Coalitional fairness in interim differential information economies*

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Abstract

In this paper we propose a concept of coalitional fair allocation in order to solve the tension that may exist between efficiency and envy-freeness when the equity of allocations is evaluated at the *interim* stage and agents are asymmetrically informed.

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1 Introduction

The problem of a fair distribution of resources among agents has been widely investigated and many notions of fair allocation have been adopted to evaluate equity. Since Foley [3], one of the most extensively studied concept is the one according to which an allocation is fair if it is envy-free and efficient. Under an analysis conducted on the individualistic level, an allocation is said to be envy-free if each individual prefers to keep his bundle rather than to receive the bundle of some other agent. Stronger notions of equity require that bundle comparisons are allowed between groups of agents and lead to coalitional fairness notions. Varian [9] introduces the concept of coalitional fair (c-fair) allocations attributing it originally to Vind [10]. Gabszewicz [4] also introduces c-fairness under a slightly different base. An allocation is c-fair if no coalition envies the aggregate bundle, and this is Varian's definition, or, and this is Gabszewicz approach, the net trade of some other coalition. It comes out that c-fair allocations are Pareto optimal.

At the individualistic level, it is known that in a pure exchange economy fair allocations always exist (see Theorem 2.3 in [9]). This is so simply because any competitive allocation that results from an equal sharing of the total initial endowment is fair. On the other hand, also c-fair allocations do always exist. Moreover, despite of differences due to technical requirements about the measures of potentially envious coalitions, c-fair allocations introduced in both papers [9] and [4] provide a complete characterization of competitive market equilibria (see also [12] and [14] for analogous results).

Envy-freeness may be incompatible with efficiency when production is allowed as well as agents are asymmetrically informed at the time of contracting. Recent papers by de Clippel [2] and Gajdos and Tallon [5] exhibit the tension between the concepts of envy-freeness and efficiency in models

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that explicitly encompasses uncertainty. The incompatibility may occur if the allocation is judged before or after the realization of uncertainty, as well as according to an interim stage evaluation.

The aim of this paper is to identify the coalitional fairness notion à la Gabszewicz as a suitable criterion to evaluate allocations on an equitable basis when agents are asymmetrically informed. As in [11], we assume that the true state of nature is commonly known at the time of implementing the contracts, focusing on interim fairness notions. The straightforward extension to differential information economies of the c-fairness criterion introduced by Varian cannot serve the same purpose since, when uncertainty is involved and agents are asymmetrically informed, a fair allocation (and, a fortiori, a c-fair one) may not exist, as shown by de Clippel [2].

Therefore, we extend to our context the coalitional fairness notion introduced by Gabszewicz [4], so that an allocation is qualified c-fair if it is not possible to find an alternative allocation such that in each state of nature a coalition of agents can redistribute among its members the net trade of a disjoint coalition and each of them, using his own private information, is better off. Our main result shows that this set of c-fair allocations is non-empty.

We do it by proving that c-fair allocations of a differential information economy correspond to c-fair allocations of an auxiliary Arrow-Debreu exchange economy with uncertainty and symmetric information. Agents of the auxiliary economy are defined adapting the idea used by Harsanyi [7] to define Bayesian games. A type-agent is a couple (t, E), where t is an agent and E is an event of his information partition. The future state of the fictitious economy is uncertain, but each type-agent has no private information. Moreover, since contracts are contingent on the future state of the economy, standard Arrow-Debreu equilibrium notions can be applied. We adapt this representation in the framework of mixed markets. We prove that there is a natural correspondence between the original and the auxiliary economy. A correspondence that preserves c-fair allocations and constrained market equilibria. This allows us to conclude that the set of c-fair allocations is non-empty, since it contains the set of constrained market equilibria. This inclusion may be strict in general and becomes an equivalence under additional assumptions.

The paper is organized as follows. In Section 2 we present the model, describe main ideas and state our results. Proofs are collected in the final section.

2 Coalitional fairness in differential information economies

We consider an exchange economy $\mathcal{E}$ with uncertainty and differential information. Uncertainty about nature is, as usual, depicted by means of a probability space $(\Omega, \mathcal{F}, \pi)$, where Ω is a finite set representing all possible states of nature. The set of all the events is given by the field $\mathcal{F}$; while the common prior π describes the relative probability of the states. We assume, without loss of generality, that $\pi(\omega) > 0$ for each $\omega \in \Omega$.

