn is larger than the one in (5) which, in turn, is equal to (2). That is, when peripheral nodes are in the set of primary defaults (either with or without the central node), the magnitude of the minimum shock sufficient to induce the defaults of all banks in the star-shaped network is always larger than the magnitude of the corresponding shock in the complete network. Conversely, if the initial shock involves the central node only, then the risk of a complete system meltdown is the same in both networks.

This result is explained by the fact that the central node provides some shelter to the peripherals banks which are not in the set of primary defaults, since it absorbs part of the losses with its own equity and diverts another part of the flow of losses towards the holders of its customer deposits h_c. Conversely, when the central node is the only node in the set of primary default the shelter role cannot be played and the exposure to systemic risk of the two network structures is the same. To sum up, the complete meltdown of a star-shaped interbank network is less likely to occur than the meltdown of a complete interbank network for any probability distribution over the set of possible external shocks.[16]

4 Decentralized Interbank Withdrawals

In this section we remove the assumption of centralized banks withdrawal decisions and show under which conditions decentralized withdrawal decisions can replicate the efficient network. We still consider the same network structures, but with two alternative individual withdrawal decisions of the banks:

- *Selective withdrawal*: banks in liquidity deficit withdraw deposits first from the

[16]Notice that it is possible to increase the first threshold of a star-shaped network by re-allocating the equity endowments. If every bank in the periphery transfer an amount x of capital to the central node, then e_c would increase by an amount $(n-1)x$. At the same time the amount me_p would decrease by mx. Since $m \leq (n-1)$, then τ_1^s in equation (9) increases. Therefore, in order to increase the resiliency of the central node (and increase the first threshold of contagion) peripheral banks should be relatively less capitalized than the bank acting as center (which is a systemic bank).

neighbors that have a liquidity surplus, if any.

- *Pro-rata withdrawal*: banks in liquidity deficit withdraw the same quota of deposits from all their neighbors.

The selective withdrawal in the one considered in the traditional micro-banking literature (see, for example, Allen and Gale [5]) since it is consistent with the observability of the bank's liquidity shock (contrary to the depositor's shock which is private information). The selective withdrawal in the interbank market implements the efficient allocation since it guarantees the same feasibility conditions attained by the social planner. The pro-rata withdrawal is more a behavioral assumption (not consistent with the observability of the bank's liquidity shock) and it would be unable to implement the efficient allocation in the micro-banking literature.

Following Section 3.1, we consider the effects of the two decentralized withdrawal policies and find, for each network structure, the minimum interbank deposit that supports the full reallocation of liquidity. This upper bound is achieved or not, depending on the withdrawal policies undertaken by banks.

When interbank withdrawals are selective, we have the following

Proposition 8 *Assume banks make selective interbank withdrawal decisions. Then: i) in a star-shaped interbank liquidity flow network L^s, full coverage of liquidity risk is achieved with $c \geq \overline{\delta}$; ii) in a complete interbank liquidity flow network L^s, full coverage of liquidity risk is achieved with $c_{ij} \geq \frac{n-1}{n} 2\overline{\delta}$; iii) in an incomplete regular interbank liquidity flow network L^r, full coverage of liquidity risk is not guaranteed.*

Therefore, if interbank withdrawals follow a selective criteria, both the star-shaped and the complete decentralized networks achieve the full coverage of liquidity risk with the amount of interbank deposits established in Section 3.1. Hence, all the results presented in Sections 3.2 and 3.3 carry over to networks with decentralized withdrawal decisions when these occur in a selective manner. In the case of incomplete network instead, the full coverage of liquidity risk in not guaranteed in a network with uncoordinated withdrawal decisions.

When interbank withdrawals are made pro-rata, we have the following

Proposition 9 *Assume banks make pro-rata withdrawal decisions. Then: i) in a star-shaped interbank liquidity flow network L^s, full coverage of liquidity risk is achieved with c*

$\geq 2\overline{\delta}$; ii) in a complete interbank liquidity flow network L^s, full coverage of liquidity risk is achieved with $c_{ij} \geq \frac{n-1}{n} 2\overline{\delta}$; iii) in an incomplete regular interbank liquidity flow network L^r, full coverage of liquidity risk is not guaranteed.

If banks follow a pro-rata withdrawal policy the complete and the incomplete networks deliver the same solution of the selective withdrawal policy. However, the star-shaped network obtains the full coverage of liquidity risk with an higher amount of interbank deposit than in the efficient network analyzed in Section 3.1. If interbank deposits are withdrawn pro-rata, banks in the star-shaped network hold an amount of interbank deposits higher than what they have to hold in a complete network.

The central node in the star-shaped network however can easily enforce interbank withdrawals in a selective manner (and not pro-rata) since it is the only node that has more than one interbank deposits and therefore it is the only one that decides on the withdrawal policy. The star-shaped network avoids any coordination failures in the withdrawing policy. The star-shaped network seems to be an optimal device to enforce the selective withdrawal policy, allowing each bank to hold the efficient amount of interbank deposits.

5 Conclusions

In this paper we investigate the efficiency of different interbank network structures in providing coverage of liquidity risk. We compare the performance of three kind of networks: star-shaped, complete and incomplete regular networks. For this purpose, we represent such interbank liquidity networks as flow networks and apply some results of flow network theory. We find that star-shaped networks achieve a complete transfer of liquidity from banks in surplus to banks in deficit with the smallest amount of interbank deposits held by each bank which is equal to the possible liquidity deficit. Complete networks instead achieve the full coverage of liquidity risk if each bank holds an amount of deposits which is roughly twice the amount hold in the star-shaped network. Finally, incomplete regular networks seem to be rather ineffective in transferring liquidity. These networks provide complete insurance against liquidity risk only if i) each bank is connected to at least half of the banks in the network, and ii) banks' withdrawals are coordinated. Even under these conditions, liquidity insurance in incomplete networks requires interbank deposits which are larger than the ones required in complete and star-shaped networks.

