Endogenous lifetime, accidental bequests and economic growth

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Abstract This paper introduces unintentional bequests in a closed economy overlapping generations model à la Chakraborty [1]. We show that poverty traps due to scarce public investments in health can exist. However, and most important, the existence of unintentional bequests makes the health tax rate to play a prominent role in determining the stability of the equilibrium in rich economies. Indeed, non-monotonic dynamics, Neimark-Sacker bifurcations and deterministic chaos can occur depending on the size of the public health system.

Keywords Accidental bequests; Endogenous lifetime; Health; OLG model

JEL Classification C62; I18; J18; O4

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1. Introduction

The demographic variables, namely fertility and longevity, have been recognised to be an important determinant of the process of economic growth and development (see, amongst many others, Becker and Barro [2]; Mason [3]; Barro and Becker [4]; de la Croix and Licandro [5]; Barro and Sala-i-Martin [6]; Fogel [7], [8]; Galor and Weil [9], [10]; Galor and Moav [11]; de la Croix and Doepke [12], [13]; Moav [14]; Kraay and Raddatz [15]). Indeed, the causes and the economic consequences of the reduction in both birth and mortality rates, observed in several developed countries in the recent decades (see, e.g., Livi-Bacci [16]), have led economists to deeply inquire about the interrelationships between demographic and macroeconomic outcomes. One reason for this being the tremendous policy consequences that the steadily reducing number of young (active) people in total population as well as the steadily raising number of elderly can produce in the near future.[1] Indeed, there exists a burgeoning macroeconomic theoretical literature centred on growth models with overlapping generations (OLG) that tries to shed light on the nature and causes of economic development by assuming either exogenous fertility (Chakraborty [1]; Chakraborty and Das [20]; Bhattacharya and Qiao [21]; de la Croix and Ponthière [22]; Leung and Wang [23]) or endogenous fertility (Blackburn and Cipriani [24]) together with endogenous lifetime.

In particular, Chakraborty [1] introduces endogenous risky lifetime in the standard two-period OLG model by Diamond [25] and considers a market for annuities (of the deceased persons) where savings are intermediated through mutual funds. He assumes that the probability of surviving from the first period (youth) to the next (old age) depends on the individual health status, which is

[1] As an example, we may think about the provision of public pensions, which are mainly organised on a pay-as-you-go basis in several European countries: i.e., the income of current workers is taxed away by the government to finance the benefits received by current pensioners. Indeed, there exists extensive debates between economists to find appropriate ways to reform the social security system (see, e.g., Boeri et al. [17], [18]; Cigno [19]) because of concerns due to the so-called population ageing.

augmented through the financing of public investments in health. He shows that poverty traps due to scarce health investments can actually exist, because a shorter life span acts as a disincentive to save and accumulate capital further on. Chakraborty and Das [20] build on a model with human capital, private health expenditure and intentional bequests (i.e., inter vivos transfers) to study the problem of persistent inequality between rich and poor countries. Bhattacharya and Qiao [21] assume that the individual lifetime is dependent of the health status which is, in turn, augmented by private health investments accompanied by a tax-financed public health program, and show that the economy may be exposed to endogenous fluctuations and even chaotic motions when the private and public inputs in the longevity function are fairly complementary. Leung and Wang [23] study a model with a private system of health care services and find that saving and health care are complementary. From a normative point of view, instead, de la Croix and Ponthière [22] show that the steady-state Golden Rule of capital accumulation in an economy with endogenous lifetime is lower than that of the standard Diamond’s [25] one. Blackburn and Cipriani [24] show in a model with endogenous fertility and endogenous longevity and where individuals accumulate human capital through education, that two regimes of development can exist: the former being characterised by low income, high fertility and a relatively low length of life, the latter by high income, low fertility and a relatively high length of life. Their model accords with the empirical evidence of the Demographic Transition.

The distinctive feature of the introduction of longevity in the basic OLG model by Diamond [25] is the treatment of saving of the deceased persons. There are two polar cases, which obviously also embody intermediate ones: (*i*) perfect annuities markets, i.e., savings are fully annuitized, which implies that old survivors will benefit not only from their own past saving plus interest, but also from the saving plus interest of those who have deceased, and savings are intermediated through mutual funds, which invest these savings and guarantee a return factor, determined by the interest factor (that prevails in the capital market) earned by the fund on its investments divided by the longevity rate, to the surviving old insured inhabitants of economy; (*ii*) no annuities markets, as in

Abel [26], i.e., savings of the deceased persons become accidental or unintentional bequests[2] to their progeny. While Chakraborty [1] assumes the former case, in this paper we focus on the latter one. This implies a noteworthy feature that indeed deserves attention: the delayed levels of the longevity rate also matter, because the dynamics of the economy is characterised by a two-dimensional non-linear system instead of the one-dimensional system that describes the dynamics in the model by Chakraborty [1].