There is a finite number of private goods, so that the commodity space is $\mathbb{R}^{\ell}_{+}$.

As in Gabszewicz [4] we refer to mixed markets by considering a complete, finite measure space $(T, \mathcal{T}, \mu)$ as space of agents. T is the set of agents, $\mathcal{T}$ is the σ-field of all eligible coalitions, whose economic weight on the market is given by the measure μ. An arbitrary finite measure space of agents makes us deal simultaneously with the case of discrete economies, non-atomic economies as well as economies that may have atoms. Indeed, discrete economies are covered by a finite set T with a counting measure μ. Atomless economies are analyzed by assuming that $(T, \mathcal{T}, \mu)$ is the Lebesgue measure space with $T = [0, 1]$. Finally, mixed markets are those for which T is composed by two sets: T_0 and T_1, where T_0 is the atomless sector that describes an ocean of uninfluential agents and T_1 a set of atoms.

Each agent $t \in T$ is characterized by:

- A private information field $\mathcal{F}_t$ generated by a partition Π_t of Ω. The interpretation is usual: if the prevailing state is ω, agent t observes the unique element $\Pi_t(\omega)$ of Π_t containing ω or, in other words, he is informed that the prevailing state is in the event $\Pi_t(\omega)$. His beliefs are described from π by Bayesian updating. Agents may be not equally informed concerning the true state of nature when they write contracts, but when consumption takes place, the realized state of nature is commonly known. Since Ω is finite, there exists a finite collection $\{\Pi_i\}_{i\in\{1,\ldots,N\}}$ of partitions of the set Ω. For each $i \in \{1, \ldots, N\}$, let $T(i) = \{t \in T: \ \Pi_t = \Pi_i\}$ be the information type set. $T(i)$ is assumed to be measurable and of positive measure[1].

- A state-dependent utility function representing his preferences:

$$\begin{aligned} u_t: \ \Omega \times \mathbb{R}^\ell_+ &\rightarrow \mathbb{R} \\ (\omega, x) &\rightarrow u_t(\omega, x). \end{aligned}$$

  We assume that for all ω, $u_t(\omega, \cdot)$ is strictly increasing, continuous and concave, and the mapping $(t, x) \mapsto u_t(\omega, x)$ is $\mathcal{T} \otimes \mathcal{B}$-measurable, where $\mathcal{B}$ is the σ-field of Borel subsets of $\mathbb{R}^\ell_+$.

- An initial endowment of physical resources represented by the function

$$e_t: \ \Omega \rightarrow \mathbb{R}^\ell_+.$$

Thus, summing up the economy $\mathcal{E}$ is modeled by the following collection:

$$\mathcal{E} = \big\{(\Omega, \mathcal{F}, \pi); \ (T, \mathcal{T}, \mu); \ \mathbb{R}^\ell_+; \ (\mathcal{F}_t, u_t, e_t)_{t\in T}\big\}.$$

Decisions are taken today about the way to redistribute the endowments when the state will be common knowledge. Therefore, incentive and measurability constraints are irrelevant.

An *allocation* is a function $x : \Omega \times T \rightarrow \mathbb{R}^\ell_+$ such that $x(\omega, \cdot)$ is integrable for all $\omega \in \Omega$. If, for each $\omega \in \Omega$,

$$\int_T x_t(\omega) d\mu \leqslant \int_T e_t(\omega) d\mu,$$

then the allocation x is said to be *feasible.*

Given an event E, the interim expected utility of agent t for some allocation x conditional on the event E is given by

$$V_t(x_t|E) = \sum_{\omega\in\Omega} u_t(\omega, x_t(\omega))\pi(\omega|E) = \sum_{\omega\in E} u_t(\omega, x_t(\omega))\frac{\pi(\omega)}{\pi(E)}.$$

Definition 2.1. *A feasible allocation y Pareto dominates an allocation x if almost all agents, given their own private information, prefer y over x in each state, i.e.,*

$$V_t(y_t|\Pi_t(\omega)) > V_t(x_t|\Pi_t(\omega))$$

for almost all $t \in T$ and each $\omega \in \Omega$.