The benefits of holding small interbank deposits lies in the containment of the risk of financial contagion. We argue that star-shaped networks are the least exposed to counterparty risk and systemic risk thanks to the smallest interbank deposits holding. Since counterparty risk faced by a bank grows with the size of its interbank exposures, star-shaped networks are the less exposed to counterparty risk. Finally, star-shaped networks are less exposed than complete networks to the risk of systemic events thanks to the shelter role of the central node.

6 Appendix

In order to characterize the carrying capacity of the different interbank flow networks (in Lemma 1, 2 and 3), we apply the *minimum cut - maximum flow* theorem by Ford and Fulkerson:

Theorem 1 (Ford and Fulkerson 1956) *In every flow network, the maximum total value of a flow equals the minimum capacity of a cut.*

Proof of Lemma 1. This lemma can be established with informal arguments, under the assumption that the central node withdraws deposits only from banks in liquidity surplus (see proof of Proposition 8). However, we present the more general proof.

Let L^s be a star-shaped interbank liquidity network that is hit by a symmetric liquidity shock $(\overline{\delta}, -\overline{\delta})$ and let Ω^+ (Ω^-) be the set of banks that face a positive (negative) change of their customer deposits. Let $(U, \overline{U})$ be a cut of L^s and recall that, by definition, $S \subseteq U$ and $T \subseteq \overline{U}$. Let $X = U \backslash S = \Omega \cap U$ be the set of banks in U and, correspondingly, let $Y = \overline{U} \backslash T = \Omega \cap \overline{U}$ be the set of banks in $\overline{U}$. Let x^- be the number of deficit banks in X, i.e. $x^- = |\Omega^- \cap U|$, and y^+ be the number of surplus banks in Y, i.e. $y^+ = |\Omega^+ \cap \overline{U}|$. Recall that the capacity $\Gamma(U, \overline{U})$ of a cut is the sum of the capacities of its forward links, i.e. the links starting in U and ending in $\overline{U}$, which – in the case at hand – are: i) the links going from X into Y, with weight c; ii) the links going from deficit nodes in X into their sinks in T, with weight $\overline{\delta}$; and iii) the links going from the source nodes in S into the surplus banks in Y, with weight $\overline{\delta}$. The number of links starting in X and ending in Y is equal to the cardinality of X, $|X|$, if the central node ω_c is in Y, otherwise it is equal to $|Y|$ if ω_c is in X. Thus the capacity of a cut $(U, \overline{U})$ of L^s is:

$$\Gamma(U, \overline{U}) = |X|\, c + x^- \overline{\delta} + y^+ \overline{\delta} \text{ if } \omega_c \in Y, \tag{11}$$

$$\Gamma(U,\overline{U}) = |Y|\, c + x^{-}\overline{\delta} + y^{+}\overline{\delta} \text{ if } \omega_c \in X. \tag{12}$$

The partitions (X,Y) of Ω that minimize the second and the third addenda of equation (11) are i) for $|X| \leq n/2$, the ones where X is composed solely by surplus nodes, and ii) for $|X| > n/2$, the ones where X includes the set of surplus nodes. Correspondingly, the partitions (X,Y) of Ω that minimize equation (12) are i) for $|Y| \leq n/2$, the ones where Y is composed solely by deficit nodes, and ii) for $|Y| > n/2$, the ones where Y includes the set of deficit nodes. Then let the cut $(\widetilde{X},\widetilde{Y})$ of Ω be such that: $\widetilde{X} \subseteq \Omega^{+}$, for $\left|\widetilde{X}\right| \leq n/2$, and $\widetilde{X} \supset \Omega^{+}$, for $\left|\widetilde{X}\right| > n/2$, which implies $\widetilde{Y} \subseteq \Omega^{-}$, for $\left|\widetilde{Y}\right| \leq n/2$, and $\widetilde{Y} \supset \Omega^{-}$, for $\left|\widetilde{Y}\right| > n/2$. Since we are seeking the partition that minimizes $\Gamma(U,\overline{U})$, we focus on such partitions $(\widetilde{X},\widetilde{Y})$ of Ω. Note that, for $\left|\widetilde{X}\right| \leq n/2$, $y^{+} = (n/2 - \left|\widetilde{X}\right|)$ and $x^{-} = 0$, while for $\left|\widetilde{X}\right| > n/2$, $y^{+} = 0$ and $x^{-} = (\left|\widetilde{X}\right| - n/2)$. Under such a restriction, rewrite equation (11) as:

$$\Gamma(U,\overline{U}) = \left|\widetilde{X}\right| c + y^{+}\overline{\delta} = \left|\widetilde{X}\right| c + \left|n/2 - \left|\widetilde{X}\right|\right| \overline{\delta}.$$

Thus, if $\omega_c \in Y$: i) for $c = \overline{\delta}$, $\Gamma(U,\overline{U}) = \frac{n}{2}\overline{\delta}$ for all $\left|\widetilde{X}\right|$; ii) for $c < \overline{\delta}$, $\Gamma(U,\overline{U})$ is minimal – and equal to $\frac{n}{2}c$ – for $\left|\widetilde{X}\right| = n/2$, iii) for $c > \overline{\delta}$, $\Gamma(U,\overline{U})$ is minimal – and equal to $\frac{n}{2}\overline{\delta}$ – for $\left|\widetilde{X}\right| = n$ and for $\left|\widetilde{X}\right| = 0$.