The main object of the present paper, therefore, is to assess the role that public investments in health can play on economic growth and stability in an OLG à la Chakraborty [1] but assuming unintentional bequests rather than a market for annuities. While in the model by Chakraborty [1] both the low and high steady state are always locally asymptotically stable with monotonic trajectories, we show here that the existence of unintentional bequests leaved by the deceased persons makes the financing of health care services responsible of the existence of non-monotonic dynamics, Neimark-Sacker bifurcations and deterministic chaos.

The rest of the paper is organised as follows. Section 2 presents the model. Section 3 analyses the dynamics of the our two-dimensional economy. Section 4 concludes.

2. The model

2.1. Individuals

Consider a general equilibrium OLG closed economy populated by perfectly rational and identical two-period lived individuals. Life is divided between youth and old age (as in Diamond [25]). The former is a working time fixed with certainty, while the latter is a retirement time whose length is uncertain. Population is fixed and constant at N in every period. We assume that the typical agent

[2] Other major bequest motives are altruism and exchange. While there is no consensus on which motive dominates (see, e.g., Altonji et al. [27]), Hurd [28] argues that bequests are largely accidental.

in each generation is either dead or alive at the beginning of the retirement period with probability $1-\pi$ and π, respectively. When she is young, the representative individual of generation t is endowed with one unit of labour inelastically supplied to firms, while receiving the wage w_t (used for consumption and saving purposes). Moreover, the government collects wage income taxes at the constant rate $0<\tau<1$ to finance a balanced-budget public health programme. Since agents do not know when they will die, additional unintentional bequests may occur.[3] If the typical agent of generation t dies at the onset of old age (with probability $1-\pi_t$), his accumulated savings are bequeathed in full to his heirs. To keep the representative agent formulation tractable, the bequests

$$b_{t+1}=(1-\pi_t)(1+r^e{}_{t+1})s_t\,, \tag{1}$$

where s_t is saving and $r^e{}_{t+1}$ the expected interest rate accrued from time t to time $t+1$, are assumed to be equally divided among all the young persons in every generation.[4] Therefore, the budget constraint of both the young and old of generation t read, respectively, as:

$$c_{1,t}+s_t=w_t(1-\tau)+b_t\,, \tag{2.1}$$

$$c_{2,t+1}=(1+r^e{}_{t+1})s_t\,, \tag{2.2}$$

where $c_{1,t}$ and $c_{2,t+1}$ are young-age consumption and old-age consumption. Eq. (2.1) implies that bequests are equally allocated across all members within a generation.

By taking factor prices and bequests as given, the representative individual of generation t chooses how much to save out of her disposable income to maximise the lifetime utility function

$$U_t=\ln(c_{1,t})+\pi_t\ln(c_{2,t+1})\,, \tag{3}$$

subject to Eqs. (2). The first order conditions for an interior solution of the problem are given by:

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[3] Note that, different from Chakraborty [1], our model is developed by assuming unintentional bequests without markets for annuities.

[4] This means that the bequest dependent wealth distribution is uniform, as in Hubbard and Judd [29]. This assumption allows us to conduct a representative agent analysis to specifically focus on the effects of changes in longevity.

$$\frac{c_{2,t+1}}{c_{1,t}} \cdot \frac{1}{\pi_t} = 1 + r_{t+1}. \tag{4.1}$$

Eq. (4.1) equates the marginal rate of substitution between consumption when young and when old to the interest factor determined on the capital market. From Eq. (4.1) it is clear that in an economy with accidental bequests, individuals take into account the effects of longevity on the inter-temporal substitution between consumption when young and when old, that is they take into account the social benefit of an increase in individual longevity. In particular, an increase in longevity makes convenient to postpone consumption in the future. This represents a first difference between an economy with accidental bequests and an economy with perfect annuities market. Indeed, in the latter case, each individual does take into account the (social) benefits of an increase in public healthcare investments on (individual) health and longevity.

Combining Eqs. (2.1), (2.2) and (4), we obtain the saving function as follows:

$$s_t = \frac{\pi_t}{1 + \pi_t}\left[w_t(1-\tau) + b_t\right], \tag{4.2}$$

where b_t is determined by the one-period backward Eq. (1).

2.2. Firms

Identical firms act competitively on the market. At time t, the homogeneous output Y_t is produced by combining capital (K_t) and labour ($L_t = N$ in equilibrium) through the constant returns to scale Cobb-Douglas technology $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$, where $A > 0$ is a scale parameter and $0 < \alpha < 1$ the output elasticity of capital. Since capital totally depreciates at the end of each period and output is sold at unit price, profit maximisation implies that factor inputs are paid their marginal products, that is:

$$r_t = \alpha A k_t^{\alpha - 1} - 1, \tag{5}$$

$$w_t = (1-\alpha) A k_t^{\alpha}, \tag{6}$$

where $k_t := K_t / N$ is the per young stock of capital.