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[1] This assumption implies that the following correspondence has measurable graph: $\Pi : T \rightarrow 2^{\mathcal{F}}$ defined by $\Pi(t) = \Pi_t$. It means that the set $G_\Pi = \{(t, \mathcal{E}) : \mathcal{E} \in \Pi_t\}$ belongs to the product σ-algebras $\mathcal{T} \otimes \mathcal{B}(2^{\mathcal{F}})$, where $\mathcal{B}$ denotes the Borel σ-algebra.

A feasible allocation is *efficient* (or *Pareto optimal*) if it is not Pareto dominated by any other feasible allocation (for a similar definition see [13]).

The two notions of coalitional fair allocations of Varian [9] and Gabszewicz [4] differ for what follows: Varian requires that a c-fair allocation must be such that no coalition envies the aggregate bundle of any other coalition of the same or smaller size. According to Gabszewicz's definition, different groups of agents compare their net trades without any requirement on the measure of the potentially envious coalition. In both cases, differently from individual fairness, the efficiency is implicitly satisfied, and the existence is guaranteed under standard assumptions.

To the best of our knowledge, coalitional fairness of allocations has not been widely studied in economies involving uncertainty and asymmetric information[2]. Our goal is to address the following question: do coalitional fair allocations still exist when agents are asymmetrically informed?

Consider first the natural extension to differential information economies of the Varian c-fair notion according to which we agree that an allocation x is c-fair if it is efficient[3] and also envy-free in the sense that it is not possible to find an alternative allocation y such that, for each state $\omega \in \Omega$, there exist two coalitions $S_1(\omega)$ and $S_2(\omega)$ for which

$$\begin{array}{ll} (i) & \mu(S_1(\omega)) \geq \mu(S_2(\omega)) \\ (ii) & t \in S_1(\omega) \ \Rightarrow \ t \in S_1(\omega') \text{ for all } \omega' \in \Pi_t(\omega) \\ (iii) & V_t(y_t|\Pi_t(\omega)) > V_t(x_t|\Pi_t(\omega)), \qquad \text{for almost all } t \in S_1(\omega) \\ (iv) & \displaystyle\int_{S_1(\omega)} y_t(\omega) d\mu \leq \int_{S_2(\omega)} x_t(\omega)\, d\mu, \end{array}$$

with $\mu(S_1(\omega)) > 0$ in at least one state of nature.

For each state of nature $\omega \in \Omega$, $S_1(\omega)$ may be interpreted as the coalition which envies at state ω the aggregate bundle of the coalition $S_2(\omega)$. Thus, it is natural to require that if $t \in S_1(\omega)$ for some state ω, that is t is envious in state ω, then t is also envious in each state that he cannot distinguish.

Obviously, in the case of perfect information, the above definition reduces to the standard one given by Varian [9]. Moreover, an allocation with all properties just above described is interim fair according to [2]. Hence, we can't guarantee the existence in general of such an allocation due to the fact that interim fair allocation may form an empty set. For this reason we shall focus our attention to coalitional fairness of net trades, extending to differential information economies the Gabszewicz notion of c-fair allocation.

Before doing this, for our sake of completeness, we explicitly show an example of a differential information economy for which it is impossible to reconcile efficiency and the envy-freeness described above.

Example. Consider an economy with: two equiprobable states of nature $\Omega = \{a, b\}$; three asymmetrically informed agents $\{1, 2, 3\}$, with $\Pi_1 = \Pi_3 = \{\{a, b\}\}$ and $\Pi_2 = \{\{a\}, \{b\}\}$; only one good. In each state ω, agents equally share the total initial endowment $e(\omega)$: $e_i(a) = \frac{e(a)}{3} = 4$ and $e_i(b) = \frac{e(b)}{3} = 9$. Moreover, for all $\omega \in \Omega$, $u_t(\omega, x) = x$ for $t = 2, 3$ and $u_1(\omega, x) = \sqrt{x}$. Even if the economy described above satisfies all the standard assumptions, the set allocations that are

[2]See [6] for an analysis on coalitional fairness notion in differential information economies in which agents receive no signal at the time of contracting.

[3]Differently from the perfect information notions, we will explicitly require Pareto efficiency due to the free disposal condition imposed on allocations.

simultaneously efficient and envy-free is empty.