By the same token, rewrite equation (12) as:

$$\Gamma(U,\overline{U}) = \left|\widetilde{Y}\right| c + x^{-}\overline{\delta} = \left|\widetilde{Y}\right| c + \left|n/2 - \left|\widetilde{Y}\right|\right| \overline{\delta}.$$

Thus, if $\omega_c \in X$: i) for $c = \overline{\delta}$, $\Gamma(U,\overline{U}) = \frac{n}{2}\overline{\delta}$ for all$\left|\widetilde{Y}\right|$; ii) for $c < \overline{\delta}$, $\Gamma(U,\overline{U})$ is minimal – and equal to $\frac{n}{2}c$ – for $\left|\widetilde{Y}\right| = n/2$, iii) for $c > \overline{\delta}$, $\Gamma(U,\overline{U})$ is minimal – and equal to $\frac{n}{2}\overline{\delta}$ – for $\left|\widetilde{Y}\right| = n$ and for $\left|\widetilde{Y}\right| = 0$. ■

Proof of Proposition 1. Full coverage of liquidity risk is achieved if the interbank liquidity flow network can move an amount of deposits equal to $(n/2)\overline{\delta}$ from the source nodes into the sink nodes, i.e. if the carrying capacity of L^s is equal to or larger than $(n/2)\overline{\delta}$. By lemma 1, such a capacity is achieved if $c \geq \overline{\delta}$ and, by the definition of carrying capacity of a flow network, a liquidity transfer equal to $(n/2)\overline{\delta}$ is achieved in L^s if interbank deposits withdrawals are coordinated. ■

Proof of Lemma 2. Let L^c be a complete interbank liquidity network that is hit by a symmetric liquidity shock, as defined above, and let Ω^{+} (Ω^{-}) be the set of banks that face a positive (negative) change of their customer deposits. Let $(U,\overline{U})$ be a cut of L^c and, as above in the proof of lemma 1, let i) $X = U\backslash S = \Omega \cap U$ be the set of banks in U; ii) $Y = \overline{U}\backslash T = \Omega \cap \overline{U}$ be the set of banks in $\overline{U}$; iii) x^- be the number of deficit banks in X; iv) y^+be the number of surplus banks in Y. Then the capacity $\Gamma(U, \overline{U})$ of a cut in L^c is

$$\begin{aligned}\Gamma(U, \overline{U}) &= |X|\,|Y|\,c_{ij} + x^-\overline{\delta} + y^+\overline{\delta} \\ &= |X|\,(n - |X|)c_{ij} + x^-\overline{\delta} + y^+\overline{\delta},\end{aligned}$$

where $|X|\,|Y|\,c_{ij}$ is the sum of the capacities of the links starting from banks in X and ending in banks in Y, the second addendum is the sum of the capacities of the links starting from deficit banks in X and ending in the sink nodes in T and, finally, the third addendum is the sum of the capacities of the links starting from the source nodes in S and ending in surplus banks in Y. The partitions (X, Y) of Ω that minimize the second and the third addenda, $x^-\overline{\delta}$ and $y^+\overline{\delta}$, are the ones where the set X is composed solely by surplus nodes, for $|X| \leq n/2$, and includes the set of surplus nodes, for $|X| > n/2$. Then let the partition $(\widetilde{X}, \widetilde{Y})$ of Ω be such that $\widetilde{X} \subseteq \Omega^+$, for $\left|\widetilde{X}\right| \leq n/2$, and $\widetilde{X} \supset \Omega^+$, for $\left|\widetilde{X}\right| > n/2$. Since we are seeking the partition that minimizes $\Gamma(U, \overline{U})$, we restrict the analysis to the partitions $(\widetilde{X}, \widetilde{Y})$ of Ω. Under this restriction, for $\left|\widetilde{X}\right| \leq n/2$, $y^+ = (n/2 - |\widetilde{X}|)$ and $x^- = 0$, while for $\left|\widetilde{X}\right| > n/2$, $y^+ = 0$ and $x^- = (\left|\widetilde{X}\right| - n/2)$. Hence we rewrite the capacity $\Gamma(U, \overline{U})$ as:

$$\Gamma(U, \overline{U}) = \left|\widetilde{X}\right|(n - \left|\widetilde{X}\right|)c_{ij} + \left|n/2 - |\widetilde{X}|\right|\overline{\delta}. \tag{13}$$

The first addendum of equation (13) is a concave parabola with two minima at the extremes of the range of $|\widetilde{X}|$, i.e. it is minimal and equal to zero for $\left|\widetilde{X}\right| = 0$ and $\left|\widetilde{X}\right| = n$. The second addendum is a piece-wise linear and convex function with minimum equal to zero for $\left|\widetilde{X}\right| = n/2$. It can be checked by inspection that, for all $c_{ij} < \overline{\delta}$, equation (13) is m-shaped, with three local minima corresponding to $\left|\widetilde{X}\right| = 0$, $\left|\widetilde{X}\right| = n/2$, and $\left|\widetilde{X}\right| = n$. More precisely:

1. for $c_{ij} = \frac{2}{n}\overline{\delta}$, i.e. for $c = \frac{n-1}{n}2\overline{\delta}$, the capacity $\Gamma(U, \overline{U})$ has three global minima, for $\left|\widetilde{X}\right| = 0$, $\left|\widetilde{X}\right| = n/2$, and $\left|\widetilde{X}\right| = n$, all equal to $\frac{n}{2}\overline{\delta}$;
2. for $c_{ij} > \frac{2}{n}\overline{\delta}$, hence for $c > \frac{n-1}{n}2\overline{\delta}$, the capacity $\Gamma(U, \overline{U})$ has two global minima, for $\left|\widetilde{X}\right| = 0$ and $\left|\widetilde{X}\right| = n$, both equal to $\frac{n}{2}\overline{\delta}$;
3. for $c_{ij} < \frac{2}{n}\overline{\delta}$, i.e. for $c < \frac{n-1}{n}2\overline{\delta}$, the capacity $\Gamma(U, \overline{U})$ has a minimum for $|\widetilde{X}| = n/2$ and equal to $\left(\frac{n}{2}\right)^2 c_{ij} = \frac{n^2}{4(n-1)}c < \frac{n}{2}\overline{\delta}$. ■

Proof of Proposition 2. By lemma 2, a complete interbank network L^c has a carrying capacity sufficient to provide full coverage from liquidity risk, i.e. equal to $(n/2)\overline{\delta}$, if

$c \geq \frac{n-1}{n} 2\overline{\delta}$. By the definition of carrying capacity, such a liquidity transfer equal to $(n/2)\overline{\delta}$ is achieved if interbank deposits withdrawals are coordinated. ■

Proof of Lemma 3. Let L^r be an incomplete regular interbank liquidity network that is hit by a symmetric liquidity shock, as defined above, and let Ω^+ (Ω^-) be the set of banks that face a positive (negative) change of their customer deposits. Let $(U, \overline{U})$ be a cut of L^r and, as above in the proofs of lemma 1 and 2, let i) $X = U \backslash S = \Omega \cap U$ be the set of banks in U; ii) $Y = \overline{U} \backslash T = \Omega \cap \overline{U}$ be the set of banks in $\overline{U}$; iii) x^- be the number of deficit banks in X; iv) y^+ be the number of surplus banks in Y. Then the capacity $\Gamma(U, \overline{U})$ of a cut in L^c is $\Gamma(U, \overline{U}) = \Gamma(X, Y) + x^-\overline{\delta} + y^+\overline{\delta}$. The first addendum is the sum of the capacities of the links that go from X into Y, the second addendum is the sum of the capacities of the links starting from deficit banks in X and ending in the sink nodes in T and, finally, the third addendum is the sum of the capacities of the links starting from the source nodes in S and ending in surplus banks in Y. Note that:

1. By the assumption of bilateral expositions among the banks in Ω, we have that $\Gamma(X, Y) = \Gamma(Y, X)$. For convenience, below we exploit the fact that, for $|X| = n/2, ..., n$, $\Gamma(X, Y)$ is equal to $\Gamma(Y, X)$ for $|Y| = n - |X|$.

2. $\Gamma(X, Y)$, for $|X| = 0, ..., n/2$, is minimal for sets X which are maximally connected, i.e. sets X such that each node in X is connected to all other nodes in X. The same applies to $\Gamma(Y, X)$ for $|Y| = 0, ...n/2$: it is minimal for sets Y in which each node is connected to all other nodes in Y.

3. x^- and y^+ are, respectively, minimal (i.e. equal to zero) for sets X composed only by surplus nodes and for sets Y composed solely by deficit nodes.

Hence, since we are seeking the cut that minimizes $\Gamma(U, \overline{U})$, we restrict our attention to partitions $(\widetilde{X}, \widetilde{Y})$ of Ω where: i) for $\left|\widetilde{X}\right| \leq n/2$, $\widetilde{X}$ is maximally connected and $\widetilde{X} \subseteq \Omega^+$; ii) for $\left|\widetilde{Y}\right| \leq n/2$, $\widetilde{Y}$ is maximally connected and $\widetilde{Y} \subseteq \Omega^-$. Under this restriction we have that, for $|X| \leq n/2$, $y^+ = (n/2 - |\widetilde{X}|)$ and $x^- = 0$, while for $|X| \geq n/2$, $x^- = (|\widetilde{X}| - n/2) = (n/2 - |\widetilde{Y}|)$ and $y^+ = 0$. Hence we have

$$\Gamma(U, \overline{U}) = \left|\widetilde{X}\right| \left[(k+1) - \left|\widetilde{X}\right|\right] c_{ij} + \left(\frac{n}{2} - \left|\widetilde{X}\right|\right) \overline{\delta} \text{ for } \left|\widetilde{X}\right| \leq n/2, \tag{14}$$

$$\Gamma(U, \overline{U}) = \left|\widetilde{Y}\right| \left[(k+1) - \left|\widetilde{Y}\right|\right] c_{ij} + \left(\frac{n}{2} - \left|\widetilde{Y}\right|\right) \overline{\delta} \text{ for } \left|\widetilde{Y}\right| \leq n/2. \tag{15}$$

It can be checked by inspection that, for all $c_{ij} < \frac{1}{k}\overline{\delta}$, both equations (14) and (15) are strictly concave, with local minima at the extremes of the range of their respective arguments. More specifically, we have that:

1. for $c_{ij} = \frac{1}{k+1-n/2}\overline{\delta}$, i.e. for $c = \frac{k}{k+1-n/2}\overline{\delta}$, i) equation (14) has two global minima, for $\left|\widetilde{X}\right| = 0$, and $\left|\widetilde{X}\right| = n/2$; and ii) equation (15) has two global minima, for $\left|\widetilde{Y}\right| = 0$, and $\left|\widetilde{Y}\right| = n/2$. Each of such minima is equal to $\frac{n}{2}\overline{\delta}$;