2.3. The public health system and endogenous lifetime

We follow Chakraborty [1] and assume that at time t health investments per young person, h_t, are financed at a balanced budget with labour income taxes levied by the government at the (constant) rate $0 < \tau < 1$, that is:

$$h_t = \tau w_t, \tag{7}$$

the right-hand side being the tax receipt.

Moreover, the survival probability at the end of youth of an individual born at t, π_t, is assumed to positively depend on the individual health status, which is in turn augmented by health investments h_t, so that $\pi_t = \pi(h_t)$. Following Blackburn and Cipriani [24] and de la Croix and Ponthière [22], we specialise this relationship with the following S-shaped function:

$$\pi_t = \pi(h_t) = \frac{\hat{\pi} \Delta h_t^{\delta}}{1 + \Delta h_t^{\delta}}, \tag{8}$$

where $\delta, \Delta > 0$, $0 < \hat{\pi} \leq 1$, $\pi'_h(h) > 0$, $\lim_{h \to \infty} \pi(h) = \hat{\pi} \leq 1$, $\pi''_{hh}(h) < 0$ if $\delta \leq 1$ and $\pi''_{hh}(h) \gtreqless 0$ for any $h \lesseqgtr h_T := \left[\frac{\delta - 1}{(1+\delta)\Delta} \right]^{\frac{1}{\delta}}$ if $\delta > 1$. The parameter $\hat{\pi}$ captures the intensity of the efficiency of health investments on longevity. A rise in $\hat{\pi}$ may be interpreted as exogenous medical advances. The parameters δ and Δ determine both the turning point of $\pi'_h(h)$ and speed of convergence of the rate of longevity up to the saturating value $\hat{\pi}$. In particular, δ measures how an additional unit of health capital is transformed into higher longevity through the health technology. If $\delta \leq 1$, $\pi(h)$ is concave for any h and, hence, no threshold effects exist so that longevity increases less than proportionally from zero up to $\hat{\pi}$ as h rises. If $\delta > 1$ the longevity function is S-shaped and threshold effects exist: i.e., longevity increases more (less) than proportionally before (after) the threshold h_T. This means that the more threshold effects are intense (high values of δ), the slower an additional unit of health capital is transformed into a higher life span when h is relatively low, while reaching the saturating value π_1 more efficiently and rapidly as h becomes larger (see, e.g., Martikainen et al. [30] and Fioroni [31] for empirical evidence). In this case δ measures the intensity of threshold effects of the accumulated health capital as an inducement to higher longevity.

2.4. Equilibrium in an economy with accidental bequests

Given the government budget Eq. (7), equilibrium the capital market can be written as:

$$k_{t+1} = s_t . \tag{9}$$

Combining Eqs. (9), (4.2), the one-period backward Eqs. (1) and (9), equilibrium implies:

$$k_{t+1} = \frac{\pi_t}{1+\pi_t}\left[w_t(1-\tau) + (1-\pi_{t-1})(1+r_t)k_t\right], \tag{10}$$

which is independent of expectations about future factor prices.

Now, using the equilibrium conditions in the factor markets Eqs (5) and (6), and knowing that through Eqs. (6)-(8) the longevity function can be expressed as $\pi_t = \pi(k_t)$, the dynamics of capital is described by the following second order nonlinear difference equation:

$$k_{t+1} = \frac{\pi(k_t)}{1+\pi(k_t)} A k_t^{\alpha}\{(1-\alpha)(1-\tau) + \alpha[1-\pi(k_{t-1})]\}, \tag{11}$$

or, alternatively

$$k_{t+1} = \frac{\hat{\pi} z_1(\tau) A k_t^{\alpha(1+\delta)}}{1+(1+\hat{\pi}) z_1(\tau) k_t^{\alpha\delta}}\left[z_2(\tau) + \alpha\frac{1+(1-\hat{\pi}) z_1(\tau) k_{t-1}^{\alpha\delta}}{1+z_1(\tau) k_{t-1}^{\alpha\delta}}\right], \tag{12}$$

where $z_1(\tau) := \Delta[\tau(1-\alpha)A]^{\delta} > 0$ and $z_2(\tau) := (1-\alpha)(1-\tau) > 0$. Fixed points of the system characterised by Eq. (12) are determined as $k_{t+1} = k_t = k_{t-1} = \bar{k}$. Indeed, they are represented by the roots of the following function:

$$F(\bar{k}) = \frac{\hat{\pi} z_1(\tau) A \bar{k}^{\alpha(1+\delta)}}{1+(1+\hat{\pi}) z_1(\tau)\bar{k}^{\alpha\delta}}\left[z_2(\tau) + \alpha\frac{1+(1-\hat{\pi})z_1(\tau)\bar{k}^{\alpha\delta}}{1+z_1(\tau)\bar{k}^{\alpha\delta}}\right] - \bar{k} = 0 . \qquad (12')$$

In the next section we study the existence and the stability properties of the fixed points of the time map Eq. (12).