The reason of the non existence of allocations that are efficient and envy-free at the same time, bases on the fact that even if agents have the same initial endowment, they may have different information and hence different budget set. Therefore the scheme used by Varian, that deduces the existence of a c-fair allocation simply by showing that any competitive equilibrium allocation resulting from an equal sharing of the total initial endowment among agents is c-fair, doesn't apply whenever agents are asymmetrically informed. In a differential information economy, a constrained market equilibrium, the proper equilibrium concept to consider, resulting from an equal sharing of the total initial endowment may not be simultaneously efficient and free of envy (just think about constrained market equilibria, certainly existing, of the economy given in the example).

Thus, we move our attention to the notion due to Gabszewicz [4], according to which different groups of agents compare their net trades without any requirement on the measure of the potentially envious coalition. A natural extension to differential information economies is proposed in the following.

Definition 2.2. *A feasible allocation x is said to be c-fair if there does not exist an alternative allocation y such that for all $\omega \in \Omega$ there exist two coalitions $S_1(\omega)$ and $S_2(\omega)$ for which*

$$\begin{array}{ll} (i) & S_1(\omega) \cap S_2(\omega) = \emptyset \\ (ii) & t \in S_1(\omega) \Rightarrow t \in S_1(\omega') \textit{ for all } \omega' \in \Pi_t(\omega) \\ (iii) & V_t(y_t|\Pi_t(\omega)) > V_t(x_t|\Pi_t(\omega)) \textit{ for almost all } t \in S_1(\omega) \\ (iv) & \displaystyle\int_{S_1(\omega)} (y_t(\omega) - e_t(\omega))\, d\mu \leq \int_{S_2(\omega)} (x_t(\omega) - e_t(\omega))\, d\mu, \end{array}$$

with $\mu\left(S_1(\omega)\right) > 0$ in at least one state of nature.

An allocation is qualified *c-fair (in the sense of Gabszewicz)* if it is not possible to find an alternative allocation such that in each state of nature a coalition of agents can redistribute among its members the net trade of a disjoint coalition and each of them, using his own private information, is better off. For each $\omega \in \Omega$, $S_1(\omega)$ is the coalition of agents envying the net trade of the disjoint coalition $S_2(\omega)$ in state ω. Thus, it is natural to require that if an agent t is envious at state ω, he is still envious in each state he cannot distinguish.

Notice that we do not require that $S_2(\omega)$ has positive measure, neither that x is efficient. Indeed, with standard arguments[4], it is easy to show that any c-fair allocation is Pareto optimal.

In pure exchange economies Gabszewicz shows that the set of c-fair allocations is non empty since it contains the set of competitive equilibria. This inclusion may be strict, it becomes an equivalence in atomless economies. Our goal is to show that, contrary to what we have seen about Varian's notion, the results obtained by Gabszewicz are still valid in differential information economies. Such results are stated below.

Theorem 2.3. *In a differential information economy, any constrained market equilibrium allocation is c-fair.*

It directly follows from the existence of constrained market equilibrium allocations (see [11]) and Theorem 2.3 that in a differential information economy, c-fair allocations exist.

Theorem 2.4. *If $\mathcal{E}$ is atomless, then constrained market equilibria are the only c-fair allocations.*

[4] It is just needed to put for all ω, the envious coalition $S_1(\omega)$ equal to the whole set of agents T and the other coalition $S_2(\omega)$ equal to the empty set.

3 Proofs

We start by detailing about the incompatibility of efficiency and envy-freeness in the previous example. Assume, indeed, that the allocation x is envy-free.

Notice that $x_1 = x_2$. For if $x_1(b) > x_2(b)$, for example, then the coalitions

$$\begin{array}{rcll} S_1(a) & = & \emptyset & S_1(b) = \{2\} \\ S_2(a) & = & \emptyset & S_2(b) = \{1\}, \end{array}$$

and the allocation

$$\begin{array}{rcll} y_1(a) & = & x_1(a) & y_1(b) = x_1(b) \\ y_2(a) & = & x_2(a) & y_2(b) = x_1(b) \\ y_3(a) & = & x_3(a) & y_3(b) = x_3(b), \end{array}$$

satisfy conditions $(i) - (iv)$ above, against envy-freeness of x.