2. for $c_{ij} > \frac{1}{k+1-n/2}\overline{\delta}$, hence for $c > \frac{k}{k+1-n/2}\overline{\delta}$, both equations (14) and (15) are minimal and equal to $\frac{n}{2}\overline{\delta}$ for, respectively, $|\widetilde{X}| = 0$ and for $|\widetilde{Y}| = 0$;

3. for $c_{ij} < \frac{1}{k+1-n/2}\overline{\delta}$, i.e. for $c < \frac{k}{k+1-n/2}\overline{\delta}$, both equations (14) and (15) are minimal and equal to $\frac{n}{2}(k+1-\frac{n}{2})c_{ij} < \frac{n}{2}\overline{\delta}$ for, respectively, $|\widetilde{X}| = n/2$ and for $|\widetilde{Y}| = n/2$. ■

Proof of Proposition 3. By lemma 3, an incomplete interbank network L^r has a carrying capacity sufficient to provide full coverage from liquidity risk, i.e. equal to $(n/2)\overline{\delta}$, if $c > \frac{k}{k+1-n/2}\overline{\delta}$. By the definition of carrying capacity, such a liquidity transfer equal to $(n/2)\overline{\delta}$ is achieved if interbank deposits withdrawals are coordinated. ■

Proof of Proposition 5. In demonstrating this result and propositions 6 and 7 below, we use a known property of network flows: for a flow defined in a flow network, the value of the net forward flow that crosses a cut is the same for all the cuts of the network. Applying this property to a financial flow network N, we have that the value of the net forward flow that crosses all cuts of N equals the value of the exogenous shock, i.e., it is equal to the flow that crosses the cut $\{A, (\Omega, H, Q)\}$. It follows that the value of the exogenous shock is equal to the forward flow that crosses the cut $\{(A, \Phi), (\Omega\backslash\Phi, Q, H)\}$, which is also the net flow across this cut, since no flow crosses it in the opposite direction. Let m be the number of primary defaults caused by a shock $m = |\Phi|$ and let $b_i \in [0, 1]$ be a parameter that measures the percentage *loss-given-default* of a node, i.e. it measures the share of the value of the liabilities issued by the i-th bank which is lost upon its default. Each of node ω_i in Φ sends 1) to the sink Q a flow equal to its own equity e_i, 2) to the sink H a flow equal to $b_i h_i$, and 3) a flow equal to $b_i d_{ij}$ to each of its $(n-m)$ creditors in $\Omega\backslash\Phi$. The shock that comes out of the source nodes is then equal to

$$me_i + \sum_{i=1}^{m} b_i h_i + \sum_{i=1}^{m} b_i d_{ij}(n-m), \tag{16}$$

where the term me_i is the value of the flow of losses going from Φ to Q, the sum $\sum_{i=1}^{m} b_i h_i$ is the flow of losses that goes from Φ to H, and the sum $\sum_{i=1}^{m} b_i d_{ij}(n-m)$ is the flow of losses going from Φ to $\Omega\backslash\Phi$. In a complete network N^c, each node in $\Omega\backslash\Phi$ receives, from its defaulting debtors, a flow of losses equal to $\sum_{i=1}^{m} b_i d_{ij}$. For default contagion to occur, this flow of losses must be larger than or equal to the absorbing capacity of a node: $\sum_{i=1}^{m} b_i d_{ij} \geq e_j$. The value of an exogenous shock that is exactly large enough to cause such a condition to be fulfilled, i.e. such that $\sum_{i=1}^{m} b_i d_{ij} = e_j$, constitutes both the first and the final threshold of contagion of a network N^c: all nodes in $\Omega\backslash\Phi$ default together if such a threshold is reached. This condition for contagion requires that $\sum_{i=1}^{m} b_i = e_j/d_{ij}$ and, substituting this value in (11), we obtain the first and final contagion thresholds of a network N^c. ■

Proof of Proposition 6. As above, we use the fact that the shock out of the source nodes is equal to the forward flow that crosses the cut $\{(A,\Phi),(\Omega\backslash\Phi,Q,H)\}$.Then, with respect to the two cases listed in this theorem, we have that:

1) if $\Phi = \omega_c$, the flow that crosses the cut $\{(A,\omega_c),(\Omega\backslash\omega_c,H,Q)\}$ is equal to

$$e_c + b_c h_c + b_c d_p(n-1),$$

where b_c is the *loss-given-default* parameter of ω_c. Contagion occurs for any shock such that $b_c d_p(n-1) \geq e_p(n-1)$. The smallest of such shocks is the one that causes $b_c d_p(n-1) = e_p(n-1)$, hence $b_c = e_p/d_p$. This condition characterizes both the first and the final threshold of contagion: if $b_c = e_p/d_p$, all agents in N^s default. Substituting $b_c = e_p/d_p$ into the above equation delivers the result.

2) if $\Phi = \{\omega_c, \omega_p | \text{for some } p \in (1,...,n-1)\}$, the flow that crosses the cut $\{(A,\Phi),(\Omega\backslash\Phi,H,Q)\}$ is equal to

$$(m-1)e_p + e_c + \sum_{p\in\Phi\backslash\omega_c} b_p h_p + b_c h_c + b_c d_p(n-m),$$

where the sum $(m-1)e_p + e_c$ is the flow of losses that goes from the set of primary defaults into the sink Q, the sum $\sum_{p\in\Phi\backslash\omega_c} b_p h_p + b_c h_c$ is the flow of losses that goes from Φ into the sink Q, and the term $b_c d_p(n-m)$ is the flow that goes from the nodes in Φ to their creditors in $\Omega\backslash\Phi$. Both first and complete contagion occur for any shock such that $b_c d_p(n-m) \geq e_p(n-m)$, hence $b_c \geq e_p/d_p$. Taking the smallest of such shocks – i.e., the ones such that $b_c = e_p/d_p$ – we obtain the first and the final thresholds:

$$\begin{aligned}\tau_1^s(N^s) &= \tau_2^s(N^s) = (m-1)e_p + e_c + \sum_{p\in\Phi\backslash\omega_c} b_p h_p + e_p\frac{h_c}{d_p} + e_p(n-m) \\ &= (n-1)e_p + e_c + e_p\frac{h_c}{d_p} + \sum_{p\in\Phi\backslash\omega_c} b_p h_p. \qquad (17)\end{aligned}$$