3. Existence and local stability of the fixed points

We now discuss existence and stability of both the zero and positive steady states of Eq. (12), starting from the analysis of $\bar{k} = 0$. The qualitative results of the model are different depending on the mutual relationship between the parameters α and δ. Moreover, a crucial role of the health tax rate τ on (local) stability is established (see Section 3.1).

From Eq. (12’) the following proposition holds.

Proposition 1. $\bar{k} = 0$ *is a fixed point of the system described by Eq. (12).*

Proof. The proof is obvious from Eq. (12). **Q.E.D.**

In order to study the local stability properties of the fixed points, starting from the case $\bar{k} = 0$, we transform the system of a single second order difference equation (12) into a system of two first order difference equations (see Azariadis [32] and Grandmont et al. [33]). Let $x_t := k_{t-1}$ be a new supporting variable. Then Eq. (12) can be written as:

$$\begin{cases} k_{t+1} = \dfrac{\hat{\pi} z_1(\tau) A k_t^{\alpha(1+\delta)}}{1+(1+\hat{\pi}) z_1(\tau) k_t^{\alpha\delta}} \left[z_2(\tau) + \alpha \dfrac{1+(1-\hat{\pi}) z_1(\tau) x_t^{\alpha\delta}}{1+z_1(\tau) x_t^{\alpha\delta}} \right]. \\ x_{t+1} = k_t \end{cases} \tag{13}$$

Local stability of the fixed points is studied by means of the linear approximation given by the Jacobian matrix of partial derivatives (J) evaluated at $\bar{k}$, which for the system (15) is:

$$J = \begin{pmatrix} J_k & J_x \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial k_{t+1}}{\partial k_t} & \dfrac{\partial k_{t+1}}{\partial x_t} \\ \dfrac{\partial x_{t+1}}{\partial k_t} & \dfrac{\partial x_{t+1}}{\partial x_t} \end{pmatrix}, \tag{14}$$

where all derivatives are evaluated at the steady state $\bar{k}$ and

$$J_k = \frac{\hat{\pi} z_1(\tau) \alpha A \bar{k}^{\alpha(1+\delta)-1}}{1+(1+\hat{\pi}) z_1(\tau) \bar{k}^{\alpha\delta}} \cdot \left[1 + \frac{\delta}{1+(1+\hat{\pi}) z_1(\tau) \bar{k}^{\alpha\delta}} \right] \cdot \left[z_2(\tau) + \alpha \frac{1+(1-\hat{\pi}) z_1(\tau) \bar{k}^{\alpha\delta}}{1+z_1(\tau) \bar{k}^{\alpha\delta}} \right] > 0, \tag{15}$$

$$J_x = \frac{-\delta z_1(\tau) \hat{\pi} \alpha^2 A \bar{k}^{\alpha(1+2\delta)-1}}{\left[1+z_1(\tau)\bar{k}^{\alpha\delta}\right]^2 \cdot \left[1+(1+\hat{\pi}) z_1(\tau) \bar{k}^{\alpha\delta}\right]} < 0. \tag{16}$$

The trace and determinant of (14) are $tr(J) = J_k > 0$ and $\det(J) = -J_x > 0$, respectively, so that the characteristic polynomial is:

$$F(\lambda) = \lambda^2 - tr(J)\lambda + \det(J), \tag{17}$$

whose discriminant $D := [tr(J)]^2 - 4\det(J)$ can either be positive or negative. Therefore, complex eigenvalues can exist and a fixed point may be periodic. As is known, bifurcation theory describes the way the topological features of the system (such as the number of stationary points or their stability) vary as parameter values changes. For a system in two dimensions, the stability conditions ensuring that both eigenvalues remain within the unit circle[5] are:

[5] If no eigenvalues of the linearised system around the fixed points of a first order discrete system lie on the unit circle, then such points are defined *hyperbolic*. Roughly speaking, at non-hyperbolic points topological features are not structurally stable.

$$\begin{cases} (i) \quad F := 1 + tr(J) + \det(J) > 0 \\ (ii) \quad TC := 1 - tr(J) + \det(J) > 0 \\ (iii) \quad H := 1 - \det(J) > 0 \end{cases}. \tag{18}$$

The violation of any single inequality in (18), with the other two being simultaneously fulfilled leads to: (*i*) a flip bifurcation (a real eigenvalue that passes through -1) when $F = 0$; (*ii*) a fold or transcritical bifurcation (a real eigenvalue that passes through $+1$) when $TC = 0$; (*iii*) a Neimark-Sacker bifurcation (i.e., the modulus of a complex eigenvalue pair that passes through 1) when $H = 0$, namely $\det(J) = 1$ and $|tr(J)| < 2$.

As regards the stability properties of the zero equilibrium, we have the following proposition.