Now, if $x_1(b) < x_2(b)$, then the coalitions

$$\begin{array}{rcll} S_1(a) & = & \{1\} & S_1(b) = \{1\} \\ S_2(a) & = & \{1\} & S_2(b) = \{2\}, \end{array}$$

and the allocation

$$\begin{array}{rcll} y_1(a) & = & x_1(a) & y_1(b) = x_2(b) \\ y_2(a) & = & x_2(a) & y_2(b) = x_2(b) \\ y_3(a) & = & x_3(a) & y_3(b) = x_3(b), \end{array}$$

would exhibit the envy of S_1 against S_2. Hence, $x_1(b) = x_2(b)$ and similarly we can show that $x_1(a) = x_2(a)$. If $x_1(b) = x_2(b) < x_3(b)$, a contradiction will appear by considering the coalitions

$$\begin{array}{rcll} S_1(a) & = & \emptyset & S_1(b) = \{2\} \\ S_2(a) & = & \emptyset & S_2(b) = \{3\}, \end{array}$$

and the allocation

$$\begin{array}{rcll} y_1(a) & = & x_1(a) & y_1(b) = x_1(b) \\ y_2(a) & = & x_2(a) & y_2(b) = x_3(b) \\ y_3(a) & = & x_3(a) & y_3(b) = x_3(b). \end{array}$$

Thus, $x_1(b) = x_2(b) \geq x_3(b)$ and, similarly, $x_1(a) = x_2(a) \geq x_3(a)$. Finally, if $x_1(b) = x_2(b) > x_3(b)$, consider the coalitions

$$\begin{array}{rcll} S_1(a) & = & \{3\} & S_1(b) = \{3\} \\ S_2(a) & = & \{3\} & S_2(b) = \{2\}, \end{array}$$

and the allocation

$$\begin{array}{rcll} y_1(a) & = & x_1(a) & y_1(b) = x_1(b) \\ y_2(a) & = & x_2(a) & y_2(b) = x_2(b) \\ y_3(a) & = & x_3(a) & y_3(b) = x_2(b), \end{array}$$

to contradict that x is envy-free. Thus, $x_1(b) = x_2(b) = x_3(b)$. Similarly, we can show that $x_1(a) = x_2(a) = x_3(a)$. Hence, from the feasibility condition it follows that for each agent, $x_t(a) = 4$ and $x_t(b) = 9$. Consider now, the following feasible allocation

$$\begin{aligned} y_1(a) &= 6.3 \quad y_1(b) = 6.4 \\ y_2(a) &= 4.1 \quad y_2(b) = 9.1 \\ y_3(a) &= 1.6 \quad y_3(b) = 11.5, \end{aligned}$$

and notice that

$$V_t(y_t|\Pi_t(\omega)) > V_t(x_t|\Pi_t(\omega)), \quad \text{for all } t \in T \text{ and all } \omega.$$

Thus, x is not efficient. $\square$

Now we prove the existence and the characterization of a c-fair allocation. For this purpose, following Wilson (see [11]) we associate to $\mathcal{E}$ an economy $\mathcal{E}^*$ with uncertainty and without asymmetric information. This allows us to deduce Theorems 2.3 and 2.4 from Gabszewicz [4].

An auxiliary economy. In the type-agent representation $\mathcal{E}^*$ of the economy $\mathcal{E}$ the uncertainty about nature is described by the measure space $(\Omega, \mathcal{F}, \pi)$ as before.

The measure space $(T^*, \mathcal{T}^*, \mu^*)$, representing the space of type agents, is constructed in the following way:

- T^* coincides with the graph of the correspondence $\Pi : T \to 2^{\mathcal{F}}$ defined by $\Pi(t) = \Pi_t$. More precisely, T^* is the set of pairs (t, E), where t is an agent and E is an event of his information partition.
- $\mathcal{T}^*$, the family of coalitions, consists of measurable subsets S^* of T^*, i. e. subsets that belong to $\mathcal{T} \otimes \mathcal{B}(2^{\mathcal{F}})$.
- The measure μ^* on $\mathcal{T}^*$ is defined as the product measure of μ and the counting measure.