For the flow conservation property, applied to the central node ω_c, we have that:

$$\begin{aligned}\sigma_c + \sum_{p\in\Phi\backslash\omega_c} b_p d_p &= e_c + b_c d_p(n-1) + b_c h_c \\ &= (n-1)e_p + e_c + e_p\frac{h_c}{d_p}\end{aligned}$$

thus

$$\sum_{p\in\Phi\backslash\omega_c} b_p = (n-1)\frac{e_p}{d_p} + \frac{e_c}{d_p} + e_p\frac{h_c}{(d)^2} - \frac{\sigma_c}{d}$$

that, substituted in equation (12), delivers the above result.

By the definition of contagion threshold, σ_c can not take on a value larger than $e_p(n-m+1) + e_p h_p/d_p$ because: i) in order to have $m-1$ peripheral nodes in Φ, the smallest possible shock born by those nodes must be equal to $e_p(m-1)$, and ii) assuming $\sum_{p\in\Phi\backslash\omega_c}\sigma_p = e_p(m-1)$, $\sigma_c = e_c + e_p(n-m) + e_p h_p/d_p$ is sufficient to induce contagion. This holds *a fortiori* if $\sum_{p\in\Phi\backslash\omega_c}\sigma_p > e_p(m-1)$. ■

Proof of Proposition 7. If $\Phi = \{\omega_p | \text{for some } p \in (1,...,n-1)\}$ and $\omega_c \notin \Phi$, the flow that crosses the cut $\{(A,\Phi),(\Omega\backslash\Phi, H, Q)\}$ is equal to

$$me_p + \sum_{p=1}^{m} b_p h_p + \sum_{p=1}^{m} b_p d_p,$$

where me_p and $\sum_{p=1}^{m} b_p h_p$ are the flows that Φ sends into Q and H, respectively, and $\sum_{p=1}^{m} b_p d_p$ is the flow that the central node ω_c receives from the defaulting nodes in Φ. The condition for the first threshold of contagion is: $\sum_{p=1}^{m} b_p d_p = e_c$, hence $\sum_{p=1}^{m} b_p = e_c/d_p$ and, substituting this into the above equation, we obtain that $\tau_1^s(N^s) = me_p + e_c(1 + h_p/d_p)$;

The second and final threshold of contagion, is set by the flow that crosses the cut $\{(A,\Phi,\omega_c),(\Omega\backslash(\Phi,\omega_c), H, Q)\}$ which is equal to

$$me_p + e_c + \sum_{p=1}^{m} b_p h_p + b_c h_c + b_c d_p(n-m-1),$$

where: $me_p + e_c = \overrightarrow{f}((\Phi,\omega_c),Q)$, $\sum_{p=1}^{m} b_p h_p + b_c h_c = \overrightarrow{f}((\Phi,\omega_c),H)$, and $b_c d_p(n-m-1) = \overrightarrow{f}(\omega_c, \Omega\backslash(\Phi,\omega_c))$. All nodes in $\Omega\backslash(\Phi,\omega_c)$ default if the central node sends to each of them a flow larger than or equal to e_p. The final threshold of contagion is equal to the smallest of such shocks: $b_c d_p = e_p$. Hence $b_c = e_p/d_p$ and

$$\tau_2^s(N^s) = (n-1)e_p + e_c + e_p\frac{h_c}{d_p} + \sum_{p=1}^{m} b_p h_p. \tag{18}$$

As above, to obtain $\sum_{p=1}^{m} b_p$, we resort to the fact that the flow that enters the central node is equal to the flow that exits from it:

$$\begin{aligned}\sum_{p=1}^{m} b_p d_p &= e_c + b_c d_p(n-1) + b_c h_c \\ &= (n-1)e_p + e_c + e_p \frac{h_c}{d_p}.\end{aligned}$$

Thus

$$\sum_{p=1}^{m} b_p = (n-1)\frac{e_p}{d_p} + \frac{e_c}{d_p} + e_p \frac{h_c}{(d_p)^2}.$$

Substituting this value in equation (13), we obtain the above result. ■

Proof of Proposition 8. Let the interbank liquidity networks at hand be hit by a symmetric liquidity shock, as defined above, and let Ω^+ (Ω^-) be the set of banks that face a positive (negative) change of their customer deposits. Suppose that the banks in deficit withdraw their interbank deposits in a selective fashion. Then:

i) In a star-shaped interbank liquidity network L^s we have that: a) if the central node in faces a liquidity deficit, each of the $(n/2) - 1$ peripheral deficit banks withdraws $\overline{\delta}$ from the central node and the latter withdraws $\overline{\delta}$ from each of the $n/2$ peripheral surplus banks; and b) if the central node has a liquidity surplus, each of the $n/2$ peripheral deficit banks withdraw $\overline{\delta}$ from the central node and the latter withdraws $\overline{\delta}$ from each of the $n/2 - 1$ peripheral surplus banks. Both these complete liquidity transfers are feasible with interbank deposits $c \geq \overline{\delta}$.

ii) In a complete interbank liquidity network L^c, each of the $n/2$ banks that face a liquidity deficit withdraws $(2/n)\overline{\delta}$ from each of the $n/2$ surplus banks. This complete liquidity transfer is feasible with interbank deposits $c_{ij} \geq (2/n)\overline{\delta}$.

iii) In an incomplete regular interbank liquidity network L^r, selective withdrawals ensure a complete re-allocation of liquidity if only if the sets of surplus banks Ω^+ and of deficit banks $\Omega^-$are maximally interconnected, i.e. if each surplus (deficit) bank is directly connected with all other surplus (deficit) bank.