Proposition 2. *(1) Let* $0 < \alpha < 1/(1+\delta)$ *hold. Then,* $\bar{k} = 0$ *is locally unstable. (2) Let* $1/(1+\delta) < \alpha < 1$ *hold. Then,* $\bar{k} = 0$ *is locally asymptotically stable.*

Proof. If $0 < \alpha < 1/(1+\delta)$, then from (15), (16) and (18) evaluated at $\bar{k} = 0$ we find that either $1 - \det(J) = -\infty < 0$ or $1 - tr(J) + \det(J) = -\infty < 0$ holds depending on whether $0 < \alpha < 1/(1+2\delta)$ or $1/(1+2\delta) < \alpha < 1/(1+\delta)$, respectively. This proves Point 1 of Proposition 2. If $1/(1+\delta) < \alpha < 1$, then from (15), (16) and (18) evaluated at $\bar{k} = 0$ we get $tr(J) = 0$ and $\det(J) = 0$. Then $1 + tr(J) + \det(J) = 1 > 0$, $1 - tr(J) + \det(J) = 1 > 0$ and $1 - \det(J) = 1 > 0$. This proves Point 2 of Proposition 2. **Q.E.D.**

As Proposition 2 shows, the zero equilibrium of the dynamic system described by Eq. (12) can either be stable or unstable depending on the relative size of the output elasticity of capital. The existence (and the stability properties) either of a single positive fixed point or multiplicity of positive fixed points strictly depends on whether $\bar{k} = 0$ is stable or unstable. In particular, below we show through numerical simulations that the following results generically hold.

Result 1. *If* $0 < \alpha < 1/(1+\delta)$*, then the dynamic system described by Eq. (12) admits two steady states* $\{0, \bar{k}\}$*, where* $0 < \bar{k}$ *, the former being locally unstable and the latter locally asymptotically stable.*

Result 2. *If* $1/(1+\delta) < \alpha < 1$*, then the dynamic system described by Eq. (12) admits either the stable zero steady state only (if the technological scale parameter* A *in the Cobb-Douglas production function is fairly low), or three steady states* $\{0, \bar{k}_1, \bar{k}_2\}$ *(if* A *is fairly high), where* $0 < \bar{k}_1 < \bar{k}_2$*, the first is locally asymptotically stable, the second is a saddle point and third may be locally asymptotically stable or unstable.*

Result 3. *If* $1/(1+\delta) < \alpha < 1$ *and* $\delta > 1$*, then the dynamics (around the highest steady state* $\bar{k}_2$*) can be non-monotonic. Moreover, depending on the size of the health tax rate* τ *, Neimark-Sacker bifurcations and deterministic chaos can occur.*

Moreover, the following proposition holds:

Proposition 3. *A flip bifurcation can never appear.*

Proof. The proof is obvious as the sign of both the trace and determinant of J , evaluated at the steady state $\bar{k}$, are $tr(J) > 0$ and $\det(J) > 0$. Therefore, $1 + tr(J) + \det(J) > 0$ always holds. **Q.E.D.**

3.1. Stability of positive steady states and bifurcations

3.1.1. Case $0 < \alpha < 1/(1+\delta)$

In this case the position of the eigenvalues of the Jacobian matrix J relative to the unit circle, evaluated at the unique positive steady state $\bar{k}$, are smaller than unity and the three conditions stated in (18) are fulfilled.[6]

3.1.2. Case $1/(1+\delta)<\alpha<1$

In this case the position of the eigenvalues of the matrix J relative to the unit circle is unclear, and the positive steady state $\bar{k}_2$ can either be stable or unstable. Since the three conditions in (18) cannot easily be treated in a neat analytical form when they are evaluated at the positive steady states $\bar{k}_1$ and $\bar{k}_2$, then in order to illustrate Results 2 and 3 above, we resort to numerical simulations to show that Neimark-Sacker bifurcations and deterministic chaos can occur when threshold effects of health investments on longevity exist. For doing this, we choose the following configurations of parameters: $\hat{\pi}=0.99$, $\Delta=1$, $\delta=20$, $\alpha=0.4$ (which may be considered as an average value between the values of the output elasticity of capital in developed and developing countries, which, according to, e.g., Kraay and Raddatz [15], are $\alpha=0.33$ and $\alpha=0.5$, respectively)[7] and $A=4.5$. Figures 1 represents the bifurcation diagram for τ with respect to the steady state values of the variable k.

[6] This case in uninteresting from a dynamical point of view and thus we do not present numerical examples (which are of course available upon request) in such a case.

[7] See Gollin [34] and Kehoe and Perri [35] for estimates on the output elasticity of capital in developed countries.