Each type agent (t, E) is characterized by:

- A state-dependent utility function $u_{(t,E)}$ defined as

$$u_{(t,E)}(\omega, x) = \begin{cases} u_t(\omega, x) & \text{if } \omega \in E \\ 0 & \text{otherwise.} \end{cases}$$

- An initial endowment of physical resources $e^*_{(t,E)}$ defined as

$$e^*_{(t,E)}(\omega) = \begin{cases} e_t(\omega) & \text{if } \omega \in E \\ 0 & \text{otherwise.} \end{cases}$$

Type-agents decide today about the way to redistribute their endowments when the state will be common knowledge. The presence of uncertainty in this economy, as well as the possibility of writing contracts that are contingent on the future state of the economy, allows us to apply the standard notion of competitive and c-fair allocations. We shall rewrite, for reader convenience, the main equilibrium notions in the economy $\mathcal{E}^*$.

An *allocation* in the fictitious economy $\mathcal{E}^*$ is a function $x^* : \Omega \times T^* \to I\!R^\ell_+$ such that $x^*(\omega, \cdot)$ is integrable for all $\omega \in \Omega$. x^* is *feasible* if

$$\int_{T^*} x^*_{(t,E)}(\omega) d\mu^* \leq \int_{T^*} e^*_{(t,E)}(\omega) d\mu^* \qquad \text{for all } \omega \in \Omega.$$

where $x^*_{(t,E)} : \Omega \to I\!R^\ell_+$ represents the bundle that the type-agent (t, E) receives under the allocation x^*, in each state ω.

Definition 3.1. *Let S^* be a coalition with positive measure, i.e. $\mu^*(S^*) > 0$. An allocation y^* Pareto dominates x^* for S^* if for almost all $(t, E) \in S^*$*

$$V_{(t,E)}(y^*_{(t,E)}) > V_{(t,E)}(x^*_{(t,E)}),$$

where $V_{(t,E)}(x^*_{(t,E)}) = \sum_{\omega \in \Omega} u_{(t,E)}(\omega, x^*_{(t,E)}(\omega))\pi(\omega)$.

Definition 3.2. *A feasible allocation x^* is said to be c-fair in the economy $\mathcal{E}^*$ if there do not exist an alternative allocation y^*, two coalitions S^*_1 and S^*_2, such that*

(1) $\mu^*(S^*_1) > 0, S^*_1 \cap S^*_2 = \emptyset$

(2) $V_{(t,E)}(y^*_{(t,E)}) > V_{(t,E)}(x^*_{(t,E)})$ *for almost all* $(t, E) \in S^*_1$

(3) $\int_{S^*_1} (y^*_{(t,E)}(\omega) - e^*_{(t,E)}(\omega))\, d\mu^* \leq \int_{S^*_2} (x^*_{(t,E)}(\omega) - e^*_{(t,E)}(\omega))\, d\mu^*$ *for all* $\omega \in \Omega$.

Definition 3.3. *An allocation x^* is a competitive equilibrium in the economy $\mathcal{E}^*$ if it is feasible and there exists a price system $p : \Omega \to I\!R^\ell_+$ such that*

$$x^*_{(t,E)} \in \arg \max_{y^*_{(t,E)} \in B_{(t,E)}(p)} V_{(t,E)}(y^*_{(t,E)})$$

where for all $(t, E) \in T^*$

$$B_{(t,E)}(p) = \left\{ y^*_{(t,E)} | \sum_{\omega \in \Omega} p(\omega) \cdot y^*_{(t,E)}(\omega) \leq \sum_{\omega \in \Omega} p(\omega) \cdot e^*_{(t,E)}(\omega) \right\}.$$

It is easy to construct a natural isomorphism between the economies $\mathcal{E}$ and $\mathcal{E}^*$.

Given an allocation $x \in \mathcal{E}$, its type-agent representation is the allocation x^* such that for each (t, E) in T^*

$$x^*_{(t,E)}(\omega) = \begin{cases} x_t(\omega) & \text{if } \omega \in E \\ 0 & \text{otherwise.} \end{cases}$$

Given an allocation $x^* \in \mathcal{E}^*$, its associated allocation x in the original economy $\mathcal{E}$ is such that for each t in T and each ω in Ω

$$x_t(\omega) = x^*_{(t,\Pi_t(\omega))}(\omega).$$

We show below that any c-fair allocation in the economy $\mathcal{E}$ corresponds to a c-fair allocation in the associated type-economy $\mathcal{E}^*$ and vice versa.