If this condition holds, then each surplus (deficit) bank is bilaterally connected to $k + 1 - n/2$ deficit (surplus) banks, where k is the degree of the nodes in L^r. A complete liquidity transfer is achieved as each deficit bank withdraws $\frac{1}{k+1-n/2}\overline{\delta}$ from the $k + 1 - n/2$ surplus banks to which is connected.

If, conversely, the above condition does not hold, then there is in L^r i) a non empty set $\underline{\Omega}^+$ composed of surplus bank connected to more than $k + 1 - n/2$ deficit banks, and ii) a non empty set $\overline{\Omega}^+$ composed of surplus bank connected to fewer than $k+1-n/2$ deficit banks. As the deficit banks withdraw their deposits from surplus banks, the banks in $\underline{\Omega}^+$ face a liquidity shortage while the banks in $\overline{\Omega}^+$ still have a surplus of liquidity. Then, to prevent a complete transfer of liquidity, it is sufficient that there is one bank in $\underline{\Omega}^+$ with no direct links with banks in $\overline{\Omega}^+$. Such a bank, facing a liquidity shortage, withdraws deposits from all its neighbors (since it has no neighbor with spare liquidity), including the ones in Ω^-, which are (formerly) deficit banks with no spare liquidity and no deposits left in surplus banks. Thus, part of the initial liquidity shortage faced by the banks in Ω^- remains within such set of banks, while an equal amount of spare liquidity remains in the hands of surplus banks.

In sum, in an incomplete regular interbank liquidity network L^r, selective withdrawals do not guarantee a complete re-allocation of liquidity because there is no guarantee that i) the sets of surplus banks Ω^+ and of deficit banks $\Omega^-$are maximally interconnected, and ii) all banks in $\underline{\Omega}^+$ are connected to at least one bank in $\overline{\Omega}^+$. ■

Proof of Proposition 9. Let the interbank liquidity networks at hand be hit by a symmetric liquidity shock, as defined above, and let Ω^+ (Ω^-) be the set of banks that face a positive (negative) change of their customer deposits. Suppose that the banks in deficit withdraw their interbank deposits in a pro-rata fashion. Then:

i) Let L^s be a star-shaped interbank liquidity network and suppose that the central node ω_c has a liquidity surplus. The $n/2$ peripheral banks that face a liquidity deficit equal to $\overline{\delta}$, the $\omega_p \in \Omega^-$, withdraw $\overline{\delta}$ from ω_c. The central node, then, faces a deficit equal to $(\frac{n}{2}-1)\overline{\delta}$ and withdraws $\frac{(\frac{n}{2}-1)}{n-1}\overline{\delta}$ from each of the peripheral nodes ω_p in $\Omega \backslash \omega_c$. Then the peripheral nodes in Ω^- face a deficit equal to $\frac{(\frac{n}{2}-1)}{n-1}\overline{\delta}$ and withdraw the same amount from ω_c. The latter, in turn, faces a deficit equal to $\frac{(\frac{n}{2}-1)^2}{n-1}\overline{\delta}$ and withdraws $\left(\frac{\frac{n}{2}-1}{n-1}\right)^2\overline{\delta}$ from each of the peripheral nodes. Now the peripheral nodes in Ω^- face a deficit equal to $\left(\frac{\frac{n}{2}-1}{n-1}\right)^2\overline{\delta}$ and withdraw the same amount from ω_c. At this point, ω_c faces a deficit equal to $\frac{(\frac{n}{2}-1)^3}{(n-1)^2}\overline{\delta}$ and withdraws $\left(\frac{\frac{n}{2}-1}{n-1}\right)^3\overline{\delta}$ from each of the peripheral nodes. Thus, each ω_p in Ω^- faces a deficit equal to $\left(\frac{\frac{n}{2}-1}{n-1}\right)^3\overline{\delta}$ and, in turn, withdraws the same amount from ω_c, and so on and so forth. In sum, at the end of such a recursive process of withdrawals, the peripheral nodes in Ω^- withdraw from ω_c a total amount of deposits equal to $\overline{\delta}\sum_{x=0}^{\infty}\left(\frac{\frac{n}{2}-1}{n-1}\right)^x = \frac{n-1}{n}2\overline{\delta}$, while the central node withdraws $\overline{\delta}\sum_{x=0}^{\infty}\left(\frac{\frac{x}{2}-1}{x-1}\right)^x = \overline{\delta}$ from each peripheral node. These withdrawals achieve the complete re-allocation of liquidity, form surplus to deficit banks, and are feasible for interbank deposits $c \geq \frac{n-1}{n} 2\overline{\delta}$.