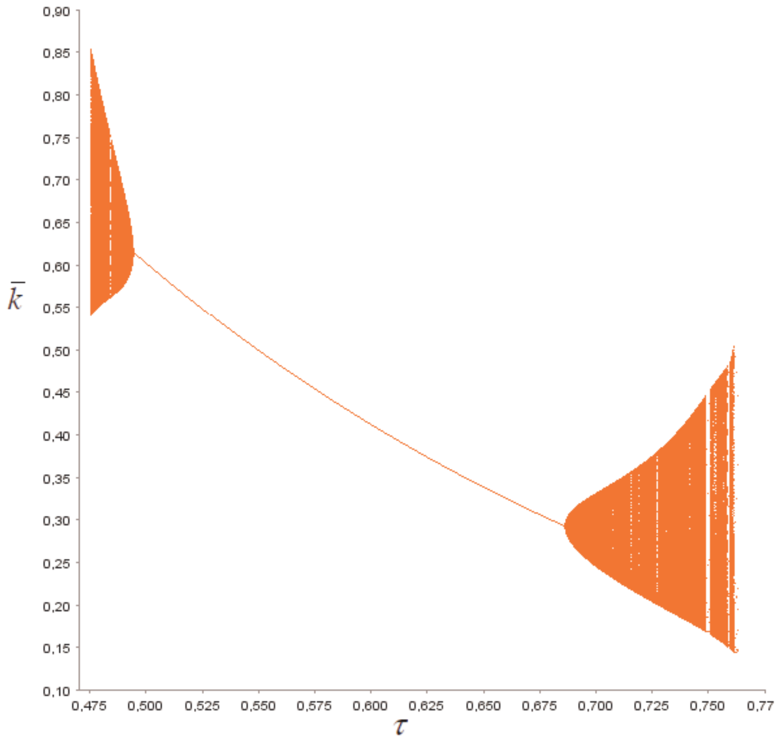

Figure 1. Bifurcation diagram for τ ($k_0 = x_0 = 1$): an enlarged view for $0.473 < \tau < 0.77$ and $0.1 < \bar{k} < 0.9$.

Figure 1 clearly reveals that a double Neimark-Sacker bifurcation exists when the health tax rate varies. In fact, when $\tau = 0.4945$ we get $1 + tr(J) + \det(J) = 2.9613 > 0$, $1 - tr(J) + \det(J) = 1.0403 > 0$, $1 - \det(J) = 0$ and $tr(J) = 0.9604$. Moreover, a second Neimark-Sacker bifurcation occurs when $\tau = 0.6859$, corresponding to which $1 + tr(J) + \det(J) = 2.7186 > 0$, $1 - tr(J) + \det(J) = 1.2827 > 0$, $1 - \det(J) = 0$ and $tr(J) = 0.7179$.

Simulations (not reported in the paper for economy of space) reveal that when $\delta = 1$ (i.e. no threshold effects of health investments on longevity exist), Results 1 and 2 resembles Point (i) of Proposition 1 by Chakraborty [1, p. 126] in a model with a perfect market for annuities. This means that there exist: (*i*) one locally asymptotically stable steady state (as in Diamond [25]) when $0 < \alpha < 1/2$, and (*ii*) two locally asymptotically stable steady states $\{0, \bar{k}_2\}$ when $1/2 < \alpha < 1$.[8]

If we slightly change the initial conditions from $k_0 = x_0 = 1$ to $k_0 = x_0 = 0.2$, Figures 2 (bifurcation diagram) and 3 (the largest Lyapunov exponent plotted against the parameter τ) reveal that deterministic chaos occur.

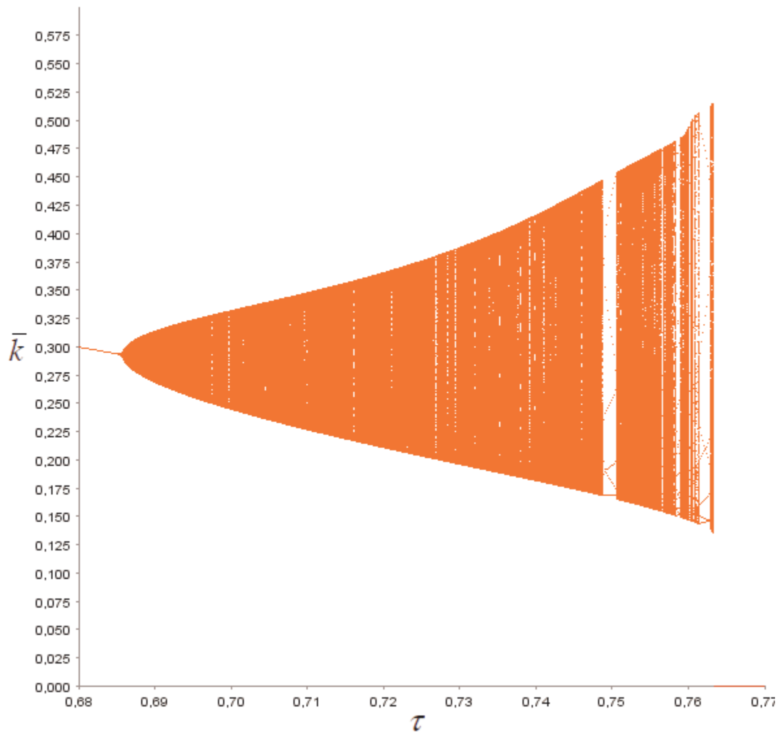

Figure 2. Bifurcation diagram for τ ($k_0 = x_0 = 0.2$): an enlarged view for $0.68 < \tau < 0.77$ and $0 < \bar{k} < 0.6$.