Proposition 3.4. *If x is a c-fair allocation for $\mathcal{E}$, then the corresponding allocation x^* is c-fair for $\mathcal{E}^*$*
Conversely, if x^* is a c-fair allocation for $\mathcal{E}^*$, then the corresponding allocation x is c-fair for $\mathcal{E}$.

PROOF: It is easy to show that if x is a feasible allocation for the economy $\mathcal{E}$, the associated allocation x^* is feasible in $\mathcal{E}^*$, and vice versa. Let x be a c-fair allocation for $\mathcal{E}$ and assume on the contrary that the corresponding allocation $x^* \in \mathcal{E}^*$ is not c-fair. Since, x^* is feasible in $\mathcal{E}^*$, this means that there exist an assignment y^* and two coalitions S_1^* and S_2^* such that

(1) $\mu^*(S_1^*) > 0$, $S_1^* \cap S_2^* = \emptyset$

(2) $V_{(t,E)}(y^*_{(t,E)}) > V_{(t,E)}(x^*_{(t,E)})$ for almost all $(t,E) \in S_1^*$

(3) $\int_{S_1^*} (y^*_{(t,E)}(\omega) - e^*_{(t,E)}(\omega))\, d\mu^* \leq \int_{S_2^*} (x^*_{(t,E)}(\omega) - e^*_{(t,E)}(\omega))\, d\mu^*$ for all $\omega \in \Omega$.

Notice that without loss of generality we can assume that for all $(t,E) \in T^*$, $y^*_{(t,E)}(\omega) = 0$ for all $\omega \notin E$.
Let us consider the allocation y such that

$$y_t(\omega) = \begin{cases} y^*_{(t,\Pi_t(\omega))}(\omega) & \text{if } (t,\Pi_t(\omega)) \in S_1^* \\ 0 & \text{otherwise.} \end{cases}$$

Let us define for each $\omega \in \Omega$, the sets

$$S_1(\omega) = \{t \in T : \ (t,\Pi_t(\omega)) \in S_1^*\} \ \text{ and } \ S_2(\omega) = \{t \in T : \ (t,\Pi_t(\omega)) \in S_2^*\}.$$

Then, for $i = 1,2$ $S_i(\omega)$ is a measurable subset of T, since it coincides with the projection over T of the measurable subset of T^* defined by $S_i^* \cap \{(t,\Pi_t(\omega)) : t \in T\}$[5]. Clearly, if $t \in S_1(\omega)$ then $t \in S_1(\omega')$ for each $\omega' \in \Pi_t(\omega)$, simply because, $(t,\Pi_t(\omega)) = (t,\Pi_t(\omega'))$ for any $\omega' \in \Pi_t(\omega)$.

Moreover, for all $\omega \in \Omega$, $S_1(\omega) \cap S_2(\omega) = \emptyset$ and, from (1), $\mu(S_1(\omega)) > 0$ in at least one state of nature. Then we get

$$\int_{S_1(\omega)} (y_t(\omega) - e_t(\omega))\, d\mu = \int_{S_1^*} (y^*_{(t,E)}(\omega) - e^*_{(t,E)}(\omega))\, d\mu^* \leq$$

$$\leq \int_{S_2^*} (x^*_{(t,E)}(\omega) - e^*_{(t,E)}(\omega))\, d\mu^* = \int_{S_2(\omega)} (x_t(\omega) - e_t(\omega))\, d\mu.$$

This contradicts the assumption that x is c-fair.

We now prove the converse. Let x^* be a c-fair allocation for $\mathcal{E}^*$ and assume on the contrary that the corresponding allocation x is not c-fair. Since x is feasible in $\mathcal{E}$, this means that there exists an assignment y such that, for all $\omega \in \Omega$, one can find $S_1(\omega)$ and $S_2(\omega)$ for which

(1) $S_1(\omega) \cap S_2(\omega) = \emptyset$

(2) $t \in S_1(\omega) \ \Rightarrow \ t \in S_1(\omega')$ for all $\omega' \in \Pi_t(\omega)$

(3) $V_t(y_t|\Pi_t(\omega)) > V_t(x_t|\Pi_t(\omega))$ for almost all $t \in S_1(\omega)$

(4) $\int_{S_1(\omega)} (y_t(\omega) - e_t(\omega))\, d\mu \leq \int_{S_2(\omega)} (x_t(\omega) - e_t(\omega))\, d\mu,$

with $\mu(S_1(\omega)) > 0$ in at least one state of nature.