Suppose now that the central node ω_c faces a liquidity deficit. The $(n/2)-1$ peripheral banks that face a liquidity deficit equal to $\overline{\delta}$, the $\omega_p \in \Omega^-$, withdraw $\overline{\delta}$ from ω_c. The central node, then, faces a deficit equal to $\frac{n}{2}\overline{\delta}$ and withdraws $\frac{n/2}{n-1}\overline{\delta}$ from each of the peripheral nodes ω_p in $\Omega \backslash \omega_c$. Then the $n/2$ peripheral nodes in Ω^- face a deficit equal to $\frac{n/2}{n-1}\overline{\delta}$ and withdraw the same amount from ω_c. The latter, in turn, faces a deficit equal to $\frac{(\frac{n}{2}-1)}{n-1}\frac{n}{2}\overline{\delta}$ and withdraws $\frac{(\frac{n}{2}-1)}{(n-1)^2}\frac{n}{2}\overline{\delta}$ from each of the peripheral nodes. At this point, the peripheral nodes in Ω^- face a deficit equal to $\frac{(\frac{n}{2}-1)}{(n-1)^2}\frac{n}{2}\overline{\delta}$ and withdraw the same amount from ω_c. Then, ω_c faces a deficit equal to $\frac{(\frac{n}{2}-1)^2}{(n-1)^2}\frac{n}{2}\overline{\delta}$ and withdraws $\frac{(\frac{n}{2}-1)^2}{(n-1)^3}\frac{n}{2}\overline{\delta}$ from each of the peripheral nodes. Thus, each ω_p in Ω^- faces a deficit equal to $\frac{(\frac{n}{2}-1)^2}{(n-1)^3}\frac{n}{2}\overline{\delta}$ and, in turn, withdraws the same amount from ω_c, and so forth and so on. In sum, at the end of such a recursive process of withdrawals, the peripheral nodes in Ω^- withdraw from ω_c a total amount of deposits equal to $\overline{\delta}+\overline{\delta}\sum_{x=0}^{\infty}\frac{(\frac{n}{2}-1)^x}{(n-1)^{x+1}}\frac{n}{2} = 2\overline{\delta}$, while the central node withdraws $\overline{\delta}\sum_{x=0}^{\infty}\frac{(\frac{n}{2}-1)^x}{(n-1)^{x+1}}\frac{n}{2} = \overline{\delta}$ from each peripheral node. These withdrawals achieve the complete reallocation of liquidity, from surplus to deficit banks, and are feasible with interbank deposits $c \geq 2\overline{\delta}$.

ii) In a complete interbank liquidity network L^c, each $\omega_i \in \Omega^-$, i.e. each of the $n/2$ banks hit by a liquidity deficit, withdraws $\frac{1}{n-1}\overline{\delta}$ from each of the remaining $n-1$ banks in $\Omega \backslash \omega_i$. Then, the banks in Ω^+ remain in surplus, while each bank ω_i in Ω^- faces now a deficit equal to $\left(\frac{n}{2}-1\right)\frac{1}{n-1}\overline{\delta}$ and, consequently, withdraws $\left(\frac{n}{2}-1\right)\frac{1}{(n-1)^2}\overline{\delta}$ from each bank in $\Omega \backslash \omega_i$. Then, the banks in Ω^+ still have a liquidity surplus, while each bank ω_i in Ω^- faces a deficit equal to $\left(\frac{n}{2}-1\right)^2\frac{1}{(n-1)^2}\overline{\delta}$ and, consequently, withdraws $\left(\frac{n}{2}-1\right)^2\frac{1}{(n-1)^3}\overline{\delta}$ from each bank in $\Omega \backslash \omega_i$, and so forth and so on. At the end of this recursive process of withdrawals, each bank in Ω^- has withdrawn $\overline{\delta}\sum_{x=0}^{\infty}\left(\frac{\frac{n}{2}-1}{n-1}\right)^x\frac{1}{n-1} = \frac{2}{n}\overline{\delta}$. These withdrawals achieve the complete re-allocation of liquidity, form surplus to deficit banks, and are feasible with interbank deposits $c_{ij} \geq 2\frac{n-1}{n}\overline{\delta}$.

iii) In an incomplete regular interbank liquidity network L^r, pro-rata withdrawals ensure a complete re-allocation of liquidity if only if the sets of surplus banks Ω^+ and of deficit banks $\Omega^-$are maximally interconnected, i.e. if each surplus (deficit) bank is directly connected with all other surplus (deficit) bank.

If this condition holds, then each surplus (deficit) bank is bilaterally connected to $k+1-n/2$ deficit (surplus) banks, where k is the degree of the nodes in L^r. A complete liquidity transfer is achieved as the each deficit bank withdraws all its deposits, both from the $k+1-n/2$ surplus banks and from the $n/2-1$ deficit banks to which is connected.

If, conversely, the above condition does not hold, then there is in L^r i) a non empty set $\underline{\Omega}^+$ composed of surplus bank connected to more than $k+1-n/2$ deficit banks, and ii) a non empty set $\overline{\Omega}^+$ composed of surplus bank connected to fewer than $k+1-n/2$ deficit banks. As each deficit bank withdraws its deposits from all its neighboring banks, the banks in $\underline{\Omega}^+$ face a liquidity shortage while the banks in $\overline{\Omega}^+$ still have a surplus of liquidity. Then, the banks in $\underline{\Omega}^+$ withdraw deposits form their neighbors, including banks in Ω^-, which are (formerly) deficit banks with no spare liquidity and no deposits left in other banks. Thus, part of the initial liquidity shortage faced by the banks in Ω^- remains within such set of banks, while an equal amount of spare liquidity remains in the hands of surplus banks.

In sum, in an incomplete regular interbank liquidity network L^r, pro-rata withdrawals do not guarantee a complete reallocation of liquidity because there is no guarantee that the sets of surplus banks Ω^+ and of deficit banks Ω^- are maximally interconnected. ■

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