[8] It should be noted that when $1/(1+\delta) < \alpha < 1$ and $\delta > 1$ both the low and high steady states in the model by Chakraborty [1] are always locally asymptotically stable with monotonic trajectories.

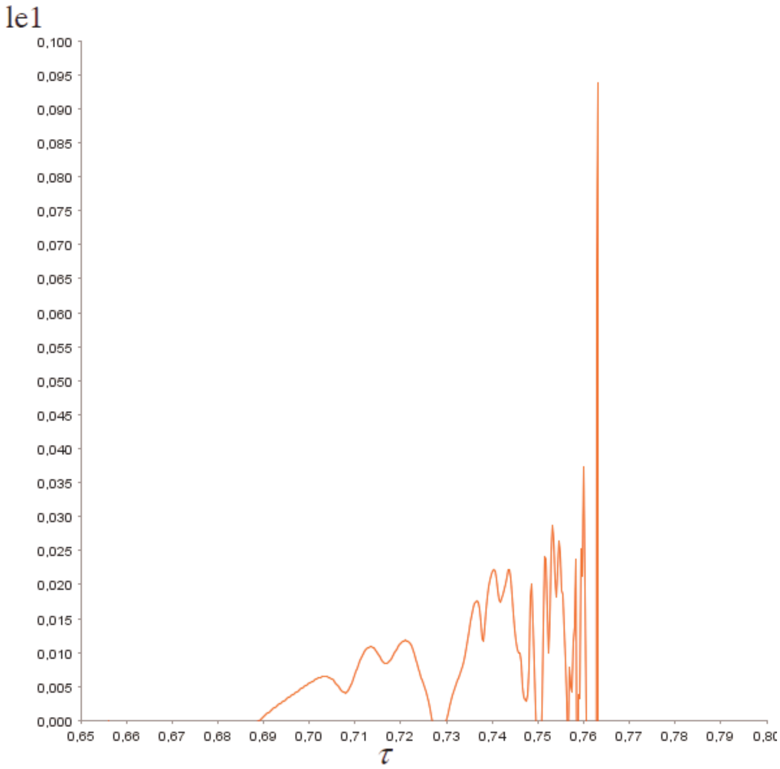

Figure 3. Largest Lyapunov exponent plotted against the parameter τ .

3.2. Sketch of Global Analysis

When two locally stable attractors coexist, trajectories may converge to one or the other, depending on the initial conditions. In our model, when the condition given in Result 2 holds, a low and a high valued attractors coexist and it becomes important to identify their basins of attraction (that is, the set of initial conditions leading to one attractor or to the other one). The study of the basins configurations is called *global analysis* and differs from the local analysis because it does not consider the linearization of the map around a steady state, while considering the map in its original nonlinear formulation. Global analysis is a mix of by analytical and numerical tools. This study is quite important because, especially in discrete-time systems, it is not always true that initial conditions closer to one attractor characterise trajectories leading towards it. In our case this means that we are not sure that when two locally stable attractors coexist, high (resp. low) initial values of the variables lead to convergence towards the high valued (resp. low valued) attractor is. In fact, basins of attractions can be characterized either by a simple or a complicated structure. In particular, they can be *connected* (that is only made up by a compact subset of the phase space containing the attractor itself) or *disconnected* (that is made up by the union of the subset of initial conditions around the attractor, called *immediate basin*, and by its disconnected preimages).

Given the complexity of our map, it is not possible to analytically detect global bifurcation values. It is still possible to produce some computer-assisted proofs by running some simulations. Let us consider now a parameters configuration leading to coexistence between two locally stable attractors (Figure 4.a). This means that the condition given in Result 2 is fulfilled. In this case, besides the locally stable steady state $\{0\}$ and the saddle point $\{\bar{k}_1\}$, there exists a locally attractive closed invariant curve originated from the Neimark-Sacker bifurcation of the steady state $\{\bar{k}_2\}$. In Figure 4.a the two basins of attraction are denoted by different colours. They are separated by the stable manifold of the saddle point and they are connected. If we increase the value of τ, for instance, we can see that the configuration of the basins changes drastically, becoming disconnected (Figure 4.b). In such a case, even starting from initial conditions closer to the closed invariant curve (like A and B) trajectories converge towards the origin. In Figure 4.c we can see, with a larger scale, the value of the health tax rate at which this global bifurcation occurs. Initially, the basin of attraction of the origin in made by the immediate basin H and by its preimages of rank 1, 2 and 3 (H^{-1}, H^{-2} and H^{-3}, respectively). Then, by increasing the value of the health tax rate, the region H and its preimages grow until they merge originated the structure of Figure 4.b (see Mira et al. [36, 37]). We leave for a more technical paper the deeper analysis of this global bifurcation. For now we only underscore that whenever two locally stable attractors coexist it is not obvious to which a generic trajectory converges, even starting very close to one of them.