[5]Since the measure space of agents is assumed to be finite and complete, the measurability of the projection $Proj_T S^*$ for each measurable subset S^* of T^* follows by the Projection Theorem (see [1, Theorem 14.84])

Then let us consider a new assignment z defined as follow, for all $\omega \in \Omega$,

$$z_t(\omega) = \begin{cases} y_t(\omega) & t \in S_1(\omega) \\ x_t(\omega) & \text{otherwise} \end{cases}$$

Hence, from (2) and (3) it follows that for each $\omega \in \Omega$, $V_t(z_t|\Pi_t(\omega)) > V_t(x_t|\Pi_t(\omega))$ if and only if $t \in S_1(\omega)$.

Consider the following subsets of T^*

$$S_1^* = \{(t,E) \in T^* : V_t(z_t|E) > V_t(x_t|E)\} = \bigcup_{\omega \in \Omega} \bigcup_{i=1}^{N} (S_1(\omega) \cap T_i) \times \{\Pi_i(\omega)\},$$

$$S_2^* = \{(t,E) \in T^* : t \in S_2(\bar{\omega}), \bar{\omega} \in E\} = \bigcup_{\omega \in \Omega} \bigcup_{i=1}^{N} (S_2(\omega) \cap T_i) \times \{\Pi_i(\omega)\}.$$

Hence S_1^* and S_2^* are measurable and $\mu^*(S_1^*) > 0$. Moreover, we can observe that $S_1^* \cap S_2^* = \emptyset$. Let us define a new allocation z^* as follows: for each $(t,E) \in T^*$ and for all $\omega \in \Omega$

$$z^*_{(t,E)}(\omega) = \begin{cases} z_t(\omega) & \text{if } \omega \in E \text{ and } (t,E) \in S_1^* \\ 0 & \text{otherwise} \end{cases}$$

Thus, for each $(t,E) \in S_1^*$

$$V_{(t,E)}(z^*_{(t,E)}) > V_{(t,E)}(x^*_{(t,E)})$$

and for each ω in Ω

$$\int_{S_1^*} (z^*_{(t,E)}(\omega) - e^*_{(t,E)}(\omega))\, d\mu^* = \int_{S_1(\omega)} (z_t(\omega) - e_t(\omega))\; d\mu = \int_{S_1(\omega)} (y_t(\omega) - e_t(\omega))\; d\mu \leq$$

$$\leq \int_{S_2(\omega)} (x_t(\omega) - e_t(\omega))\; d\mu = \int_{S_2^*} (x^*_{(t,E)}(\omega) - e^*_{(t,E)}(\omega))\; d\mu^*.$$

This contradicts the assumption that x^* is a c-fair allocation for $\mathcal{E}^*$. □

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. Let x be a constrained market equilibrium for the economy $\mathcal{E}$, we want to show that it is c-fair. First, we notice that the associated allocation x^* is a competitive equilibrium of the economy $\mathcal{E}^*$ (see [11]), which is a c-fair allocation (see [4, Proposition 1]). Then, from Proposition 3.4, it follows that x is c-fair for the economy $\mathcal{E}$. □

Remark 3.5. From [4, Proposition 2] and Proposition 3.4, it follows that the set of c-fair allocations may differ from the set of constrained market equilibria.

Proof of Theorem 2.4. Let $\mathcal{E}$ be atomless, i.e., $T_1 = \emptyset$, and x be a c-fair allocation, then by Proposition 3.4 the associated allocation x^* is c-fair and hence, by the Core-Warlas equivalence, it is a competitive equilibrium. Then, as proved by Wilson, x is a constrained market equilibrium. □

Remark 3.6. Observe that similar arguments cannot be used to deduce in differential information economies the existence of a c-fair allocation (in the sense of Varian), simply because in the type-agent economy $\mathcal{E}^*$ we lose the needed equal income property (see Theorem 4.4 in [9]). Indeed, even if $e_t(\omega) = e$ for each $t \in T$ and each $\omega \in \Omega$, in the associated economy $\mathcal{E}^*$,

$$e^*_{(t,E)}(\omega) = \begin{cases} e_t(\omega) & \text{if } \omega \in E \\ 0 & \text{otherwise,} \end{cases}$$

and hence different agents may have different initial endowment.

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