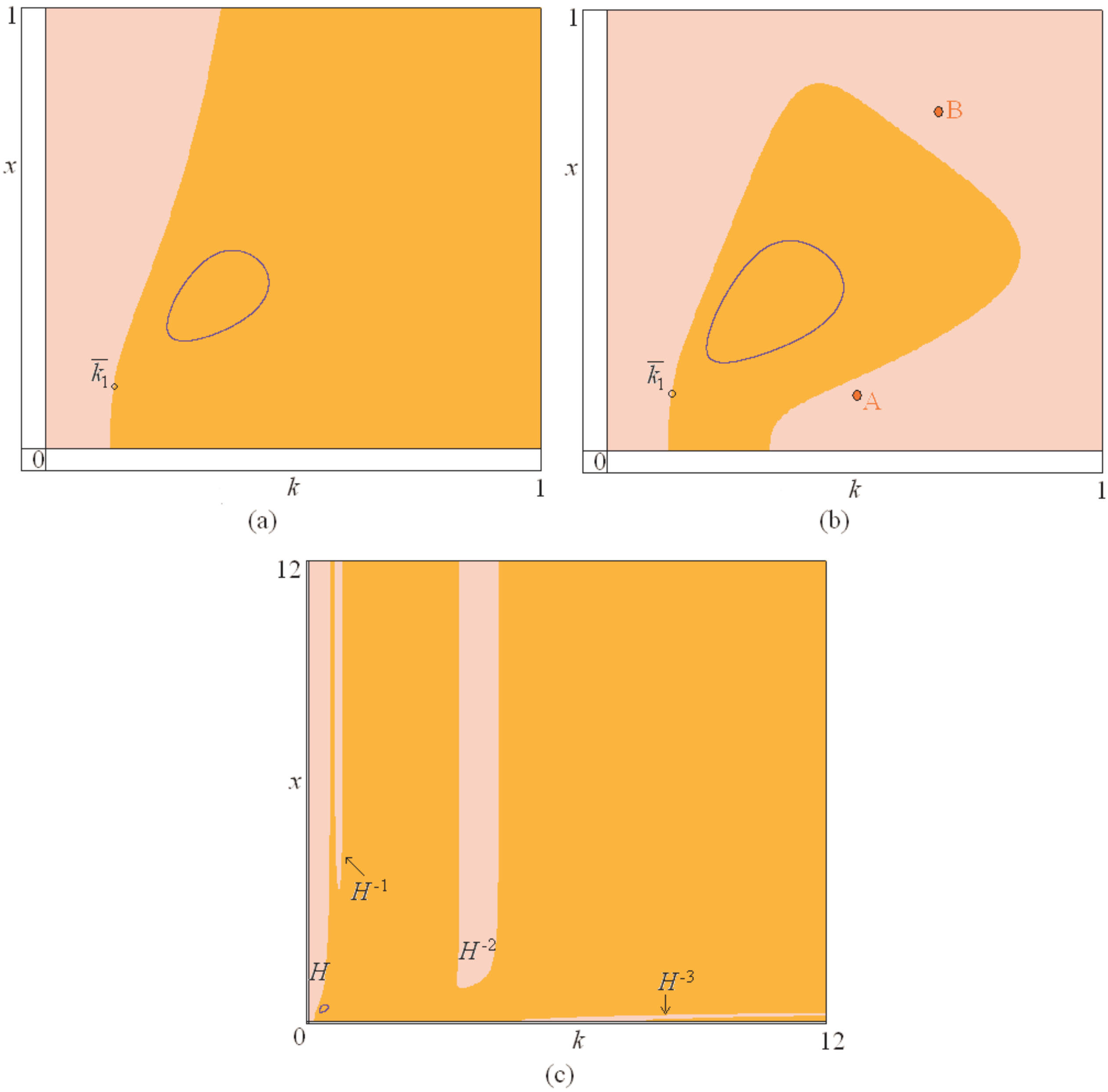

Figure 4. Basin of attraction of the steady state $\{0\}$ (in pink) and of the closed invariant curve around $\{\bar{k}_2\}$. The parameters kept fixed are $A = 5.35$, $\alpha = 0.5$, $\delta = 6$, $\Delta = 0.9$ and. The health tax rate τ is 0.7815 in (a), 0.8 in (b) and 0.7817 in (c).

4. Conclusions

We studied the dynamical and stability properties of a general equilibrium overlapping generations economy with public health investments that affect the lifetime of people. We showed that the existence of unintentional bequests, rather than a market for annuities, makes the dynamics of the economy be characterised by a two-dimensional discrete non-linear system. This introduces the possible existence of non-monotonic dynamics, Neimark-Sacker bifurcation and deterministic chaos when the health tax rate changes and threshold effects of health investments on longevity exist.

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